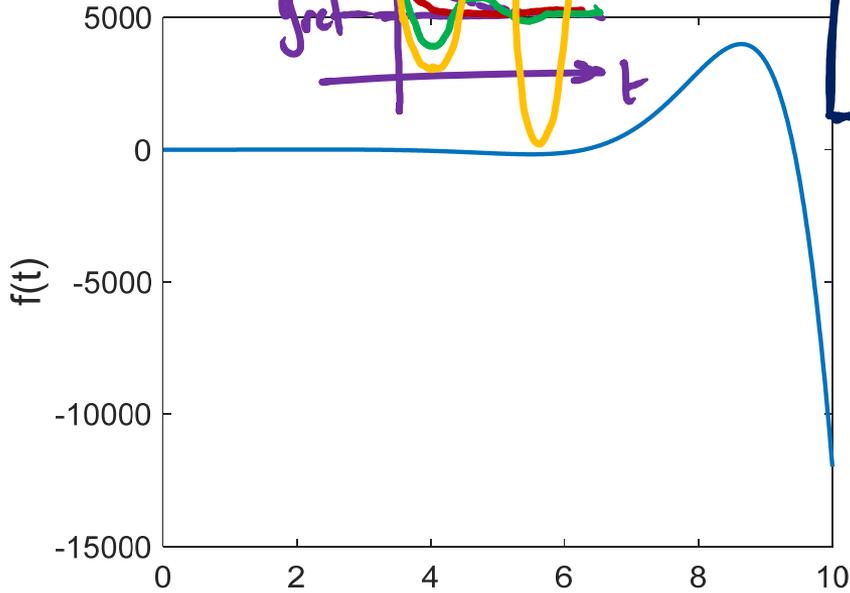
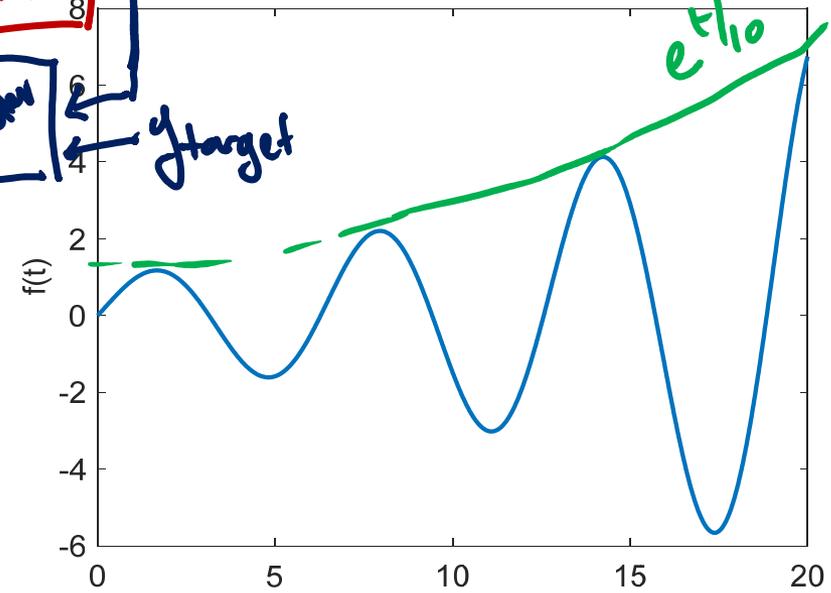
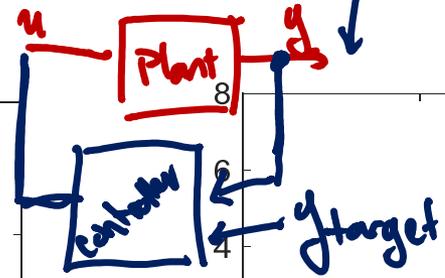


Unbounded Signals & Unstable Systems



$$f(t) = e^t \sin(t) e^{-tk} \quad k > 1$$



$$f(t) = e^{t/10} \sin(t)$$

Bounded signal $\exists B$ s.t. $|f(t)| \leq B \forall t$
 BIBO stable system: "BIBO" = bounded input, bounded output
 Always want any hardware circuit to be bounded & stable

Aside: Laplace and Fourier Revisited

Fourier Transform:

$$F(j\omega) = \int_{-\infty}^{+\infty} e^{-j\omega t} f(t) dt$$

Laplace Transform (Bilateral):

$$s = \sigma + j\omega$$

$$F(s) = \int_{-\infty}^{+\infty} e^{-st} f(t) dt$$

force unbounded signals to converge

*s = j\omega
i.e. \sigma = 0*

e^{-\sigma t} e^{-j\omega t}

if we let $s \Rightarrow j\omega$ Laplace becomes equivalent to Fourier Transform. But some signals don't have a Fourier Transform

Inverse Fourier Transform:

$$f(t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} F(j\omega) e^{j\omega t} d\omega$$

