

Announcements

- Midterm Wednesday 3/9
 - Covers coupled inductors and transformers, phasor circuit analysis and complex power
 - 3-~~X~~ problems, ~3x quiz length, Full class period for the exam
 - Lectures 1-16, HW 1-5, Quiz 1-2, Chapters 10,11, & 13
 - Recommended:
 - Review all lecture slides and in-class examples
 - Review solutions to HW problems where you missed points
 - Rework Quizzes
 - Create crib sheet
 - Practice complex numbers on your calculator

Quiz #2

- Average: 83 / Median: 85

- Time domain vs phasor domain

$$v_L(t) = A \cos(\omega t + \phi)$$

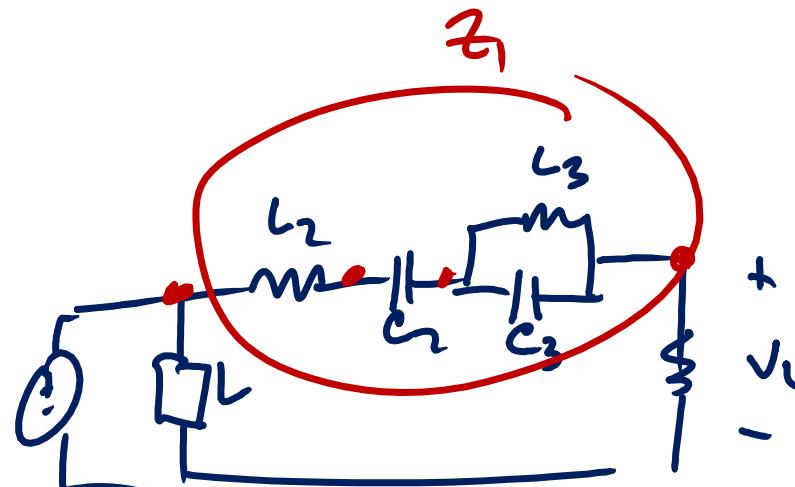
No complex #'s
no phaser

$$v_L = A \angle \phi$$

No "t" anywhere
in phaser

- Be careful with computation

- Simplify first



Midterm Problems:

1. Phasor circuit analysis with mutual inductances

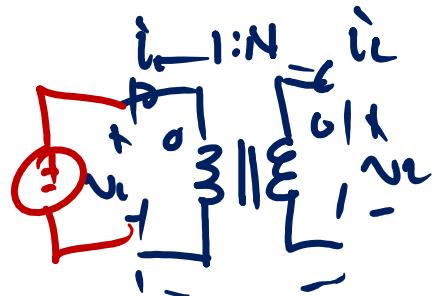
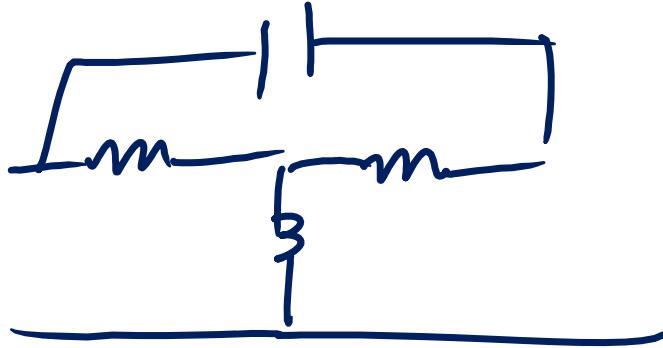
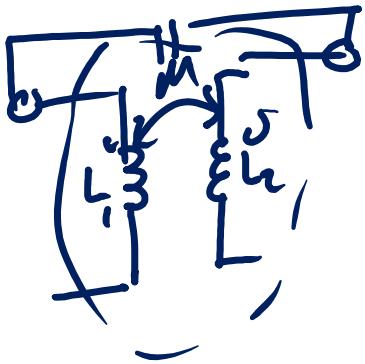
- Understand coupling coefficient and dot notation
- Solve for output signal

2. Phasor power

- Solve for matched load impedance
- Sketch circuit
- Solve S and PF

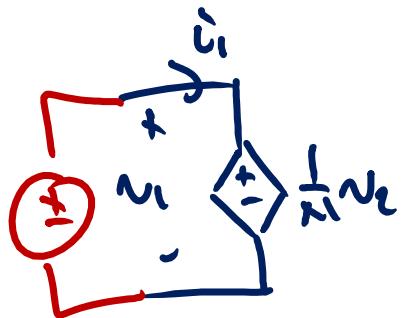
3. Phasor circuit analysis with transformer and multiple sources

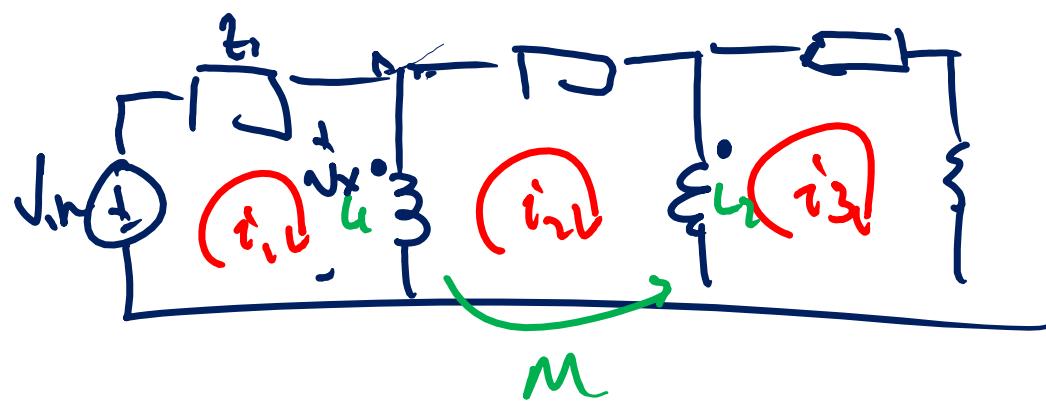
- Solve for output signal
- Multiple sources
- May or may not be at the same frequency



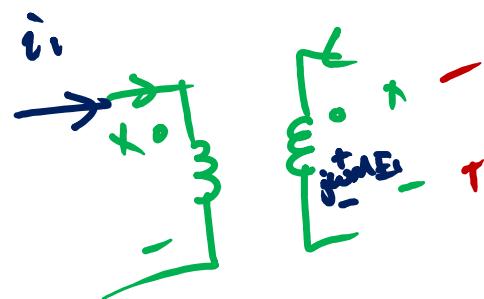
$$\frac{U_1}{i} = \frac{N_1}{N} \quad \text{and} \quad i_1 + N i_2 = 0$$

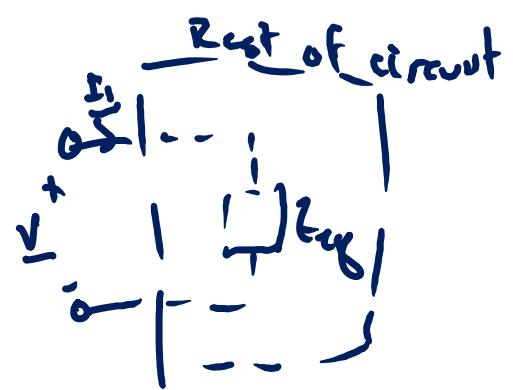
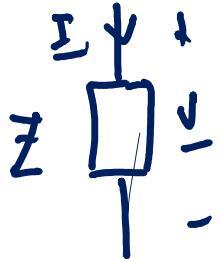
$$N_1 = \frac{N_2}{N} \quad \text{and} \quad i_2 = -\frac{i_1}{N}$$





$$\begin{aligned}
 i_1: \quad V_{in} &= \underline{I}_1 Z_1 + \underline{V}_u \\
 &= \underline{I}_1 Z_1 + j\omega L_1 (\underline{I}_1 - \underline{I}_2) + j\omega m (\underline{I}_2 - \underline{I}_3)
 \end{aligned}$$

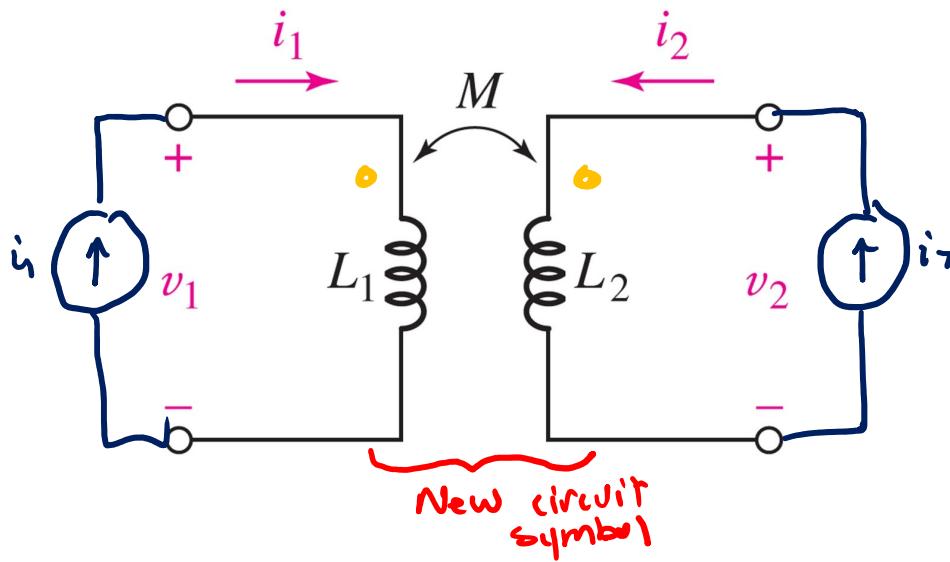




$$S = \frac{1}{2} V I^*$$

CHAPTER 13: MUTUAL INDUCTANCE

Mutual Inductance



With both sources on
Apply superposition

when $i_1 > 0$

$$\begin{cases} v_1 = L_1 \frac{di_1}{dt} \\ v_2 = M \frac{di_1}{dt} \end{cases}$$

when $i_2 > 0$

$$\begin{cases} v_1 = M \frac{di_2}{dt} \\ v_2 = L_2 \frac{di_2}{dt} \end{cases}$$

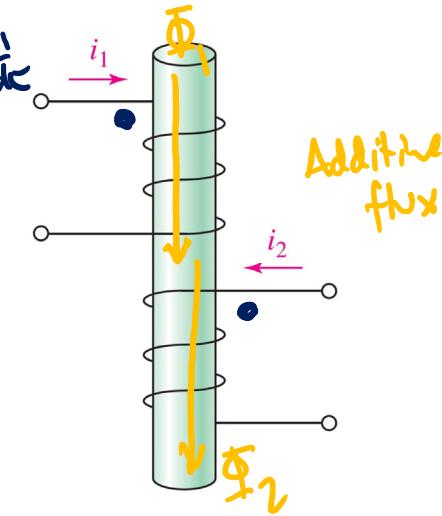
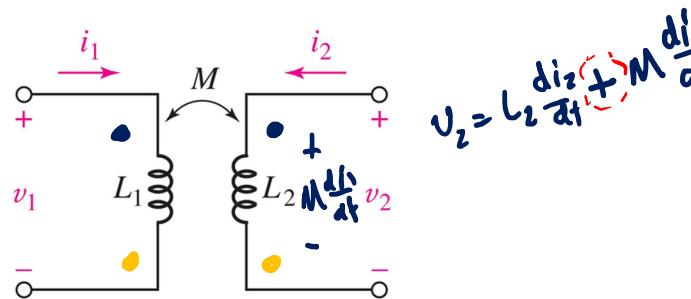
By superposition

$$\begin{cases} v_1 = L_1 \frac{di_1}{dt} \pm M \frac{di_2}{dt} \\ v_2 = \pm M \frac{di_1}{dt} + L_2 \frac{di_2}{dt} \end{cases}$$

