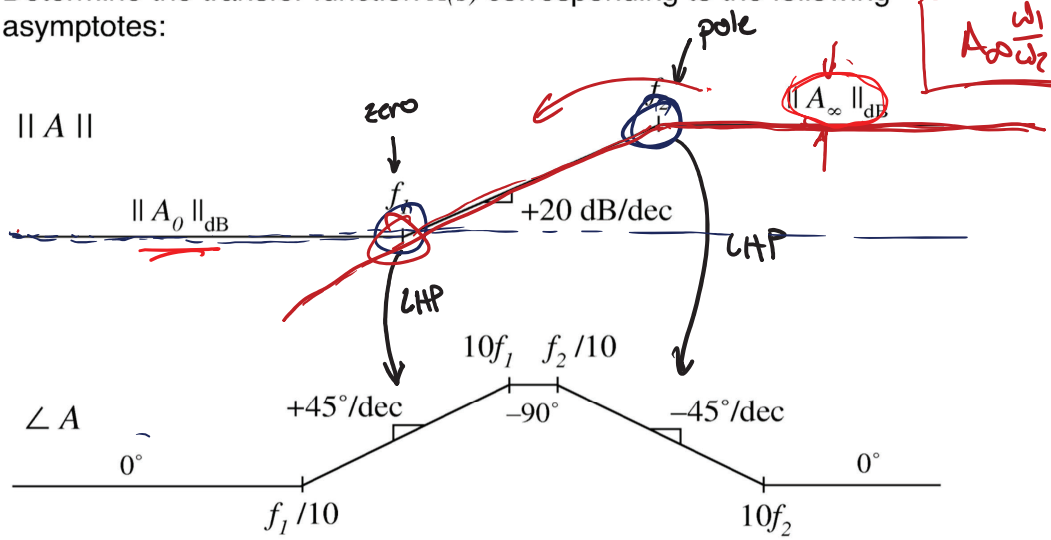
# Example 2

$A(j\omega)$

$$A(s) = A_0 \frac{\left(\frac{s}{\omega_1}+1\right)}{\left(\frac{s}{\omega_2}+1\right)} = A_\infty \frac{\left(1+\frac{\omega_1}{s}\right)}{\left(1+\frac{\omega_2}{s}\right)} = A_\infty \frac{\frac{\omega_1}{\cancel{s}}\left(1+\frac{s}{\omega_1}\right)}{\frac{\omega_2}{\cancel{s}}\left(1+\frac{s}{\omega_2}\right)}$$

$$\boxed{A_\infty \frac{\omega_1}{\omega_2} = A_0}$$

Determine the transfer function $A(s)$ corresponding to the following asymptotes:



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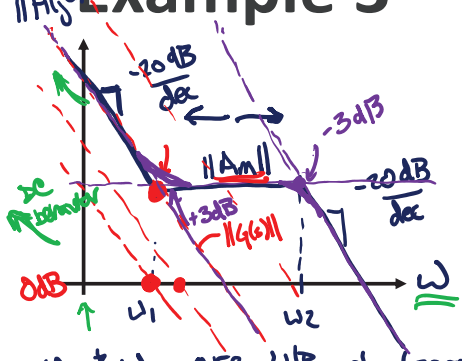
# Example 3




$\omega_1$ & $\omega_2$ are LHP poles/zeroes.

$$G(s) = \frac{1}{s}(G_x)$$

$$\|G(s)\|_{dB} = 20\log_{10}(G_x) - 20\log_{10}(\omega)$$

$$G_x = \omega_1 A_m$$

$$H(s) = A_m\omega_1 \frac{1}{s} \frac{\left(1+\frac{s}{\omega_1}\right)}{\left(1+\frac{s}{\omega_2}\right)}$$

A little easier w/ inverted

$$H(s) = A_m \frac{\left(1+\frac{\omega_1}{s}\right)}{\left(1+\frac{s}{\omega_2}\right)}$$

Bode plots → Asymptotes of s-domain function

(1) multiplication → addition in log domain

(2) addition → max() in log domain

$$1+\frac{\omega}{\omega_0} \approx \begin{cases} 1, & \omega \ll \omega_0 \\ \omega/\omega_0, & \omega \gg \omega_0 \end{cases}$$

Note: where is DC?

