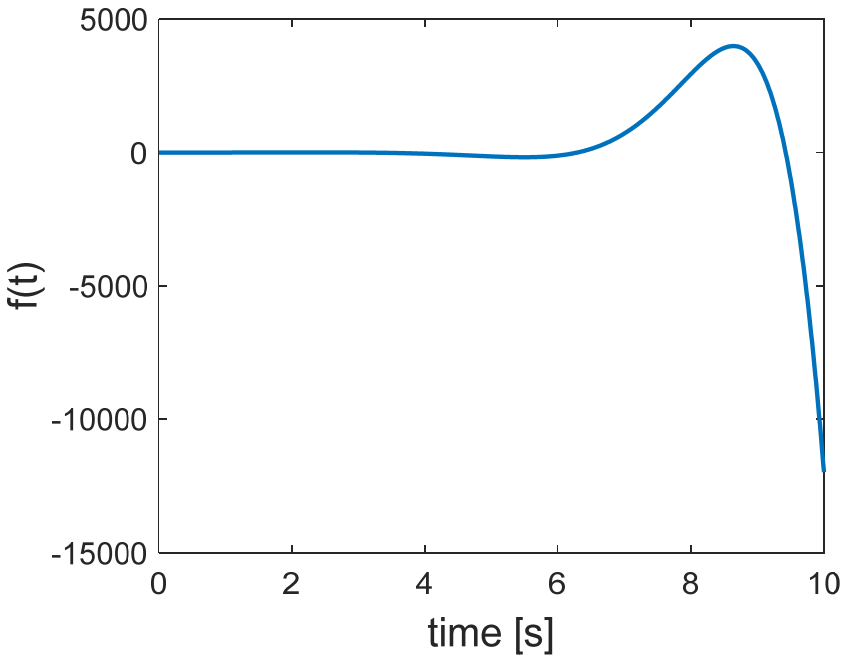
Pole Locations

$$\boxed{x = N_H + N_I}$$

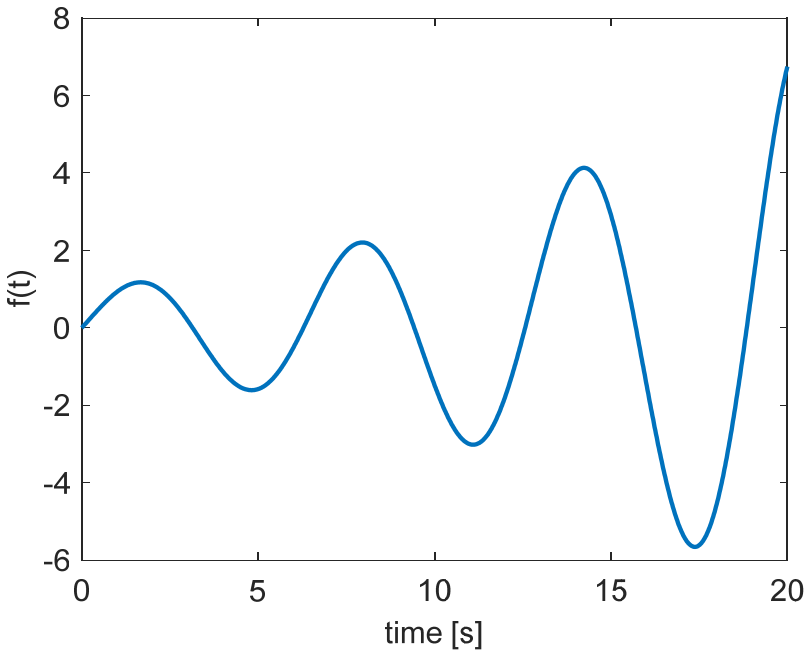
$$V_o(s) = V_I(s)H(s) = \left(\frac{\sum_{i=0}^{M_I} c_i s^i}{\sum_{i=0}^{N_I} d_i s^i}\right)\left(\frac{\sum_{i=0}^{M_H} a_i s^i}{\sum_{i=0}^{N_H} b_i s^i}\right) = \frac{(s-z_1)(s-z_2)\cdots(s-z_{M_H+M_I})}{(s-p_1)(s-p_2)\cdots(s-p_{N_H+N_I})}$$