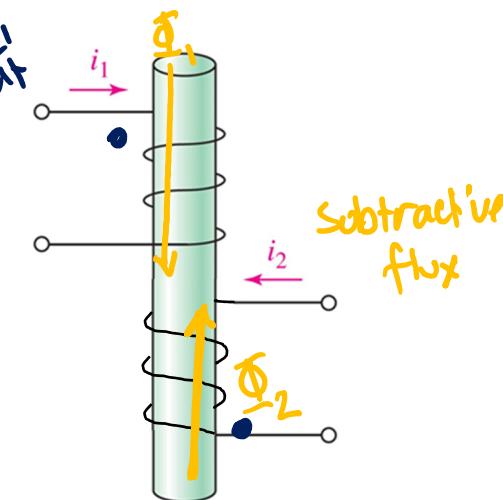
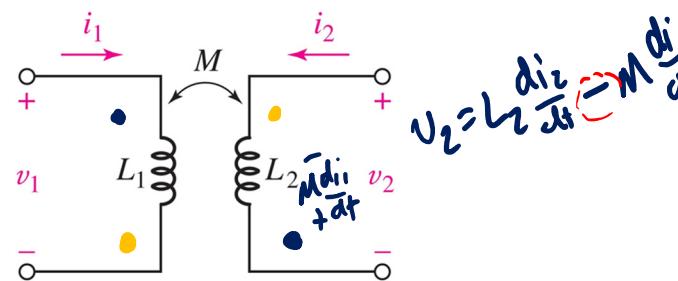
↑ ↑
sign depends on winding
polarity

Symbols and Dot Convention

* Make sure to use passive sign convention

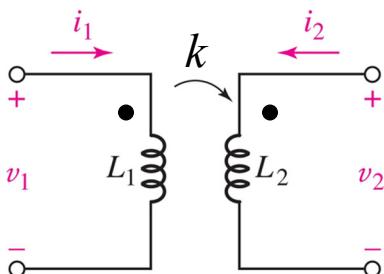
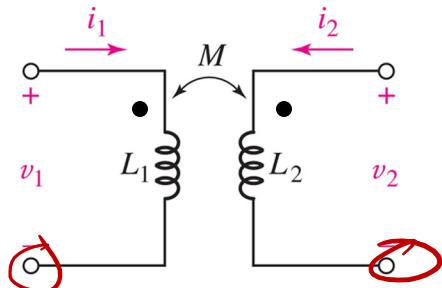


Physical: If both i_1 & i_2 enter the dotted terminals of the windings, they produce additive flux

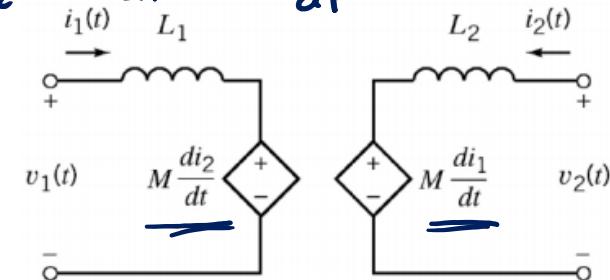


Circuit: Current flowing into the dotted terminal of one winding will produce a positive voltage at the dotted terminal of the other winding

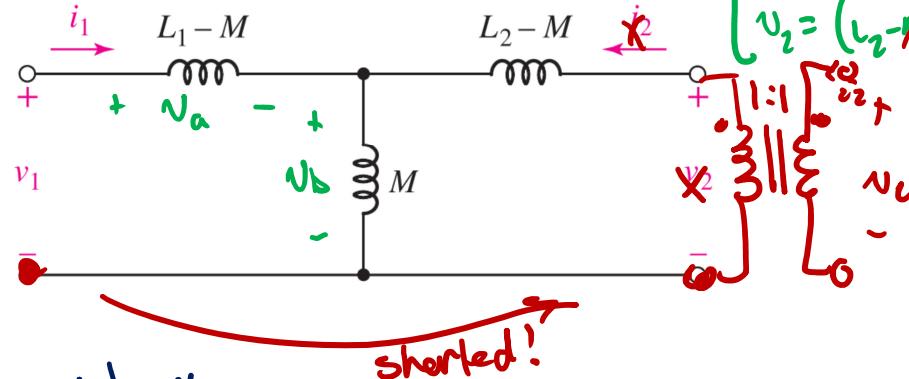
Equivalent Circuits



$$\begin{cases} v_1 = L_1 \frac{di_1}{dt} + M \frac{di_2}{dt} \\ v_2 = M \frac{di_1}{dt} + L_2 \frac{di_2}{dt} \end{cases}$$

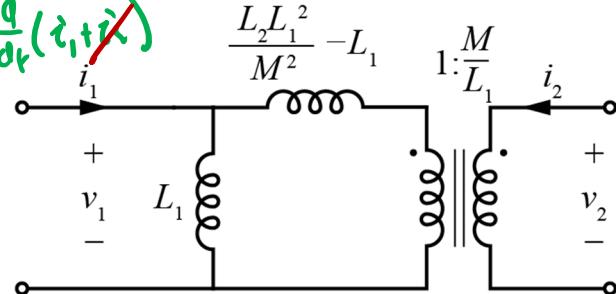


T-Network

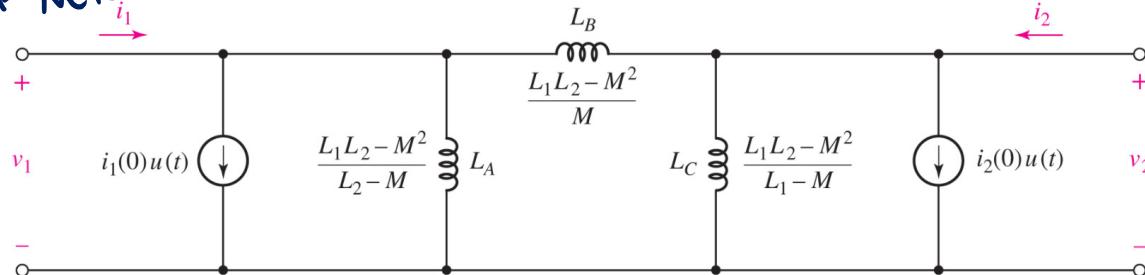


$$\begin{cases} v_1 = (L_1 - M) \frac{di_1}{dt} + M \frac{d}{dt}(i_1 + i_2) \\ v_2 = (L_2 - M) \frac{di_2}{dt} + M \frac{d}{dt}(i_1 + i_2) \end{cases}$$

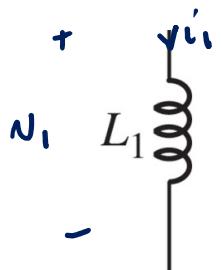
cantilever model



Π -Network



Energy Storage

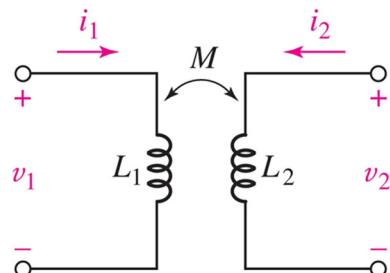
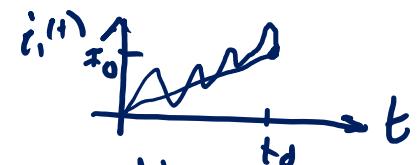


Review

When I_0 current flowing through L_1

$$E_L = \int_0^{t_0} p_L(t) dt = \int_0^{t_0} i_1(t) \cdot v_1(t) dt = \int_0^{t_0} L \underbrace{i_1(t)}_{\frac{di_1}{dt}} dt$$

$$= L \int_0^{t_0} \frac{1}{2} \left[\frac{d}{dt} i_1(t)^2 \right] dt = \boxed{\frac{1}{2} L I_0^2} = E_L$$



At t_0 $i_1(t_0) = I_{01}$ & $i_2(t_0) = I_{02}$

$$E_{12} = \int_0^t (v_1(t) i_1(t) + v_2(t) i_2(t)) dt$$

$$= \int_0^t \left(L_1 i_1(t) \frac{di_1}{dt} + M i_1(t) \frac{di_2}{dt} + L_2 i_2(t) \frac{di_2}{dt} + M i_2(t) \frac{di_1}{dt} \right) dt$$

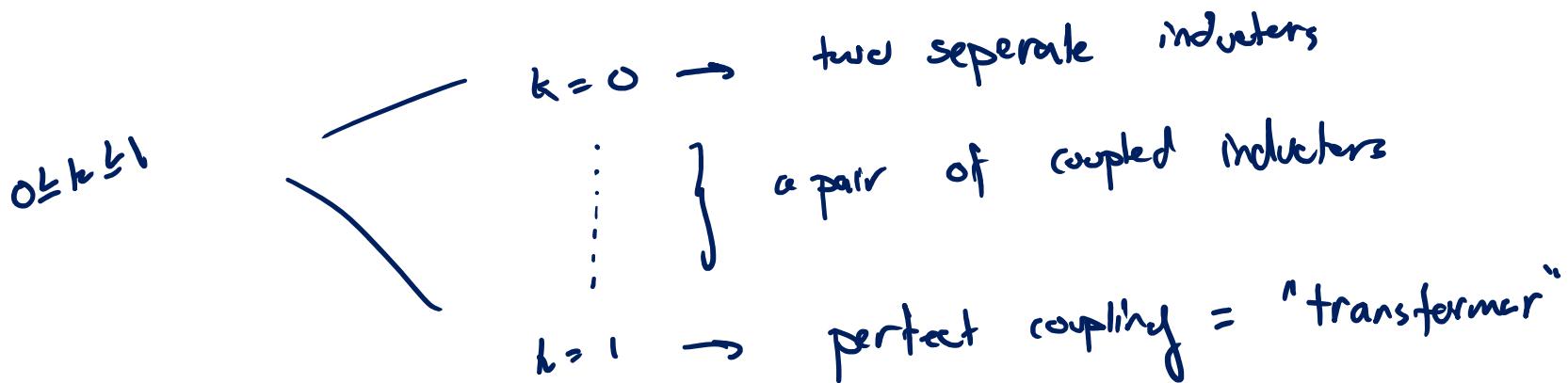
$$= \frac{1}{2} L_1 I_{01}^2 + \frac{1}{2} L_2 I_{02}^2 + \int_0^t M \left(i_1(t) \frac{di_2}{dt} + i_2(t) \frac{di_1}{dt} \right) dt$$

$$= \frac{d}{dt} (i_1 \cdot i_2)$$

$$\boxed{E_{12} = \frac{1}{2} L_1 I_{01}^2 + \frac{1}{2} L_2 I_{02}^2 + M I_{01} I_{02}}$$

Coupling Coefficient

Define $k = \frac{M}{\sqrt{L_1 L_2}}$ is the "coupling coefficient"