Inverse Laplace Transform (Bilateral):

$$f(t) = \frac{1}{2\pi j} \int_{\sigma_0 - j\infty}^{\sigma_0 + j\infty} F(s) e^{st} ds$$

\sigma_0 is any value that worked

Laplace Explanation

$$F(s) = \int_{0^-}^{+\infty} e^{-st} f(t) dt = \int_{0^-}^{+\infty} \underbrace{e^{-\sigma t}}_{\text{decay}} \underbrace{e^{-j\omega t}}_{\text{oscillation}} f(t) dt$$

$$f(t) = \frac{1}{2\pi j} \int_{\sigma_0 - j\infty}^{\sigma_0 + j\infty} \underbrace{e^{\sigma t}}_{\text{growth}} \underbrace{e^{j\omega t}}_{\text{oscillation}} F(s) ds$$

L21

Fourier Series

Assume we have some function $f(t)$ which is periodic with period $T_0 = \frac{2\pi}{\omega_0}$

$f(t) = a_0 + \sum_{k=1}^{\infty} a_k \cos(k\omega_0 t) + b_k \sin(k\omega_0 t)$

$f(t)$ can be expressed this way if:

- $f(t)$ is single-valued
- $\int_{T_0} f(t) dt$ exists
- $f(t)$ had finite discontinuities and max/min per period

for a_0 (constant / DC term)

$$a_0 = \frac{1}{T_0} \int_0^{T_0} f(t) dt$$

for a_k : find average value of $f(t) \cdot \cos(n\omega_0 t)$

$$\frac{1}{T_0} \int_0^{T_0} f(t) \cos(n\omega_0 t) dt$$

Assuming Fourier Series is valid, that is

$$\frac{1}{T_0} \int_0^{T_0} \left[a_0 + \sum_{k=1}^{\infty} a_k \cos(k\omega_0 t) + b_k \sin(k\omega_0 t) \right] \cos(n\omega_0 t) dt$$

L24

Non-periodic Waveforms: Fourier Transform

Fourier Series \rightarrow only periodic waveforms

Fourier Transform \rightarrow non-periodic signals

\rightarrow treat any non-periodic signal as a periodic signal with $T \rightarrow \infty$

Fourier Series: $C_k = \frac{1}{T} \int_{-T/2}^{T/2} f(t) e^{-jk\omega_0 t} dt$

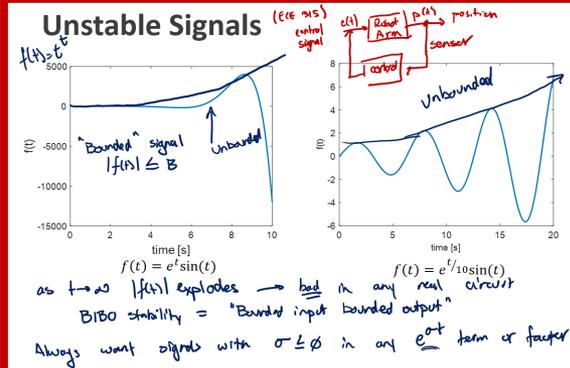
Fourier Transform: $T C_k = \int_{-\infty}^{\infty} f(t) e^{-j\omega t} dt = F(\omega)$

Fourier Series: $f(t) = C_0 + \sum_{k=-\infty}^{\infty} C_k e^{jk\omega_0 t}$

Fourier Inverse Transform: $f(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(\omega) e^{j\omega t} dt$

L30

Unstable Signals



L23

Complex Form of Fourier Series

Euler: $e^{j\omega t} = \cos(\omega t) + j \sin(\omega t)$

$$\cos(\omega t) = \frac{1}{2} (e^{j\omega t} + e^{-j\omega t})$$

$$\sin(\omega t) = \frac{1}{2j} (e^{j\omega t} - e^{-j\omega t})$$

Fourier Series

$$f(t) = a_0 + \sum_{k=1}^{\infty} a_k \left[\frac{e^{jk\omega_0 t} + e^{-jk\omega_0 t}}{2} \right] + \sum_{k=1}^{\infty} b_k \left[\frac{e^{jk\omega_0 t} - e^{-jk\omega_0 t}}{2j} \right]$$

C_k for $k > 0$ C_k for $k < 0$

$$f(t) = a_0 + \sum_{k=-\infty}^{\infty} C_k e^{jk\omega_0 t}$$

L25

Example Signal Laplace Transforms

$f(t) = u(t)$

$$\mathcal{L}\{f(t)\} = F(s) = \int_0^{\infty} e^{-st} u(t) dt = \int_0^{\infty} e^{-st} dt$$

$$= \left[-\frac{1}{s} e^{-st} \right]_0^{\infty} = \left[\frac{1}{s} - \frac{1}{s} e^{-s\infty} \right] = \frac{1}{s}$$