$$= \frac{k_1}{(s-p_1)} + \frac{k_2}{(s-p_2)} + \cdots + \frac{k_x}{(s-p_x)}$$

Unbounded Signals & Unstable Systems



$$f(t) = e^{t}\sin(t)$$



$$f(t) = e^{t/10}\sin(t)$$

Aside: Laplace and Fourier Revisited

Fourier Transform:

$$F(j\omega) = \int_{-\infty}^{+\infty} e^{-j\omega t} f(t)\, dt$$

Laplace Transform (Bilateral):

$$F(s) = \int_{-\infty}^{+\infty} e^{-st} f(t)\, dt$$

Inverse Fourier Transform:

$$f(t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} F(j\omega) e^{j\omega t} d\omega$$

Inverse Laplace Transform (Bilateral):

$$f(t) = \frac{1}{2\pi j} \int_{\sigma_0 - j\infty}^{\sigma_0 + j\infty} F(s) e^{st} ds$$

Laplace Explanation

$$F(s) = \int_{0^-}^{+\infty} e^{-st} f(t)\, dt = \int_{0^-}^{+\infty} e^{-\sigma t} e^{-j\omega t} f(t) dt$$

$$f(t) = \frac{1}{2\pi j} \int_{\sigma_0 - j\infty}^{\sigma_0 + j\infty} e^{\sigma t} e^{j\omega t} F(s)\, ds$$

L21

Fourier Series

Assume we have some function $f(t)$ which is periodic with period $T_0 = \frac{2\pi}{\omega_0}$

$$f(t) = \underbrace{a_0}_{\text{constant}} + \sum_{k=1}^{\infty} a_k \cos(k\underbrace{\omega_0}_{\omega_0 \text{ f above}} t) + b_k \sin(k\omega_0 t)$$

$f(t)$ can be expressed this way if
1. $f(t)$ is single-valued
2. $\int_{t_0}^{t_0+T_0} |f(t)|\, dt$ exists
3. $f(t)$ had finite discontinuities and max/min per period

for a_0 (constant / DC term)

$$\boxed{a_0 = \frac{1}{T_0}\int_0^{T_0} f(t)\, dt}$$

for a_k: find average value of $f(t) \cdot \cos(n\omega_0 t)$

$$\frac{1}{T_0}\int_0^{T_0} f(t) \cdot \cos(n\omega_0 t)$$

Assuming Fourier Series is valid, this is

$$\frac{1}{T_0}\int_0^{T_0} \left[a_0 + \sum_{k=1}^{\infty} a_k \cos(k\omega_0 t) + b_k \sin(k\omega_0 t) \right] \cos(n\omega_0 t)$$

L23

Complex Form of Fourier Series

Euler: $e^{j\omega t} = \cos\omega t + j \sin\omega t$

$$\begin{cases} \cos(\omega t) = \frac{1}{2}\left(e^{j\omega t} + e^{-j\omega t}\right) \\ \sin(\omega t) = \frac{1}{2j}\left(e^{j\omega t} - e^{-j\omega t}\right) \end{cases}$$

Fourier Series

$$f(t) = a_0 + \sum_{k=1}^{\infty} a_k \frac{1}{2}\left(e^{jk\omega_0 t} + e^{-jk\omega_0 t}\right) + \frac{b_k}{2j}\left(e^{jk\omega_0 t} - e^{-jk\omega_0 t}\right)$$

$$f(t) = a_0 + \sum_{k=1}^{\infty} e^{jk\omega_0 t}\underbrace{\left(\frac{a_k - b_k j}{2}\right)}_{c_k \text{ for } k>0} + e^{-jk\omega_0 t}\underbrace{\left(\frac{a_k + b_k j}{2}\right)}_{c_k \text{ for } k<0}$$

$$f(t) = a_0 + \sum_{k=-\infty}^{\infty} c_k e^{jk\omega_0 t}$$

L24

Non-periodic Waveforms: Fourier Transform

Fourier Series → only periodic waveforms

Fourier Transform → non-periodic signals

→ treat any non-periodic signal as a periodic signal with $T \to \infty$

Fourier Series: $c_k = \frac{1}{T}\int_{-T/2}^{T/2} f(t) e^{-jk\omega_0 t}\, dt$

Fourier Transform: $Tc_k = \boxed{\int_{-\infty}^{\infty} f(t) e^{-j\omega t}\, dt = F(\omega)}$