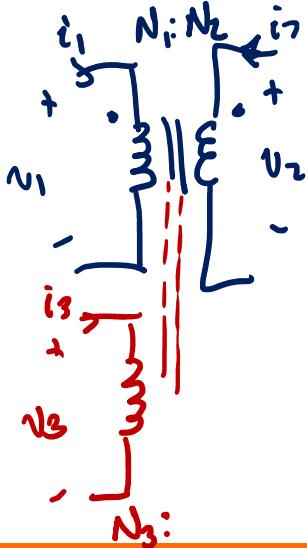
Ideal Transformer

$\hookrightarrow L_1 \text{ & } L_2$ are "brgc"

$\hookrightarrow h=1$

- Recall: Inductors (\neq transformers) cannot have DC voltage applied so current goes to ∞
- $N = L \frac{di}{dt} \rightarrow i = \frac{1}{L} \int V dt$
 - Materials: core materials saturate at high flux causing inductance to drop to nearly zero.
 - $V = N \frac{d\phi}{dt} \rightarrow$ Need to have time-varying (non-DC signals)

When L_1 & L_2 are "large" i_1 & i_2 are negligible \neq no energy



$$V_1 = N_2 \frac{N_1}{N_2}$$

$$V_1 i_1 + N_2 i_2 = \phi$$

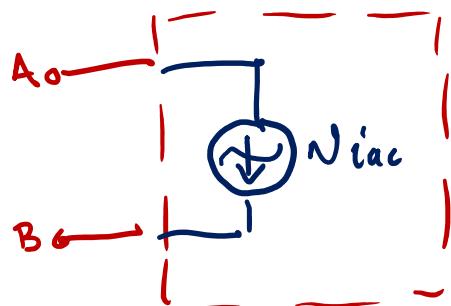
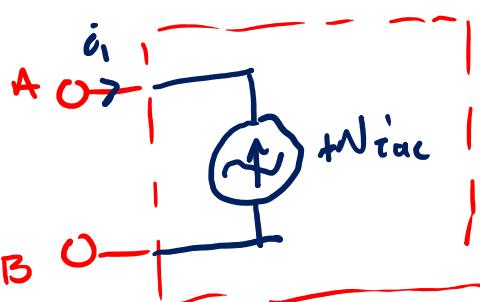
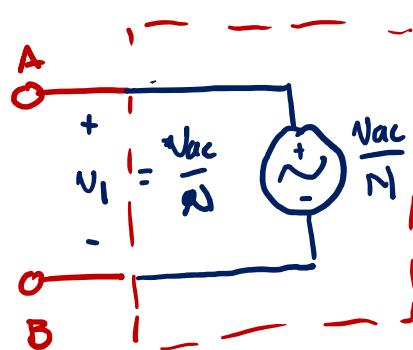
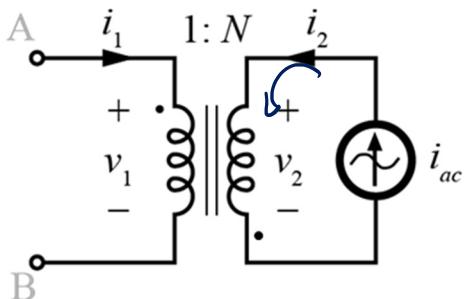
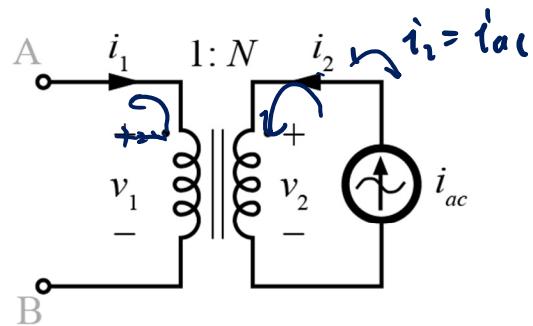
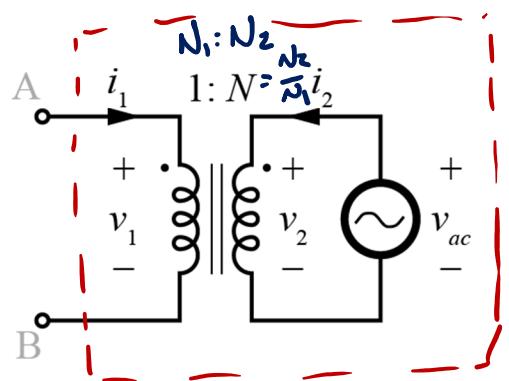
$$V_2 \frac{N_1}{N_2} i_1 + V_2 i_2 = \phi$$

$$N_1 i_1 + N_2 i_2 = \phi$$

$$\frac{V_1}{N_1} = \frac{V_2}{N_2} = \frac{V_3}{N_3}$$

$$i_1 N_1 + i_2 N_2 + i_3 N_3 = 0$$

Transformer Reflection



$$\frac{N_1}{N_2} = \frac{N_2}{N_1}$$

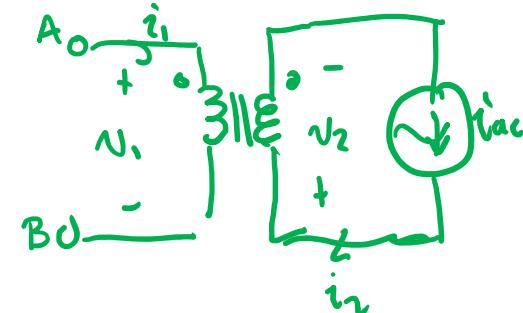
$$N_1 i_1 + N_2 i_2 = \phi$$

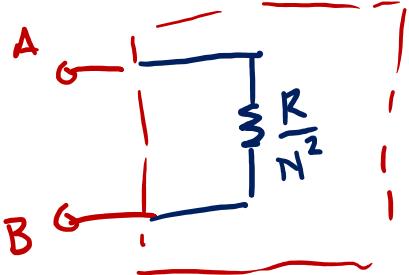
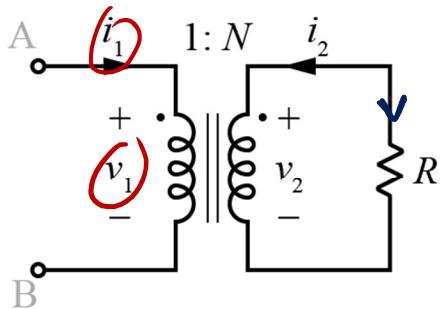
$$\frac{v_1}{1} = \frac{v_2}{N}$$

$$1 \cdot i_1 + N \cdot i_2 = \phi$$

$$i_1 + N i_2 = \phi$$

$$i_1 = -N i_2$$



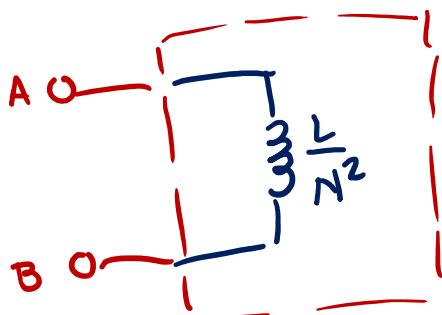
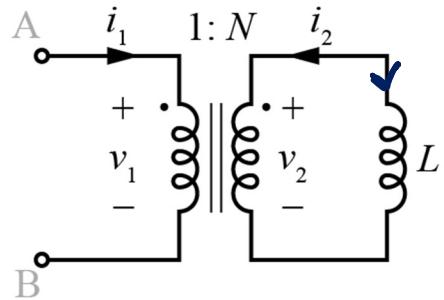


$$v_1 = \frac{v_2}{N} + i_1 + N i_2 = 0$$

$$v_2 = (-i_2) R$$

$$N v_1 = \left(-\frac{i_1}{N}\right) R$$

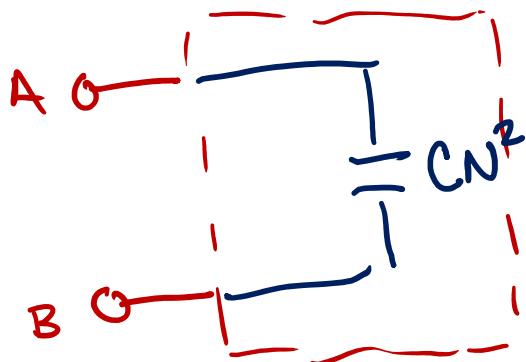
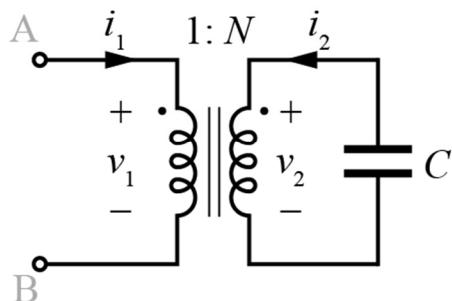
$$v_1 = i_1 \frac{R}{N^2}$$



$$v_2 = L \frac{d}{dt} (-i_2)$$

$$N v_1 = L \frac{d}{dt} \left(\frac{i_1}{N} \right)$$

$$v_1 = \frac{L}{N^2} \frac{d i_1}{d t}$$

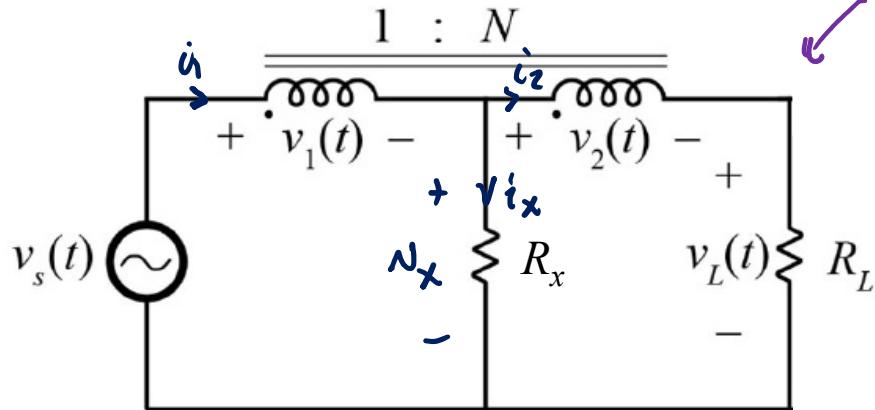


$$-i_2 = C \frac{d v_2}{d t}$$

$$\frac{i_1}{N} = C \frac{d}{d t} N v_1$$

$$i_1 = N^2 C \frac{d v_1}{d t}$$

Ideal Transformer Example



"Autotransformer" → e.g. a variac
Solve for $v_L(t)$

Transformer Equations:

$$\frac{v_1}{i_1} = \frac{v_2}{i_2} \quad \text{or} \quad i_1 + N i_2 = 0$$

KCL

Node:

$$i_2 = i_L$$

$$i_1 = i_x + i_2$$

$$-N i_2 - i_2 = i_x$$

$$-i_2(N+1) = i_x$$

$$-\frac{v_L}{R_L}(N+1) = \frac{v_x}{R_x} = \frac{v_L + v_x}{R_x}$$

$$v_x = -\frac{R_x}{R_L}(N+1)v_L - v_x$$

$$v_s = v_x + \left(\frac{1}{N+1}\right) \left(-v_L - \frac{R_x}{R_L}(N+1)v_L\right)$$

$$v_s = v_L \left[1 - \left(\frac{1}{N+1}\right) \left(1 + \frac{R_x}{R_L}(N+1)\right) \right]$$

$$v_s = v_L \left[1 - \frac{1}{N+1} - \frac{R_x}{R_L} \frac{1}{N+1} (N+1)^2 \right] \rightarrow$$

$$v_L = v_s \frac{-N}{1 + \frac{R_x}{R_L}(N+1)^2}$$

CHAPTER 10: SINUSOIDAL STEADY-STATE

Complex Numbers (Review)

$$z = x + jy \quad \begin{array}{l} \text{Rectangular form} \\ \text{Real part} \end{array} \quad = \quad \begin{array}{l} \text{Polar form} \\ \text{Imaginary part} \end{array} \quad r e^{j\theta} \quad \begin{array}{l} \text{amplitude} \\ \text{phase} \end{array}$$

$$\left\{ \begin{array}{l} |z| = r = \sqrt{x^2 + y^2} \\ \angle z = \theta = \tan^{-1}\left(\frac{y}{x}\right) \end{array} \right.$$

$$\left\{ \begin{array}{l} \operatorname{Re}\{z\} = x = r \cos \theta \\ \operatorname{Im}\{z\} = y = r \sin \theta \end{array} \right.$$