DC ↔ $\omega \to 0$

$\log(0) \to -\infty$

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# Resonant Poles

2nd order LPF

Example

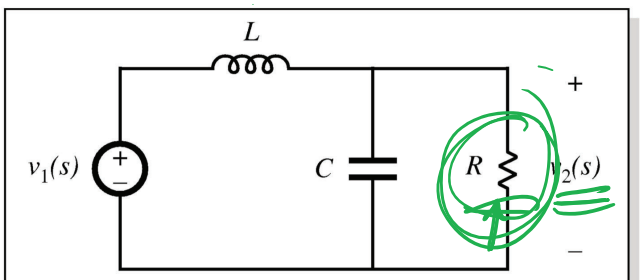
$$G(s) = \frac{v_2(s)}{v_1(s)} = \frac{1}{1 + s\frac{L}{R} + s^2 LC}$$

Second-order denominator, of the form

$$G(s) = \frac{1}{1 + a_1 s + a_2 s^2}$$

standard form for 2nd order polynomial

with $a_1 = L/R$ and $a_2 = LC$




*Two-pole low-pass filter example*

How should we construct the Bode diagram?



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# Standard Form for Complex Poles

$$\frac{2\zeta}{\omega_0} = \frac{1}{Q\omega_0} \qquad \zeta = \frac{1}{2Q}, \quad Q = \frac{1}{2\zeta}$$

$$\frac{1}{1 + a_1 s + a_2 s^2}$$

$$\begin{cases} \frac{1}{\omega_0^2} = a_2 \\ \frac{1}{Q\omega_0} = a_1 \end{cases}$$

$$G(s) = \frac{1}{1 + 2\zeta \frac{s}{\omega_0} + \left(\frac{s}{\omega_0}\right)^2}$$

zeta

or

$$G(s) = \frac{1}{1 + \frac{s}{Q\omega_0} + \left(\frac{s}{\omega_0}\right)^2}$$

$$\begin{cases} \omega_0 = \frac{1}{\sqrt{a_2}} \\ Q = \frac{\sqrt{a_2}}{a_1} \end{cases}$$

- When the coefficients of $s$ are real and positive, then the parameters $\zeta$, $\omega_0$, and $Q$ are also real and positive
- The parameters $\zeta$, $\omega_0$, and $Q$ are found by equating the coefficients of $s$
- The parameter $\omega_0$ is the angular corner frequency, and we can define $f_0 = \omega_0/2\pi$
- The parameter $\zeta$ is called the *damping factor*. $\zeta$ controls the shape of the exact curve in the vicinity of $f = f_0$. The roots are complex when $\zeta < 1$.
- In the alternative form, the parameter $Q$ is called the *quality factor*. $Q$ also controls the shape of the exact curve in the vicinity of $f = f_0$. The roots are complex when $Q > 0.5$.



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# The Q Factor

In a second-order system, $\zeta$ and $Q$ are related according to

$$Q = \frac{1}{2\zeta}$$

$Q$ is a measure of the dissipation in the system. A more general definition of $Q$, for sinusoidal excitation of a passive element or system is

$$Q = 2\pi \frac{\text{(peak stored energy)}}{\text{(energy dissipated per cycle)}}$$

For a second-order passive system, the two equations above are equivalent. We will see that $Q$ has a simple interpretation in the Bode diagrams of second-order transfer functions.



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# Magnitude Asymptotes

$s \to j\omega$

In the form $$G(s) = \frac{1}{1 + \frac{s}{Q\omega_0} + \left(\frac{s}{\omega_0}\right)^2}$$

let $s = j\omega$ and find magnitude: $$\| G(j\omega) \| = \frac{1}{\sqrt{\left(1 - \left(\frac{\omega}{\omega_0}\right)^2\right)^2 + \frac{1}{Q^2}\left(\frac{\omega}{\omega_0}\right)^2}}$$