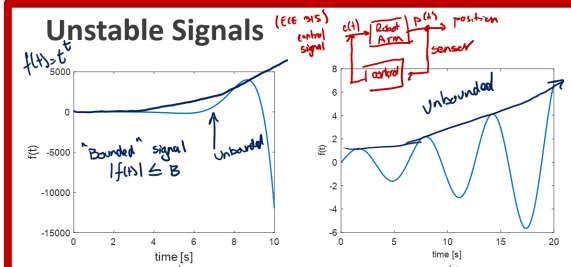
Fourier Series: $f(t) = c_0 + \sum_{k=-\infty}^{\infty} c_k e^{jk\omega_0 t}$

Fourier Inverse Transform: $\boxed{f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\omega) e^{j\omega t}\, dt}$

L30

Unstable Signals

(ECE 315) control signal



as $t \to \infty$ $|f(t)|$ explodes → bad in any real circuit

BIBO stability = "Bounded input bounded output"

Always want signals with $\sigma \le 0$ in any $e^{\sigma t}$ term or factor

L25

Example Signal Laplace Transforms

$f(t) = u(t)$

$$\mathcal{L}\{f(t)\} = F(s) = \int_0^{\infty} e^{-st} u(t)\, dt = \int_0^{\infty} e^{-st}\, dt$$

$$= \left[\frac{-1}{s} e^{-st}\right]_{t=0}^{t\to\infty} = \left[0 - \left(\frac{-1}{s}\right)\right] = \frac{1}{s}$$

$$\boxed{\mathcal{L}\{u(t)\} = \frac{1}{s}, \text{ if } Re\{s\} > 0}$$

Region of Convergence (R.O.C)

$f(t) = e^{-at}u(t)$, $a \in \mathbb{R}^+$

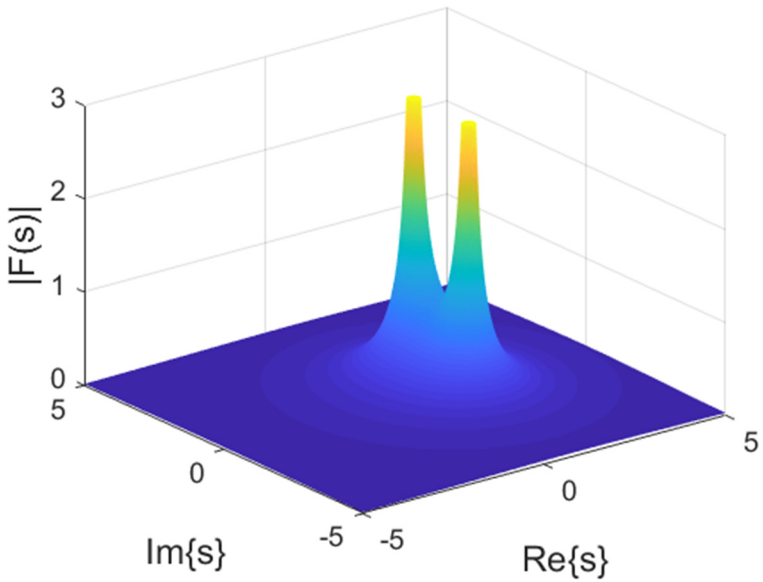
$$\mathcal{L}\{f(t)\} = \int_0^{\infty} e^{-st} e^{-at} u(t)\, dt = \int_0^{\infty} e^{-(s+a)t}\, dt$$