Euler:

$$r e^{j\theta} = r \cos \theta + j r \sin \theta$$

Complex Conjugate:

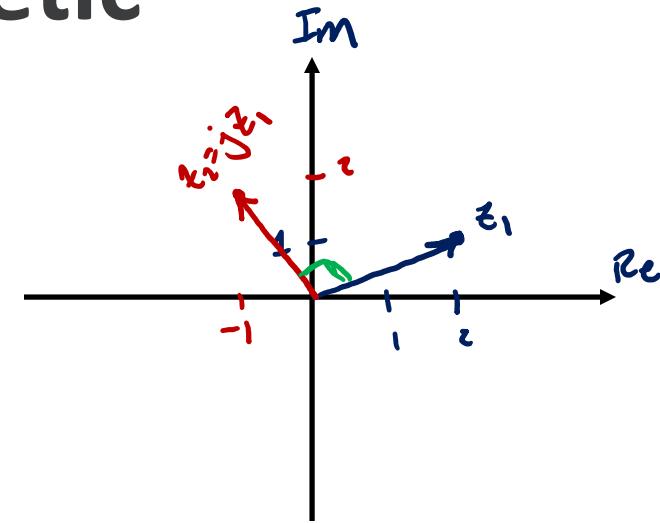
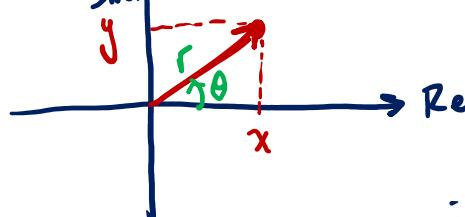
for $z = x + jy$

$$z^* = x - jy$$

Complex Number Arithmetic

Complex #'s are vectors in 2D space

$$z = x + jy \quad = r e^{j\theta}$$



ex $z_1 = 2 + j1 = \sqrt{5} e^{j\tan^{-1}(\frac{1}{2})}$

$$z_2 = j z_1 = j^2 + j^2 1 = -1 + j2$$

Multiplication by j is a 90° phase shift

$$j = 1 e^{j\pi/2}$$
$$z_1 j = \sqrt{5} e^{j\tan^{-1}(\frac{1}{2})} \cdot 1 e^{j\pi/2} = \sqrt{5} e^{j(\tan^{-1}(\frac{1}{2}) + \frac{\pi}{2})}$$

Usually:

$$z_1 + z_2 = (x_1 + x_2) + j(y_1 + y_2)$$

Multiplication

$$z_1 \cdot z_2 = (r_1 r_2) e^{j(\theta_1 + \theta_2)}$$

Phasor Notation

$$v(t) = A \cos(\omega t + \phi) = \text{Re} \{ A e^{j\omega t} e^{j\phi} \}$$

\neq ↑ ↓
phasor transform $\leftrightarrow \omega$

$$\underline{V} = A e^{j\phi} \leftrightarrow A \angle \phi$$

(short hand notation)

*bold in book
underbar in lecture*

$$i(t) = B \sin(\omega t + \theta) \\ = B \cos(\omega t + \theta - 90^\circ)$$

↓
phasor transform

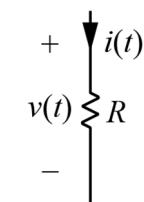
$$\underline{I} = B e^{j(\theta - \frac{\pi}{2})} \leftrightarrow B \angle (\theta - \frac{\pi}{2})$$

Comments:

- Phasor transform works for volt/current sources & signals
- Everything in the circuit must be at a single frequency ω
- Complex numbers:
 - No imaginary numbers in $v(t)$ or $i(t)$ (in time domain)
 - No time in the phasor domain \rightarrow no "t" in \underline{V} or \underline{I}

Phasor Circuit Elements

Time Domain



$$v(t) = i(t)R$$

Phasor Domain



$$v(t) = A \cos(\omega t + \phi)$$

$$i(t) = \frac{A}{R} \cos(\omega t + \phi)$$

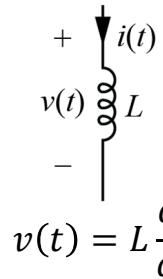
$$v(t) = i(t)R$$

phasor transform

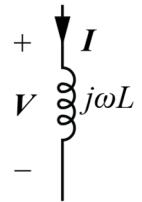
$$\underline{V} = Ae^{j\phi}$$

$$\underline{I} = \frac{A}{R} e^{j\phi}$$

$$\boxed{\underline{V} = \underline{I} R}$$



$$v(t) = L \frac{di}{dt}$$



$$i(t) = A \cos(\omega t + \phi)$$

$$v(t) = LA\omega \cos(\omega t + \phi + 90^\circ)$$

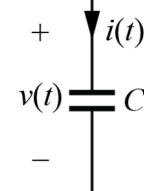
$$v(t) = L \frac{di}{dt}$$

$$\underline{I} = Ae^{j\phi}$$

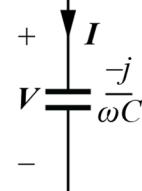
$$\underline{V} = LA\omega e^{j(\phi + \frac{\pi}{2})}$$

$$\underline{V} = j\omega L \underline{I}$$

$$\boxed{\underline{V} = j\omega L \underline{I}}$$



$$i(t) = C \frac{dv}{dt}$$



$$\rightarrow \underline{I} = C j\omega \underline{V}$$

$$v(t) = A \cos(\omega t + \phi)$$

$$i(t) = CA\omega \cos(\omega t + \phi + 90^\circ)$$

$$i(t) = C \frac{dv}{dt}$$

$$\underline{V} = Ae^{j\phi}$$

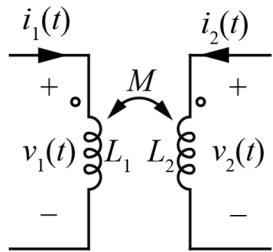
$$\underline{I} = CA\omega e^{j(\phi + \frac{\pi}{2})}$$

$$\underline{V} = j\omega C Ae^{j\phi}$$

$$\boxed{\underline{V} = \frac{-j}{\omega C} \underline{I}}$$

$$= \boxed{\underline{V} = \frac{1}{j\omega C} \underline{I}}$$

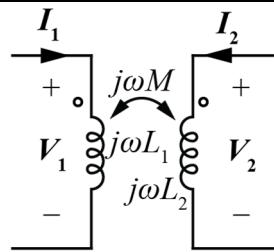
Time Domain



$$v_1(t) = L_1 \frac{di_1}{dt} + M \frac{di_2}{dt}$$

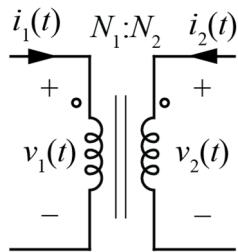
$$v_2(t) = M \frac{di_1}{dt} + L_2 \frac{di_2}{dt}$$

Phasor Domain



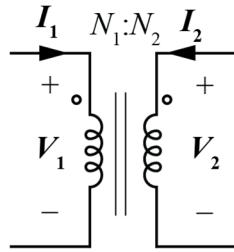
$$\underline{V}_1 = j\omega L_1 \underline{I}_1 + j\omega M \underline{I}_2$$

$$\underline{V}_2 = j\omega M \underline{I}_1 + j\omega L_2 \underline{I}_2$$



$$\frac{v_1(t)}{N_1} = \frac{v_2(t)}{N_2}$$

$$N_1 i_1(t) + N_2 i_2(t) = 0$$



$$\frac{\underline{V}_1}{N_1} = \frac{\underline{V}_2}{N_2}$$

$$N_1 \underline{E}_1 + N_2 \underline{E}_2 = \phi$$

Impedance

Phasor equivalent of ohm's law

$$\underline{V} = \underline{I} \underline{Z}$$

$$z = \text{"Impedance"} = R + jX$$

$\text{Re}\{z\} = R$, "Resistance"
 $\text{Im}\{z\} = X$, "Reactance"

$$\left\{ \begin{array}{l} z_R = R, \text{ Resistor} \\ z_L = j\omega L, \text{ Inductor} \\ z_C = \frac{-j}{\omega C}, \text{ Capacitor} \end{array} \right.$$

All have units of ohms (z, R, X)

$$Y = \text{"Admittance"} = \frac{1}{z} = G + jB$$

conductance susceptance

\rightarrow All units of Siemens

$$Y = \frac{1}{z} = \frac{1}{R+jX} \neq \cancel{\frac{1}{R} + j \frac{1}{X}}$$

No.

Yes:

$$\frac{1}{R+jX} \frac{(R-jX)}{(R-jX)} = \frac{R-jX}{R^2+X^2} = \frac{R}{R^2+X^2} - j \frac{X}{R^2+X^2}$$

Phasor Circuit Analysis

Start:

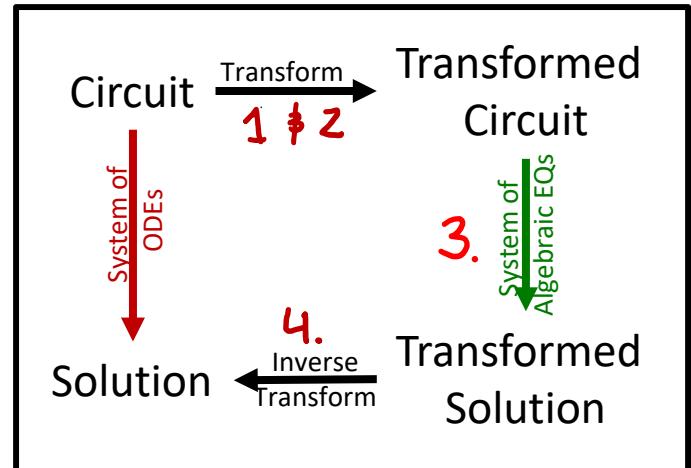
LTI circuit with single-frequency sinusoidal source(s) & want to find steady-state solution

1: Transform all sources & signals into their phasor equivalents

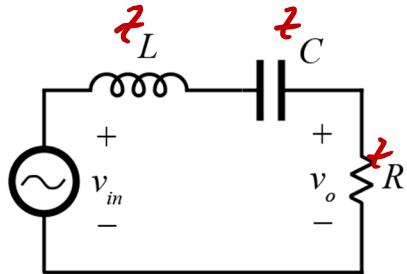
2: Transform all passives into their impedances

3: Solve the circuit
- Can use all 201 analysis techniques for DC, resistor-only circuits