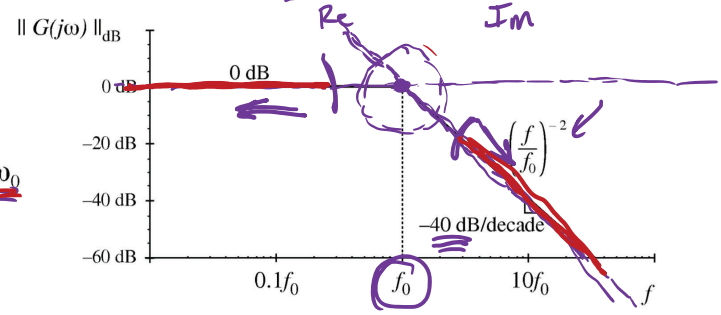
$\frac{1}{\left(\frac{\omega}{\omega_0}\right)^2}$

Re Im

Asymptotes are

$\| G \| \to 1$ for $\omega << \omega_0$

$\| G \| \to \left(\frac{f}{f_0}\right)^{-2}$ for $\omega >> \omega_0$



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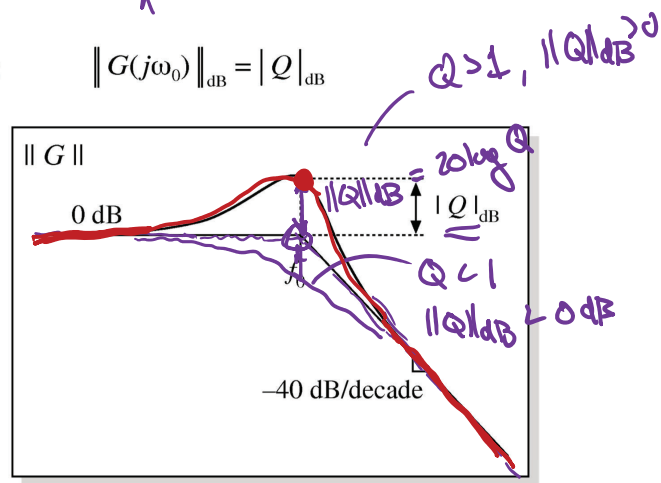
# Exact Magnitude Curve

$$\left\| G(j\omega) \right\| = \frac{1}{\sqrt{\left(1 - \left(\frac{\omega}{\omega_0}\right)^2\right)^2 + \frac{1}{Q^2}\left(\frac{\omega}{\omega_0}\right)^2}} = Q \text{ @ } \omega = \omega_0$$

At $\omega = \omega_0$, the exact magnitude is

$$\left\| G(j\omega_0) \right\| = Q \qquad \text{or, in dB:} \qquad \left\| G(j\omega_0) \right\|_{dB} = \left| Q \right|_{dB}$$

The exact curve has magnitude $Q$ at $f = f_0$. The deviation of the exact curve from the asymptotes is $\left| Q \right|_{dB}$