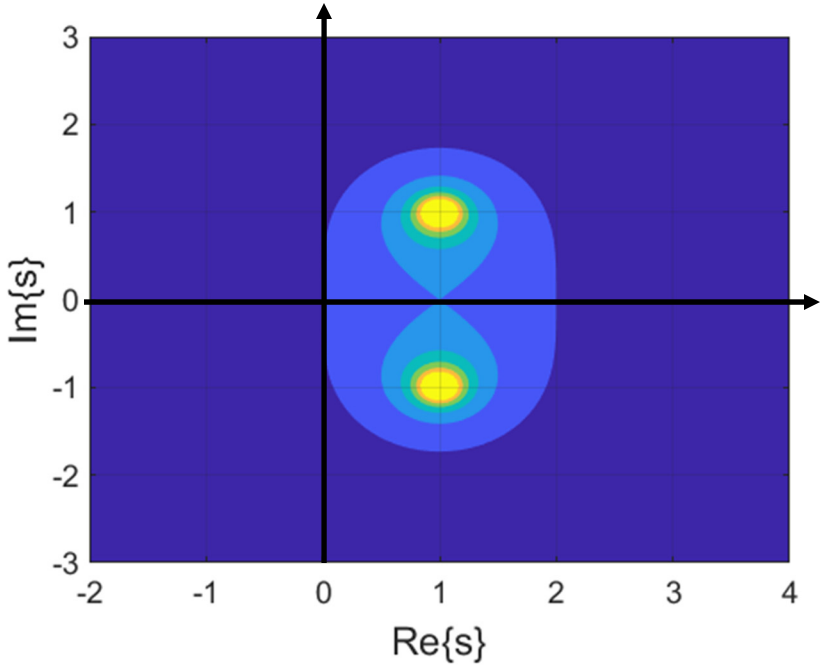
using previous

$$\boxed{\mathcal{L}\{e^{-at}u(t)\} = \frac{1}{s+a} \text{ if } Re\{s+a\} > 0}$$

Generalize: $\mathcal{L}\{f(t)e^{-at}\} = F(s+a)$ where $\mathcal{L}\{f(t)\} = F(s)$, $a \in \mathbb{R}^+$

The s-plane

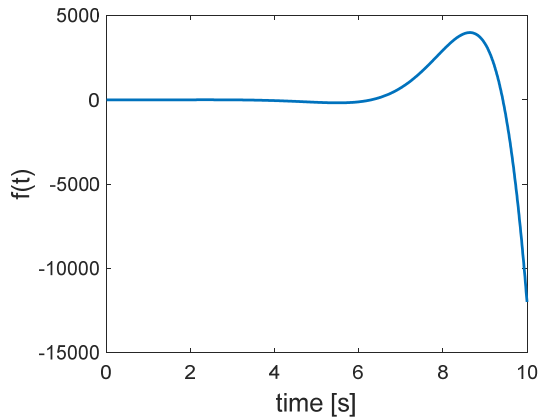


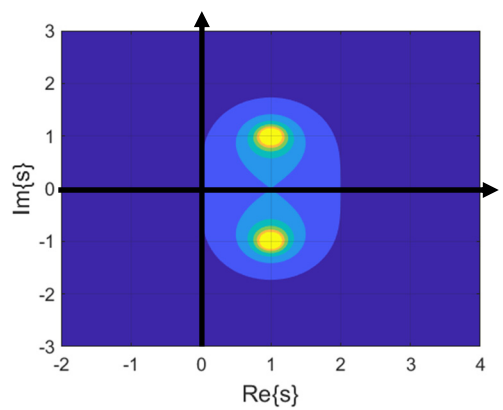


$$F(s) = \frac{1}{s^2 - 2s + 2} = \frac{1}{\big(s - (1 + j)\big)\big(s - (1 - j)\big)}$$