4: Transform phasor voltage or current back into the time domain



Resonance Example



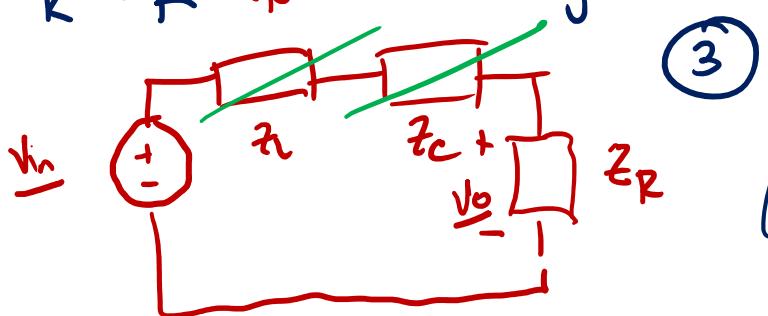
$$= 10 \cos(\omega t - 90^\circ)$$

Find $v_o(t)$ for $v_{in}(t) = 10 \sin(\omega t)$ and $\omega = 2\pi 100 \text{ kHz}$, $R = 10 \Omega$, $L = 10 \mu\text{H}$, and $C = 253 \text{ nF}$

① $v_{in}(t) \rightarrow v_{in} = 10 \angle -90^\circ \text{ or } 10 e^{j -\frac{\pi}{2}}$

$$v_o(t) \rightarrow \underline{v_o}$$

② $R \rightarrow R = z_R$



$$L \rightarrow j\omega L = z_L$$

$$C \rightarrow \frac{-j}{\omega C} = z_C$$

③

$$\underline{v_o} = \underline{v_{in}} \frac{z_R}{z_R + z_L + z_C}$$

$$R = R = 10 \Omega$$

$$z_L = j\omega L = j(2\pi 10^3)(10e-6) \\ = j(2\pi)$$

$$z_C = \frac{-j}{\omega C} = \frac{-j}{(2\pi 10^3)(253e-9)} = -j(2\pi)$$

④

$$\underline{v_o} = 10 e^{j -\frac{\pi}{2}}$$

$$v_o(t) = 10 \cos(\omega t - 90^\circ) \\ = 10 \sin(\omega t)$$

Reactance and Resonance

Batch at the end of 201

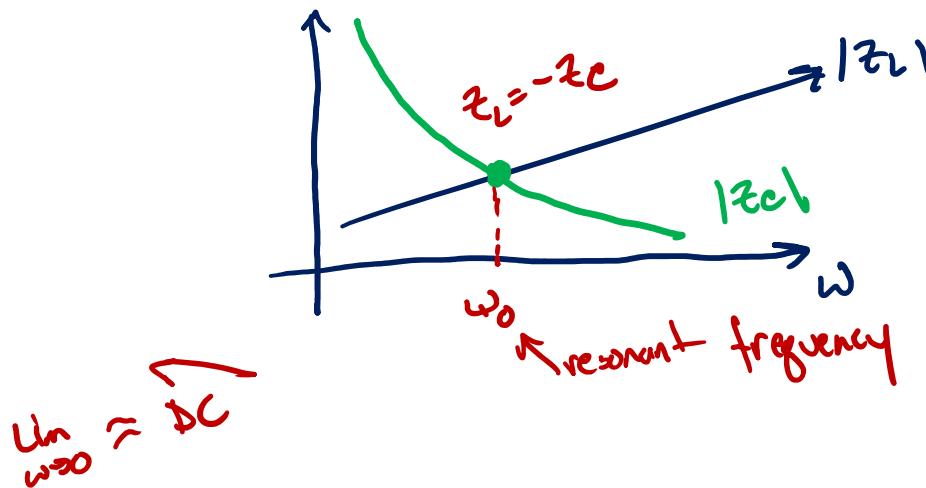
$$\omega_0 = \frac{1}{\sqrt{LC}} \rightarrow \text{resonant frequency}$$

Now:

$$z_L = j\omega L$$

$$z_C = \frac{-j}{\omega C}$$

$$z_L = -z_C$$
$$j\omega L = -\left(\frac{-j}{\omega C}\right)$$



$$\omega L = \frac{1}{\omega C}$$
$$\omega^2 = \frac{1}{LC}$$
$$\omega_0 = \frac{1}{\sqrt{LC}}$$

$$\text{as } \omega \rightarrow 0 \approx \infty$$

$$z_C \rightarrow \infty \quad \text{cap} \rightarrow \text{open}$$
$$z_L \rightarrow 0 \quad \text{inductor} \rightarrow \text{short}$$

$$\text{as } \omega \rightarrow \infty$$

$$z_C \rightarrow 0 \quad \text{cap} \rightarrow \text{short}$$
$$z_L \rightarrow \infty \quad \text{inductor} \rightarrow \text{open}$$

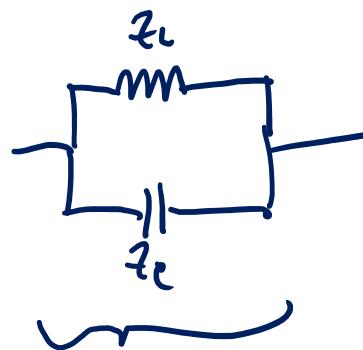
$$\textcircled{Q} \quad \omega_0 = \frac{1}{\sqrt{LC}}$$

$$Z_L = -Z_C$$



$$Z_{eq} = Z_L + Z_C$$

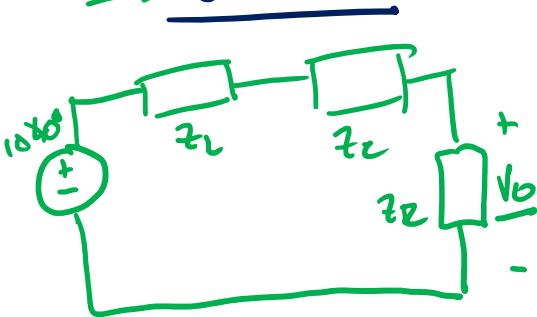
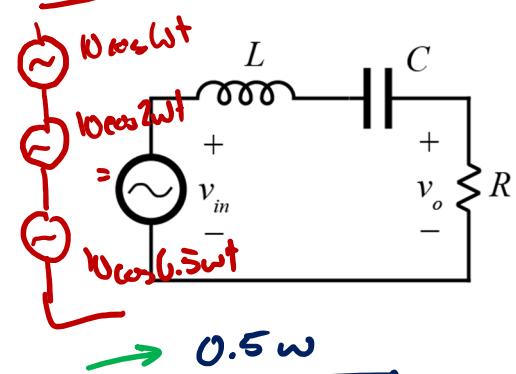
$= \phi$ @ resonance
(short)



$$Z_{eq} = \frac{Z_L Z_C}{Z_L + Z_C} \rightarrow \phi$$

$Z_{eq} \rightarrow \infty$ (open)
@ resonance

Phasor Superposition



$$\underline{V_o} = \underline{V_{in}} \frac{Z_R}{Z_L + Z_C + Z_R}$$

$$Z_L = j(0.5\omega)L \quad Z_C = \frac{j}{(0.5\omega)C}$$

$$\underline{V_o} = 0.73 \angle 43^\circ$$

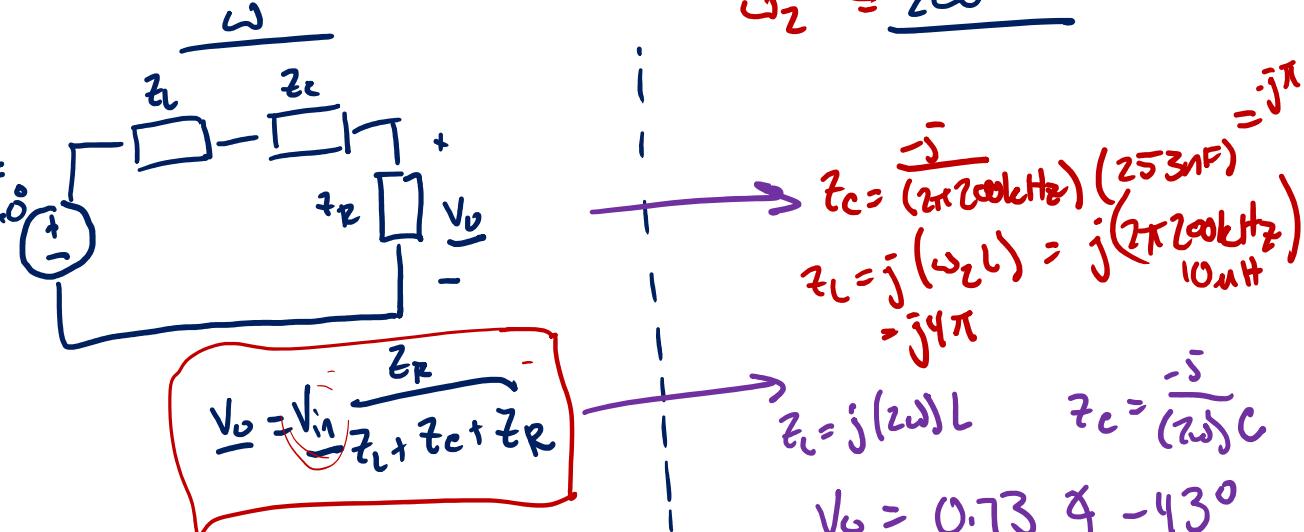
$$v_o(t) = 0.73 \cos(0.5\omega t + 43^\circ)$$

Find $v_o(t)$ for $v_{in}(t) = 10\cos(\omega t) + 18\cos(2\omega t) + 10\cos(0.5\omega t)$
 and $\omega = 2\pi 100 \text{ kHz}$, $R = 10 \Omega$, $L = 10 \mu\text{H}$, and $C = 253 \text{ nF}$

3 frequencies phaser analysis doesn't apply?