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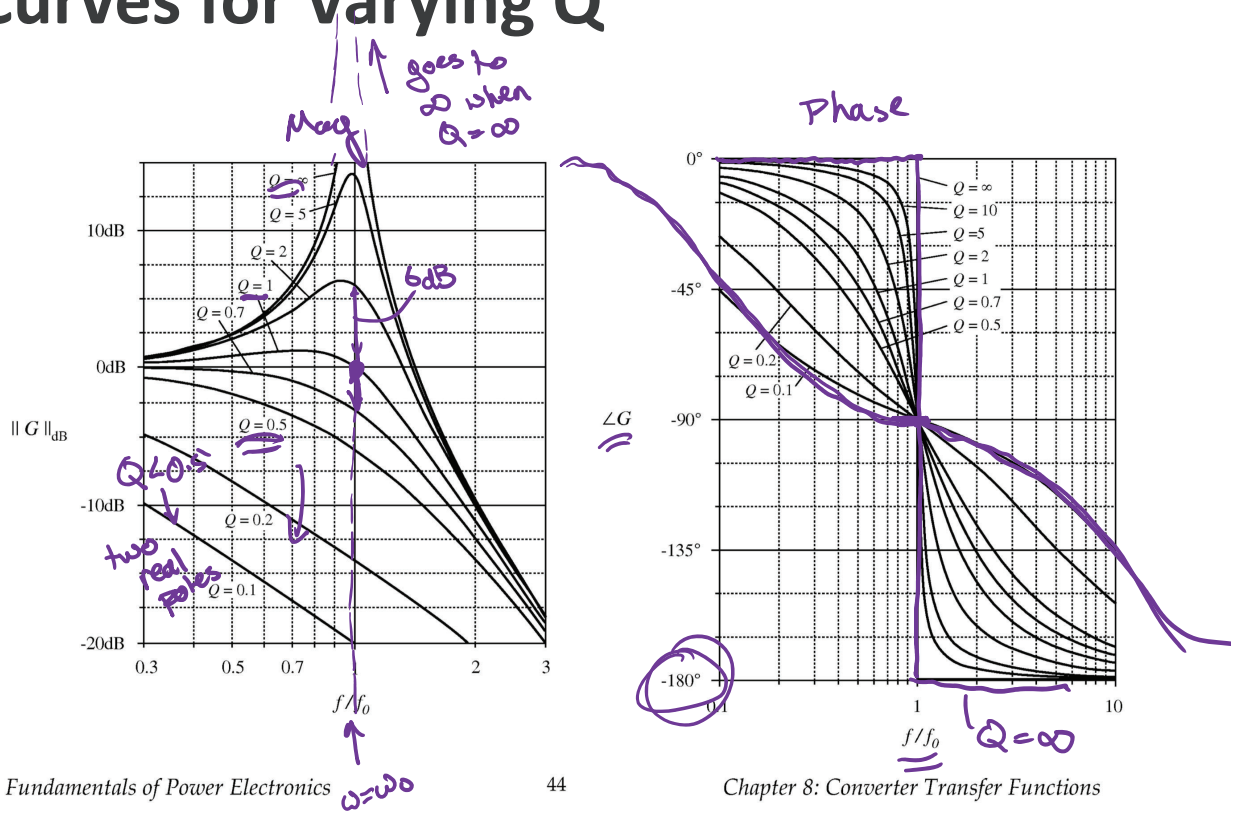
# Curves for Varying Q



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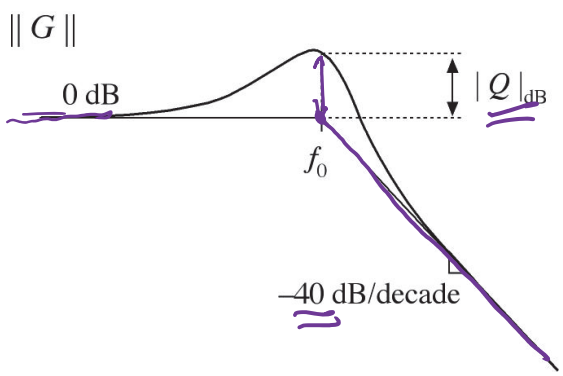
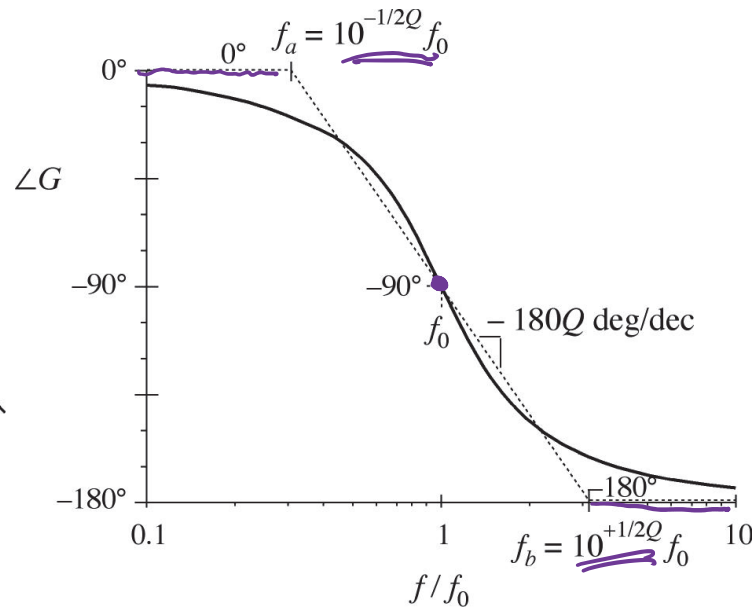
# Asymptotes for Complex Poles, $Q$>0.5

Magnitude




Phase






# Low Q Factorization

Given a second-order denominator polynomial, of the form

$$G(s) = \frac{1}{1 + a_1 s + a_2 s^2} \qquad \text{or} \qquad G(s) = \frac{1}{1 + \frac{s}{Q\omega_0} + \left(\frac{s}{\omega_0}\right)^2}$$