Example Functions

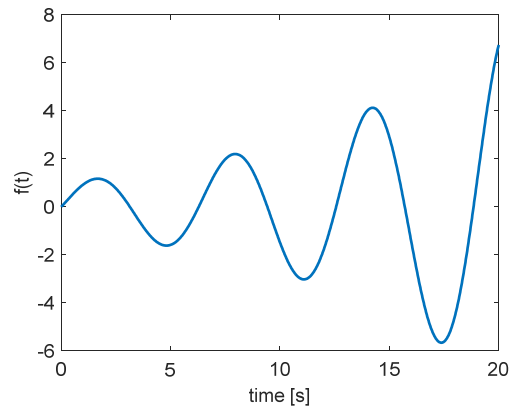
$$f(t) = e^{t}\sin(t)$$

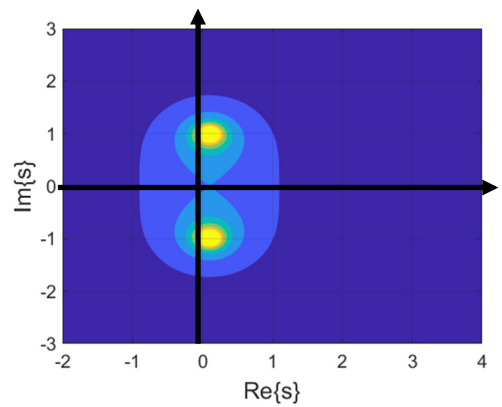




$$F(s) = \frac{1}{(s-(1+j))(s-(1-j))}$$

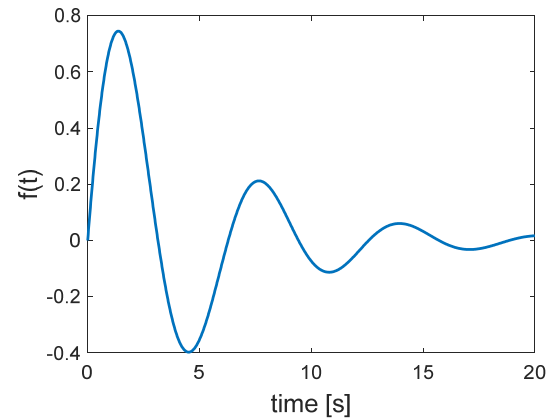
$$f(t) = e^{t/10}\sin(t)$$

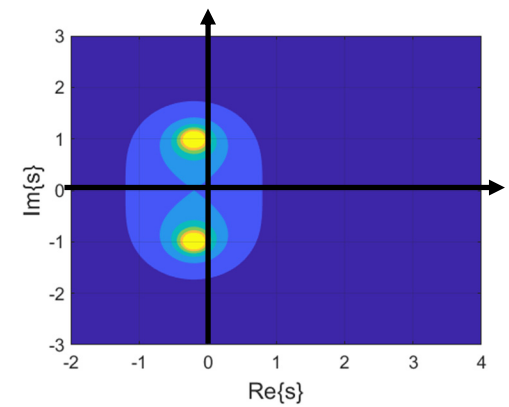




$$F(s) = \frac{1}{\left(s-\left(\frac{1}{10}+j\right)\right)\left(s-\left(\frac{1}{10}-j\right)\right)}$$