Apply superposition in the time domain

$$\omega_2 = \underline{z\omega} = 2\pi 200 \text{ kHz}$$



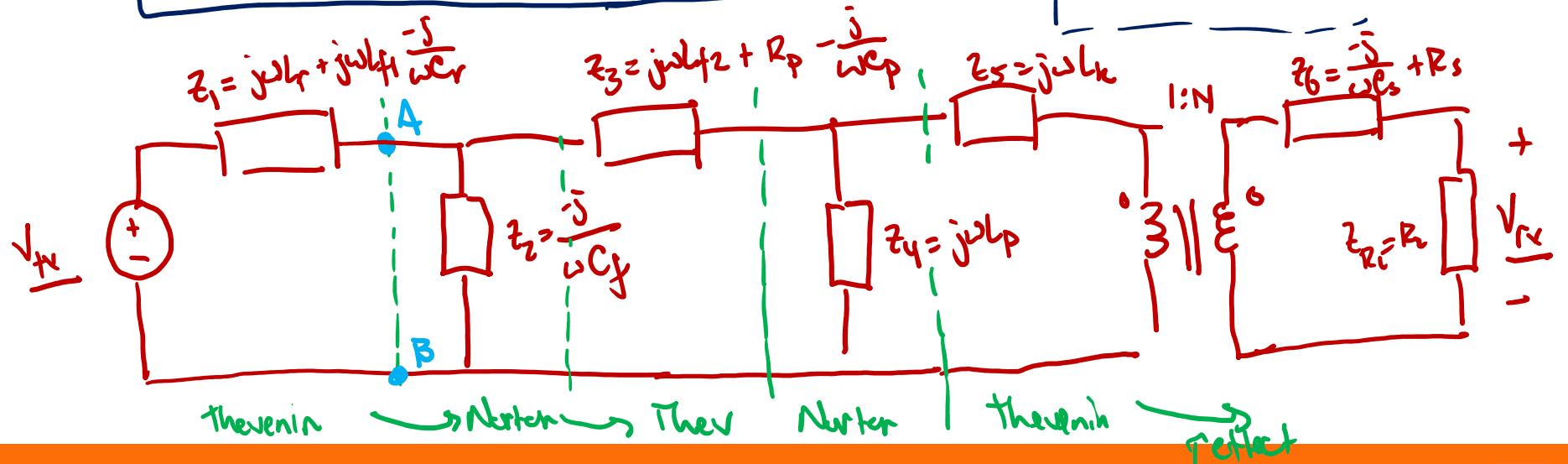
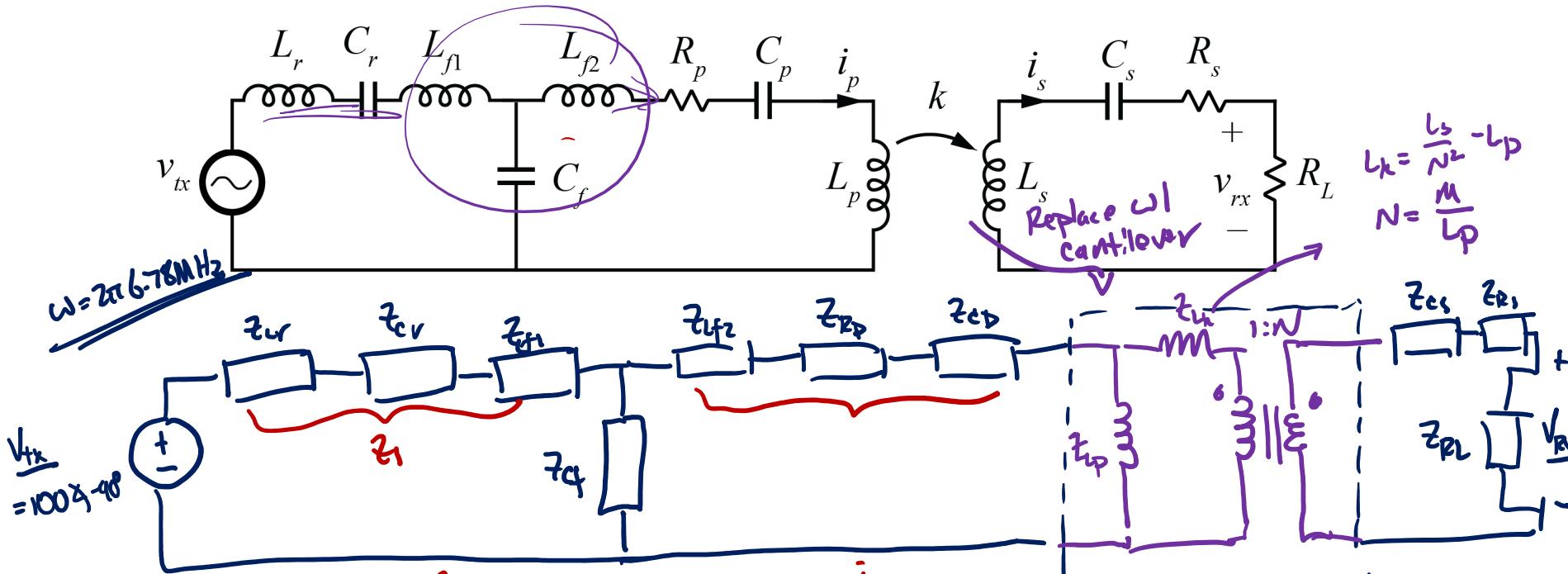
$$v_o(t) = 10 \cos(\omega t)$$

$$v_o(t) = 0.73 \cos(2\omega t - 43^\circ)$$

$$v_o(t) = 0.73 \cos(0.5\omega t + 43^\circ) + 10 \cos(\omega t) + 0.73 \cos(2\omega t - 43^\circ)$$

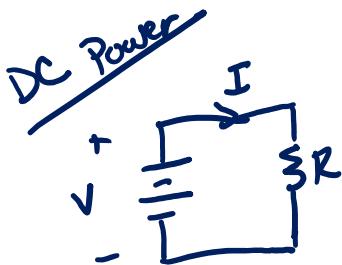
Example: WPT Problem

$$v_{tx}(t) = 100 \sin(2\pi 6.78 \text{ MHz} t)$$



CHAPTER 11: POWER IN THE SINUSOIDAL STEADY-STATE

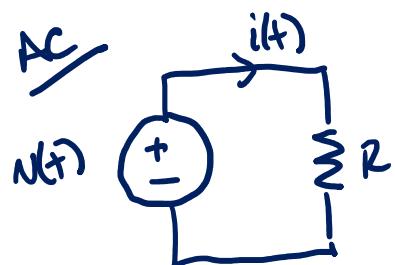
Average Power



$$P = V \cdot I \rightarrow \text{Generally true for any 2-terminal element}$$

for resistors $V = IR$

$$P_R = \frac{V^2}{R} = \underline{\underline{I^2 R}}$$



$$p(t) = \underline{\underline{v(t) \cdot i(t)}} \rightarrow \text{Generally true for any 2-terminal element}$$

for resistors $P_R(t) = \frac{v(t)^2}{R} = \underline{\underline{i(t)^2 R}}$

power calculation is not LTI

Average Power

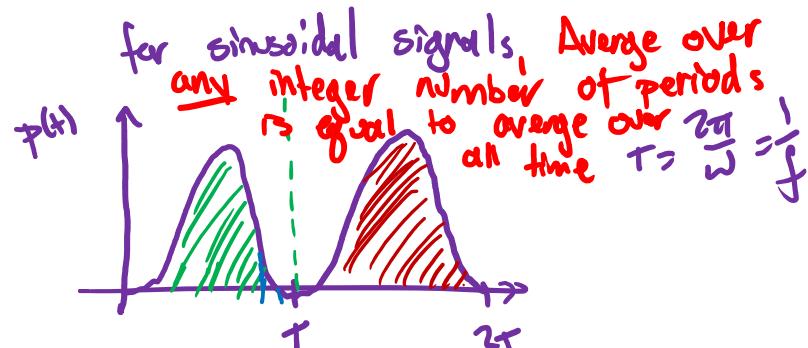
capital P
for average

Average power over some interval $[t_1, t_1 + T]$

$$P = \frac{1}{T} \int_{t_1}^{t_1+T} p(t) dt$$

Average power over all time

$$P = \lim_{T \rightarrow \infty} \frac{1}{T} \int_{-T/2}^{T/2} p(t) dt$$



Power in a Resistor

Average power in a resistor with some periodic (e.g. sinusoidal) current

$$P_R = \frac{1}{T} \int_0^T P_R(t) dt = \frac{1}{T} \int_0^T i_R(t)^2 R dt$$

$$= R \underbrace{\frac{1}{T} \int_0^T i_R(t)^2 dt}_{\text{rms}}$$

$$P_R = R \left(\sqrt{\frac{1}{T} \int_0^T i_R(t)^2 dt} \right)^2 \rightarrow P_R = \underline{(I_{\text{rms}})^2 R}$$

rms = "root mean square"

Define rms \rightarrow

$$V_{\text{rms}} = \sqrt{\frac{1}{T} \int_0^T V(t)^2 dt}$$

$$X_{\text{rms}} = \sqrt{\frac{1}{T} \int_0^T X(t)^2 dt}$$

Note: book calls this "effective" instead of rms

$$I_{\text{eff}}, V_{\text{eff}}, \Leftarrow \sqrt{V_{\text{rms}}} I_{\text{rms}}$$

RMS of a sinusoid

$$i(t) = I_A \cos(\omega t + \phi_I)$$

$$\omega = 2\pi f \quad T = \frac{1}{f} = \frac{2\pi}{\omega}$$

$$I_{rms} = \sqrt{\frac{1}{T} \int_0^T [I_A \cos(\omega t + \phi_I)]^2 dt}$$

$$I_{rms}^2 = \frac{\omega}{2\pi} I_A^2 \int_0^{2\pi/\omega} \cos^2(\omega t + \phi_I) dt$$

$$I_{rms}^2 = \frac{\omega}{2\pi} I_A^2 \frac{1}{2} \int_0^{2\pi/\omega} 1 + \cos(2\omega t + 2\phi_I) dt$$

$$I_{rms}^2 = \frac{\omega}{2\pi} I_A^2 \frac{1}{2} \left[t + \sin(2\omega t + 2\phi_I) \frac{1}{2\omega} \right] \Big|_0^{2\pi/\omega}$$

$$I_{rms}^2 = \frac{\omega}{2\pi} I_A^2 \frac{1}{2} \left[\left(\frac{2\pi}{\omega} - 0\right) + \frac{1}{2\omega} (\phi) \right]$$

$$I_{rms}^2 = \frac{I_A^2}{2}$$

Trig Identity

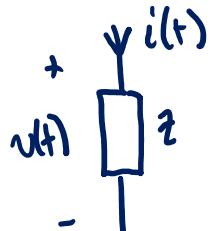
$$\cos^2(\theta) = \frac{1}{2} (1 + \cos 2\theta)$$

$$I_{rms} = \frac{I_A}{\sqrt{2}}$$

$$X_{rms} = \frac{X_A}{\sqrt{2}}$$

for sinusoidal signals

Power with Sinusoidal Sources



In steady-state for sinusoidal $i(t) \neq v(t)$ at ω , z is some impedance

$$\begin{aligned} v(t) &\rightarrow V = V_A e^{j\phi_V} \\ i(t) &\rightarrow I = I_A e^{j\phi_I} \end{aligned} \quad \left. \begin{array}{l} \\ \end{array} \right\} V = IZ \quad z = \frac{V}{I} = \frac{V_A}{I_A} e^{j(\phi_V - \phi_I)}$$

$$= \frac{V_A}{I_A} \times \frac{(\phi_V - \phi_I)}{\phi_Z}$$

power: $p(t) = i(t) \cdot v(t)$ is not LTI calculation
so go back to time domain

$$p(t) = I_A \cos(\omega t + \phi_I) V_A \cos(\omega t + \phi_V)$$

$$p(t) = I_A V_A \frac{1}{2} \left[\underbrace{\cos(2\omega t + \phi_I + \phi_V)}_{\text{sinusoidal at } 2\omega} + \underbrace{\cos(\phi_I - \phi_V)}_{\text{constant (not time-varying)}} \right]$$