When the roots are real, i.e., when $Q < 0.5$, then we can factor the denominator, and construct the Bode diagram using the asymptotes for real poles. We would then use the following normalized form:

$$G(s) = \frac{1}{\left(1 + \frac{s}{\omega_1}\right)\left(1 + \frac{s}{\omega_2}\right)}$$

This is a particularly desirable approach when $Q << 0.5$, i.e., when the corner frequencies $\omega_1$ and $\omega_2$ are well separated.

$$\omega_1 = \frac{\omega_0}{Q}\,\frac{1 - \sqrt{1 - 4Q^2}}{2} \qquad \omega_2 = \frac{\omega_0}{Q}\,\frac{1 + \sqrt{1 - 4Q^2}}{2}$$

# Corner Frequency $\omega_1$

$$\omega_1 = \frac{\omega_0}{Q} \frac{1 - \sqrt{1 - 4Q^2}}{2}$$

can be written in the form

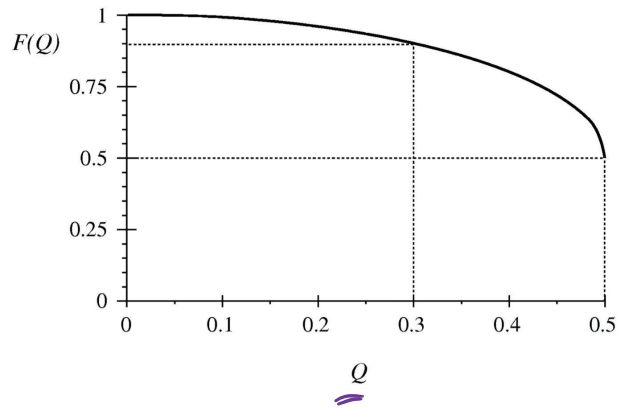
$$\omega_1 = \frac{Q\,\omega_0}{F(Q)}$$

where

$$F(Q) = \frac{1}{2}\left(1 + \sqrt{1 - 4Q^2}\right)$$

For small $Q$, $F(Q)$ tends to 1. We then obtain

$$\omega_1 \approx Q\,\omega_0 \quad \text{for } Q << \frac{1}{2}$$




For $Q < 0.3$, the approximation $F(Q) = 1$ is within 10% of the exact value.



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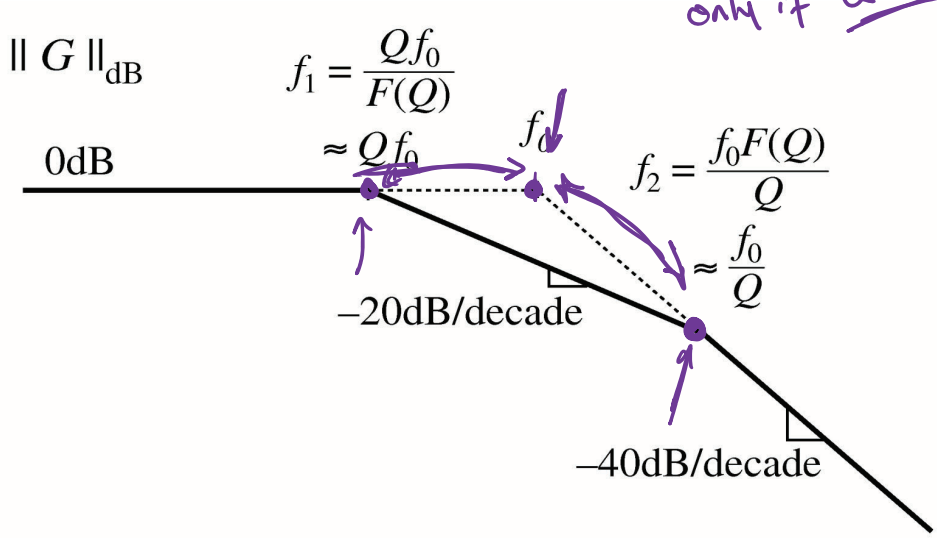
# The Low-Q Approximation

can be used only if $Q << 0.5$

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