$$f(t) = e^{-t/5}\sin(t)$$

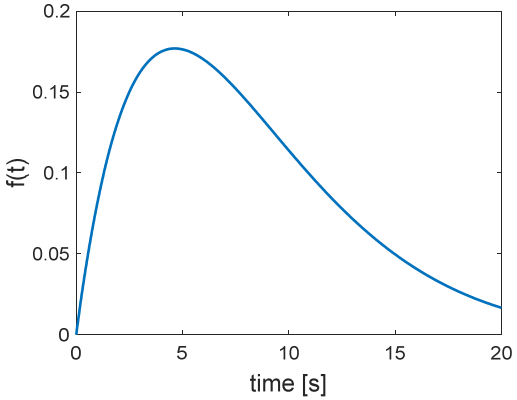


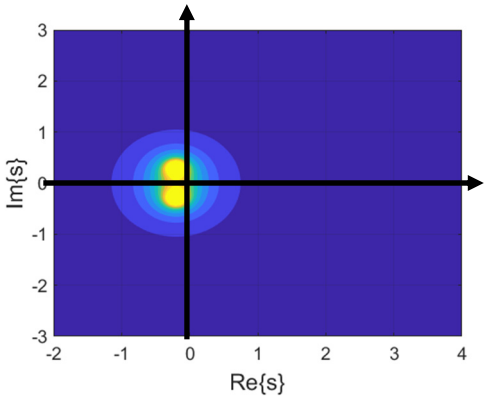


$$F(s) = \frac{1}{\left(s+\left(\frac{1}{5}+j\right)\right)\left(s+\left(\frac{1}{5}-j\right)\right)}$$

Example Functions

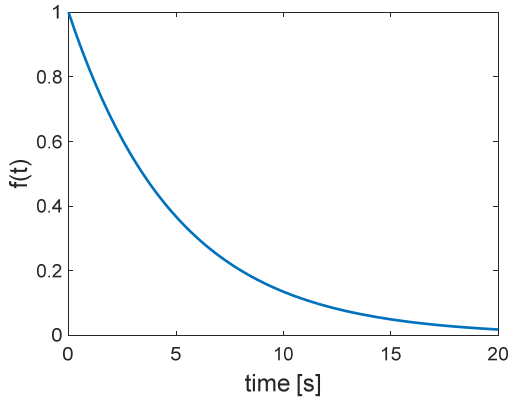
$$f(t) = e^{-t/5}\sin\left(t/10\right)$$

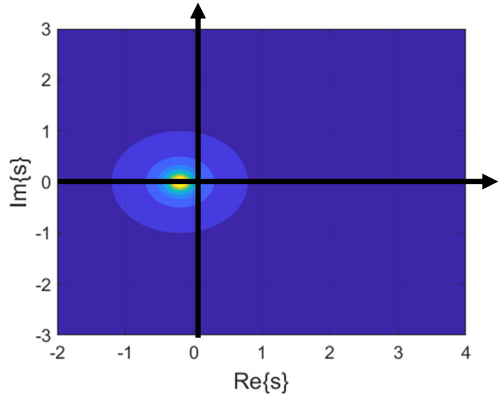




$$F(s) = \frac{1/10}{\left(s + \left(\frac{1}{5} + \frac{j}{10}\right)\right)\left(s + \left(\frac{1}{5} - \frac{j}{10}\right)\right)}$$