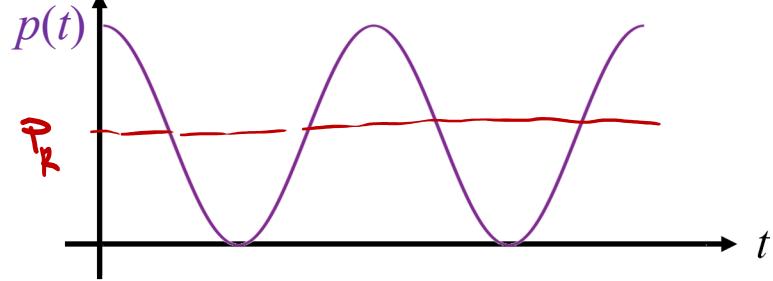
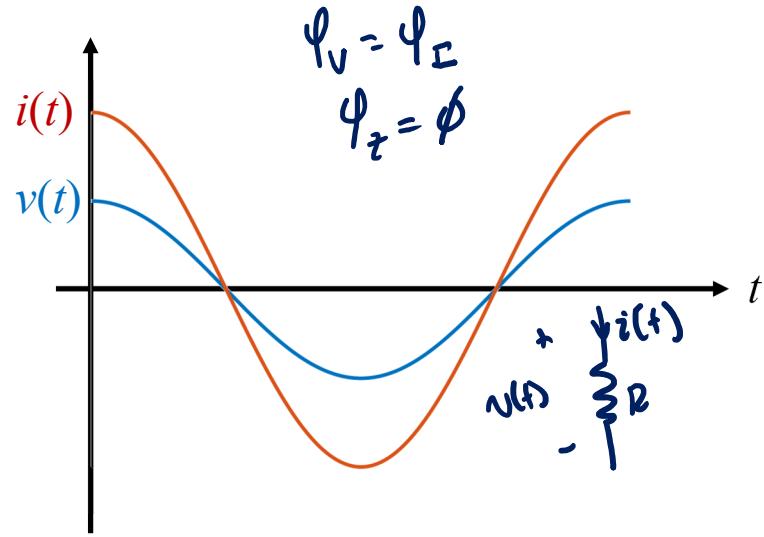
Trig Identity

$$2\cos\theta\cos\phi = \cos(\theta + \phi) + \cos(\theta - \phi)$$

Average power: $\phi_V - \phi_I$

$$\boxed{P = \frac{I_A V_A}{2} \cos(\phi_I - \phi_V)} = \frac{I_A V_A}{2} \cos(\phi_Z) = V_{rms} I_{rms} \cos(\phi_Z)$$

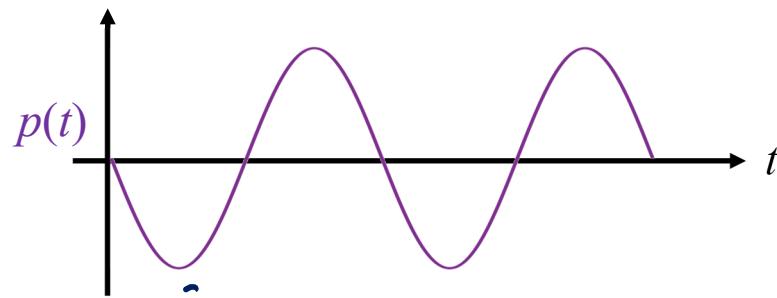
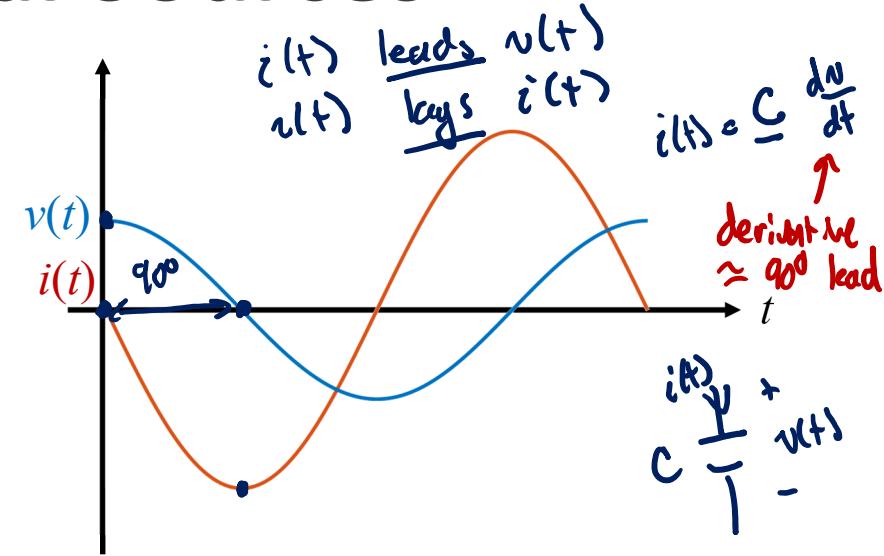
Power with Sinusoidal Sources



$$z = z_R e^{j\phi} = z_A \text{ all real}$$

so its a resistor for a resistor

$$P_R = \frac{I_A V_A}{2} \cos(\phi_z) = \frac{I_A V_A}{2} = \frac{I_{rms} V_{rms}}{2}$$



$$z_C = -\frac{j}{\omega C}$$

$$\phi_z = -90^\circ$$

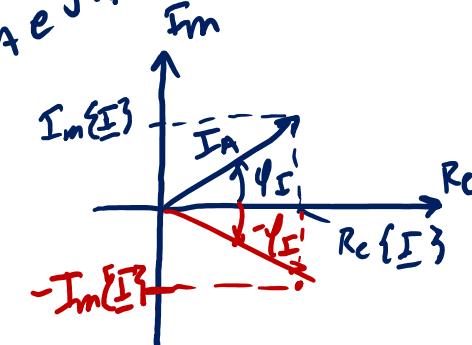
$$P_C = \frac{I_A V_A}{2} \cos(\phi_z)$$

Reactive elements (L or C) have zero average power in steady-state

Complex Power

$$\begin{aligned}
 P &= \frac{V_A I_A}{2} \cos(\varphi_V - \varphi_I) = \frac{1}{2} R_c \left\{ \underline{V} \underline{I}^* \right\} \quad \text{complex conjugate} \\
 &= \frac{1}{2} R_c \left\{ V_A e^{j\varphi_V} I_A e^{-j(-\varphi_I)} \right\} \\
 &= \frac{1}{2} R_c \left\{ V_A I_A e^{j(\varphi_V - \varphi_I)} \right\} \\
 &= \frac{1}{2} V_A I_A \cos(\varphi_V - \varphi_I)
 \end{aligned}$$

$$\begin{aligned}
 \underline{V} &= V_A e^{j\varphi_V} \\
 \underline{I} &= I_A e^{j\varphi_I}
 \end{aligned}$$



Average Power $P = \frac{V_A I_A}{2} \cos(\varphi_V - \varphi_I) = V_{rms} I_{rms} \cos(\varphi_Z) = \frac{1}{2} R_c \left\{ \underline{V} \underline{I}^* \right\} = R_c \left\{ V_{rms} I_{rms} \right\}$

what about the imaginary part of $\underline{V} \underline{I}^*$?

$$S = \frac{1}{2} \underline{V} \underline{I}^* = \begin{matrix} P \\ \uparrow \\ \text{Average Power} \\ [W] \\ \text{watts} \end{matrix} + jQ \quad \begin{matrix} \text{Reactive or Quadrature} \\ \text{Power} \\ [V \cdot A R] \\ \text{volt-amps reactive} \\ \text{"VAR"} \end{matrix}$$

Complex Power
[V · A]
volt-amps

Apparent Power & Power Factor

"Apparent Power" = $|S| = \frac{V_A I_A}{Z} = V_{rms} I_{rms}$
- maximum possible value of $P = \text{real, average power}$
for a given V_A & I_A

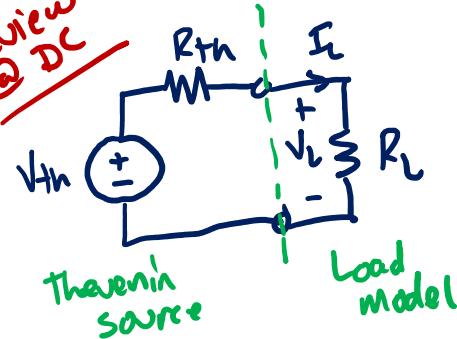
"Power Factor" $PF = \frac{P}{|S|} = \frac{P}{V_{rms} I_{rms}} = \cos(\phi_Z)$

"leading" \rightarrow capacitive
"lagging" \rightarrow inductive
current w.r.t. voltage

$$0 \leq PF \leq 1$$

Maximum Power Transfer

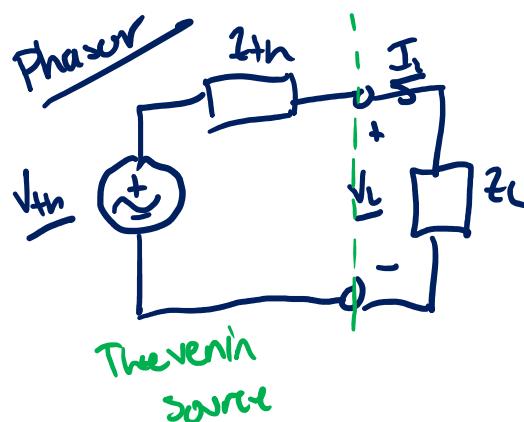
Review
@ DC



Numerator

What value of R_{th} gives maximum power to a fixed L_L ?

$$R_{th} = \phi$$

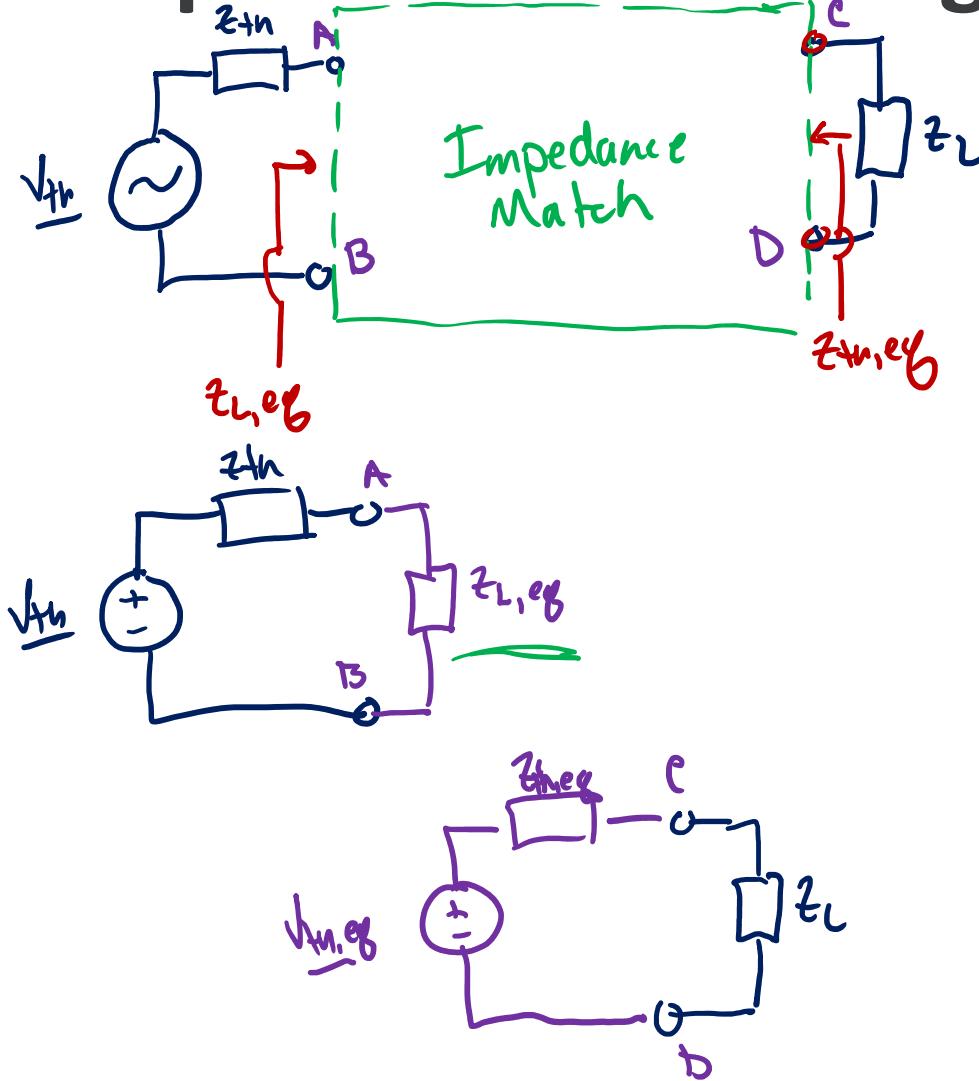


What value of Z_L results in maximum $P_L = \frac{1}{2} \operatorname{Re} \{ V_L I_L^* \}$
 Cancel out imaginary & set real parts equal