Region of Convergence: $\text{Re}\{s\} > 0$

$f(t) = e^{-at} u(t)$ $a \in \mathbb{R}^+$

$$\mathcal{L}\{f(t)\} = \int_0^{\infty} e^{-st} e^{-at} u(t) dt = \int_0^{\infty} e^{-(s+a)t} dt$$

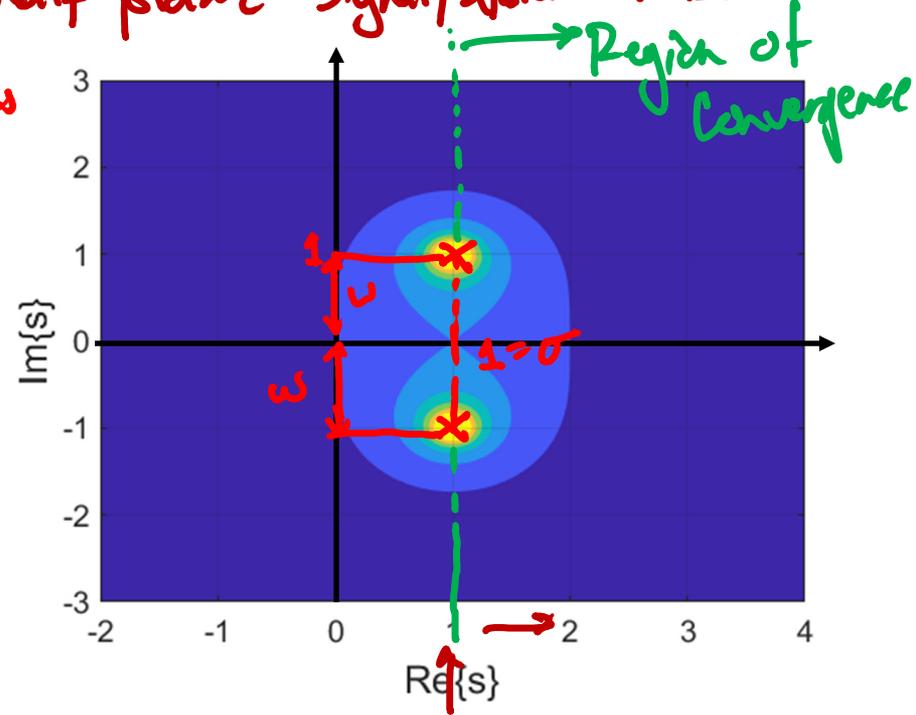
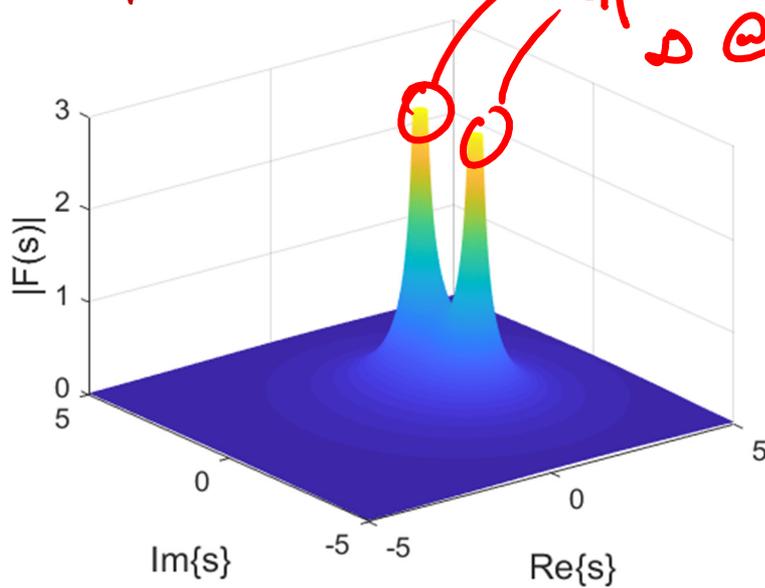
using previous: $\mathcal{L}\{e^{-at} u(t)\} = \frac{1}{s+a}$ if $\text{Re}\{s+a\} > 0$

Generalize: $\mathcal{L}\{f(t) e^{-at}\} = F(s+a)$ where $\mathcal{L}\{f(t)\} = F(s)$

The s-plane

The R.O.C of the Laplace Transform is the complex plane to the right of all poles

If all poles are in the open left half-plane signal/system is Bounded
 explodes ∞ @ poles



$$\underline{\underline{F(s)}} = \frac{1}{s^2 - 2s + 2} = \frac{1}{(s - (1 + j))(s - (1 - j))}$$

→ poles @ $s = 1 \pm j$

$f(t) = e^t \sin(t) u(t)$ → in Laplace transform must multiply by $e^{-\sigma t}$ $\sigma \geq 1$ for convergence