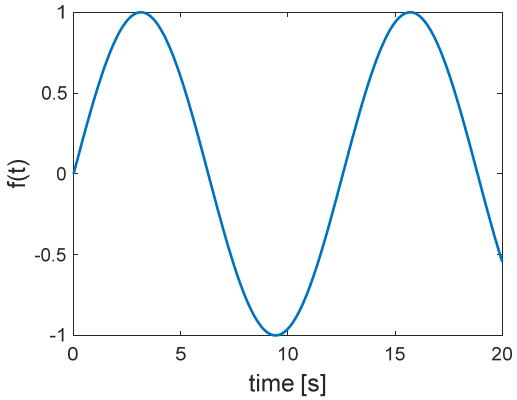
$$f(t) = e^{-t/5}$$

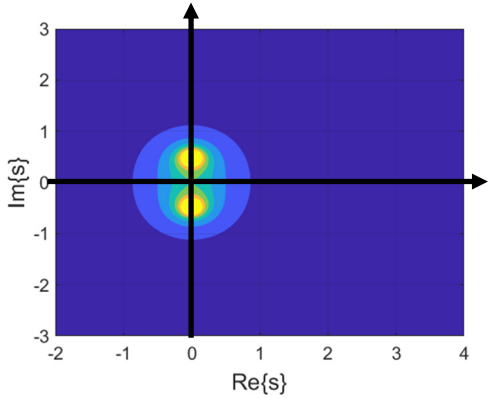




$$F(s) = \frac{1}{\left(s + \frac{1}{5}\right)}$$

$$f(t) = \sin\left(t/2\right)$$

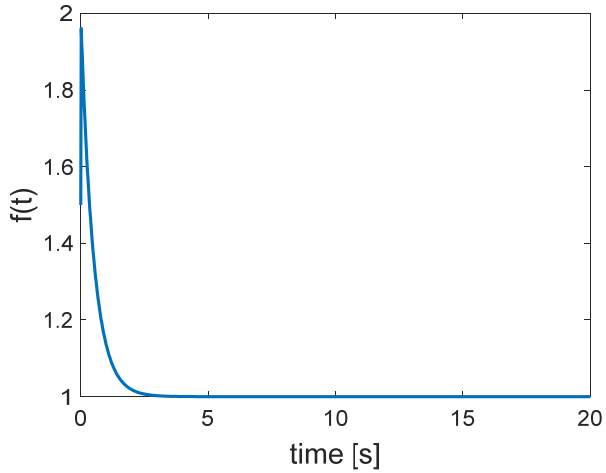


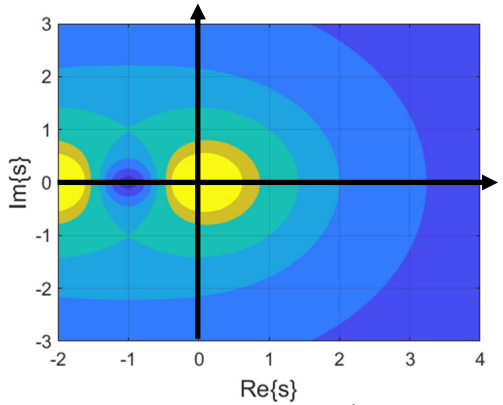


$$F(s) = \frac{1/2}{\left(s + \frac{j}{2}\right)\left(s - \frac{j}{2}\right)}$$

Example Functions

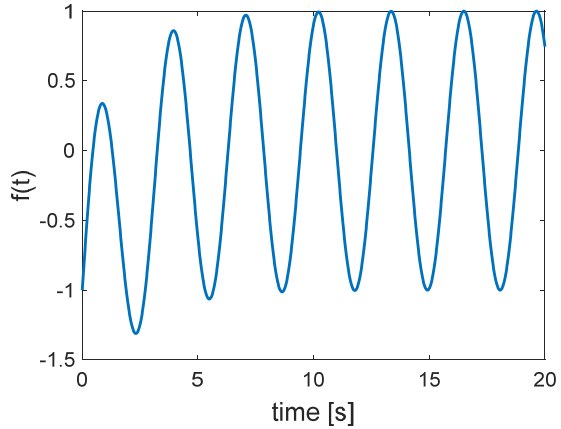