$$z_c = \underline{\operatorname{Re}\{z_{th}\}} - \operatorname{Im}\{z_{th}\}$$

$$z_L = z_{th}^*$$

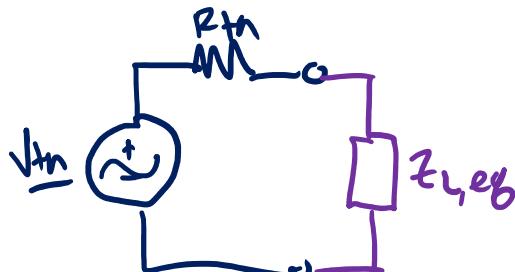
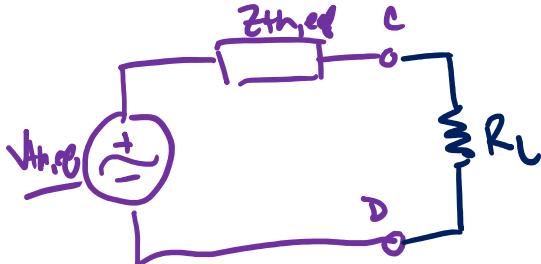
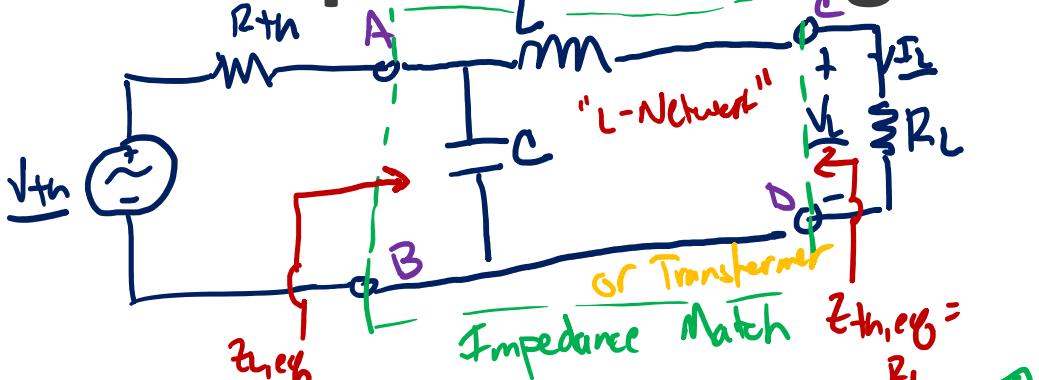
Impedance Matching



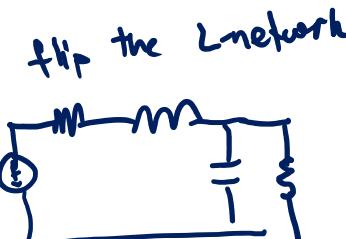
Impedance Matching Goals

- Maximize power to z_L ($z_L = z_{th,eq}^*$)
- Minimize distortion ($z_L = z_{th,eq}$)
- Maximize efficiency ($Re\{z_L\} \gg Re\{z_{th,eq}\}$)
- Minimize Q ($Im\{z_{L,eq}\} = 0$)

Example Matching Circuits



Works for $R_{th} \geq R_L$
if we had $R_L \geq R_{th}$



$$R_{th} > R_L$$

with no matching network

$$R_L < R_{th, max}$$

$$Z_L = j\omega L = jX_L$$

$$Z_C = \frac{-j}{\omega C} = jX_C$$

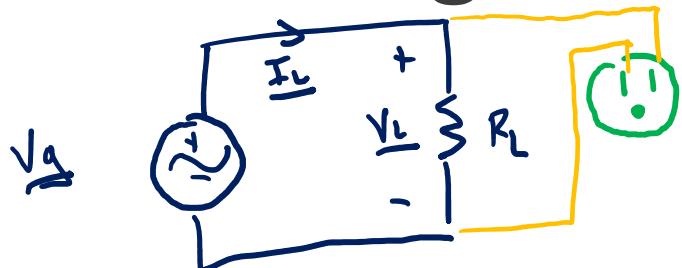
$$Z_{th,eq} = R_{th} \parallel (-jX_C) + jX_L$$

$$Z_{th,eq} = \frac{(-jX_C)R_{th}}{R_{th} - jX_C} \frac{(R_{th} + jX_C)}{(R_{th} + jX_C)} + jX_L$$

$$Z_{th,eq} = \frac{jX_C R_{th}}{R_{th}^2 + X_C^2} + j \left[X_L - \frac{jX_C R_{th}^2}{R_{th}^2 + X_C^2} \right]$$

$$\left\{ \begin{array}{l} X_C^2 = \frac{R_L R_{th}^2}{R_{th} - R_L} \\ X_L = \frac{jX_C R_{th}^2}{R_{th}^2 + X_C^2} \end{array} \right.$$

Matching Example



$$\omega = 2\pi 60 \text{ Hz}$$

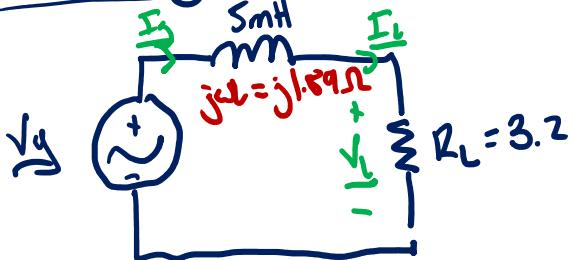
$$V_g = 170 \angle 40^\circ$$

$$R_L = 3.2 \Omega$$

(modeling 4.5kW of real power)

$$I_L = \frac{V_g}{R_L} = \frac{170 \angle 40^\circ}{3.2} = 53 \angle 40^\circ \text{ A}$$

$$S_g = \frac{1}{2} V_g I_L^* = \frac{1}{2} (170 \angle 40^\circ) (53 \angle -40^\circ) = 4.5 \text{ kW} + j0 \text{ VAR}$$



$$I_L = \frac{V_g}{R_L + j\omega L} = \frac{170 \angle 40^\circ}{3.2 + j1.89} = 45 \angle -30^\circ \text{ A}$$

$$V_L = I_L R_L = (45 \angle -30^\circ)(3.2) = 146.5 \angle -30^\circ \text{ V}$$

$$P_L = \frac{|V_L| |I_L|}{2} \cos(\phi_2) = \frac{146.5 \cdot 45}{2} = 3.3 \text{ kW}$$

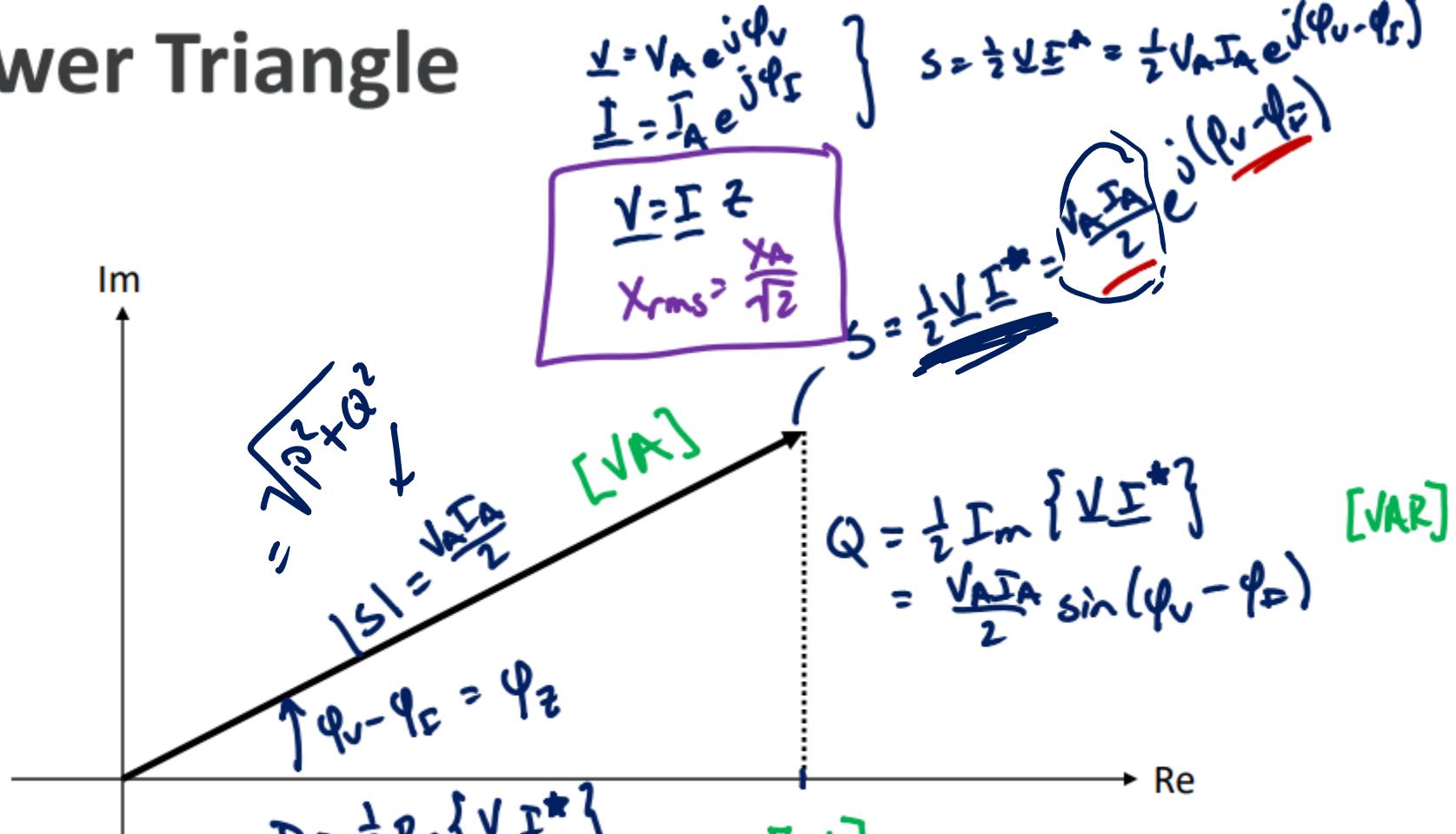
$$S_g = \frac{1}{2} V_g I_g^* = \frac{1}{2} (170 \angle 40^\circ) (45 \angle 30^\circ) = 3.8 \text{ kW} \angle 30^\circ = 3.3 \text{ kW} + j1.9 \text{ kVAR}$$

$$PF = \frac{P}{|S_g|} = \frac{3.3 \text{ kW}}{3.8 \text{ kVA}} = 0.87 \text{ lagging}$$

$$PF = \cos(\phi_L - \phi_I) = \cos(\phi_2)$$

current w.r.t. voltage
 "lagging" → inductive
 "leading" → capacitive

Power Triangle



$PF = \frac{P}{|S|} = \cos(\phi_V - \phi_I)$ (leading or lagging)
 current w.r.t. voltage
 capacitive induction