$$f(t) = -e^{-t/2} + u(t)$$

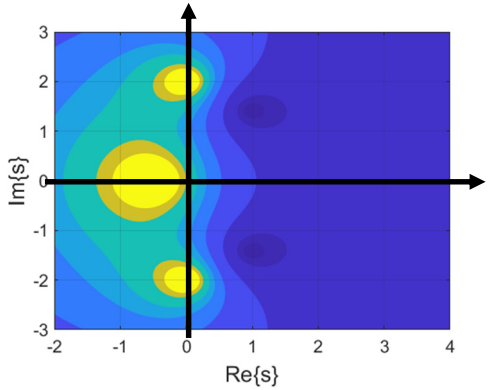




$$F(s) = 2\frac{s+1}{s(s+2)}$$

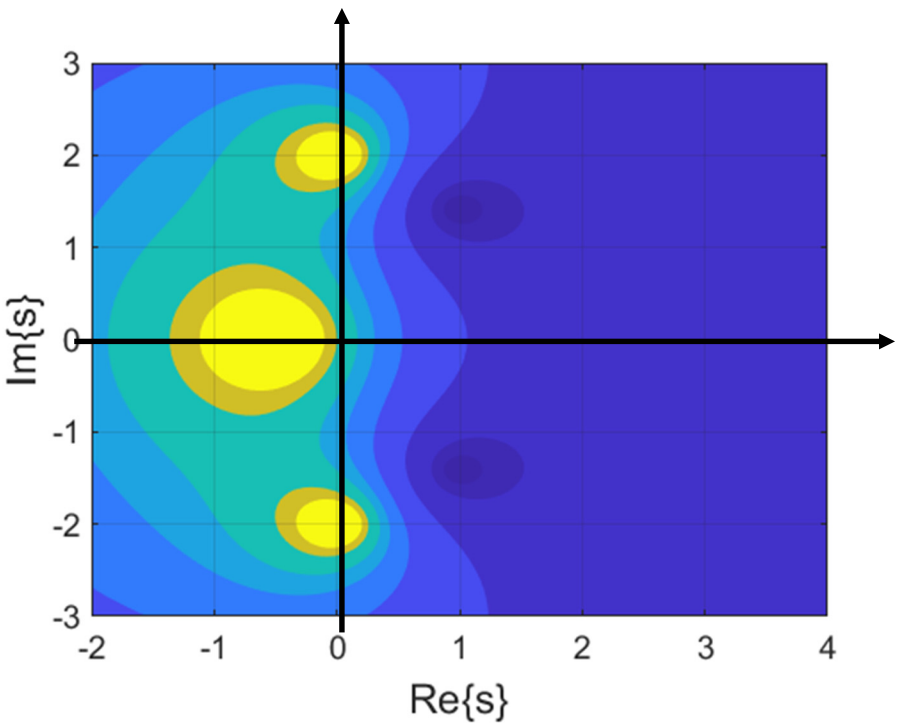
$$f(t) = -e^{-t/2} + \sin(2t)$$

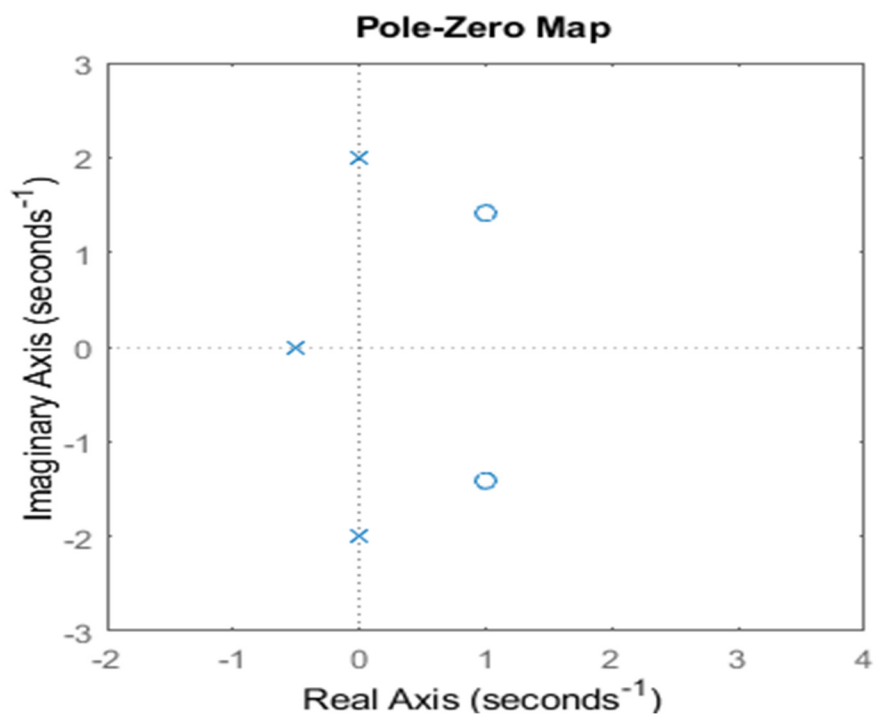




$$F(s) = -\frac{\left(s + \left(-1 + j\sqrt{2}\right)\right)\left(s + \left(-1 - j\sqrt{2}\right)\right)}{\left(s + {}^{1}/_{2}\right)(s + j2)(s - j2)}$$

Pole-Zero Map



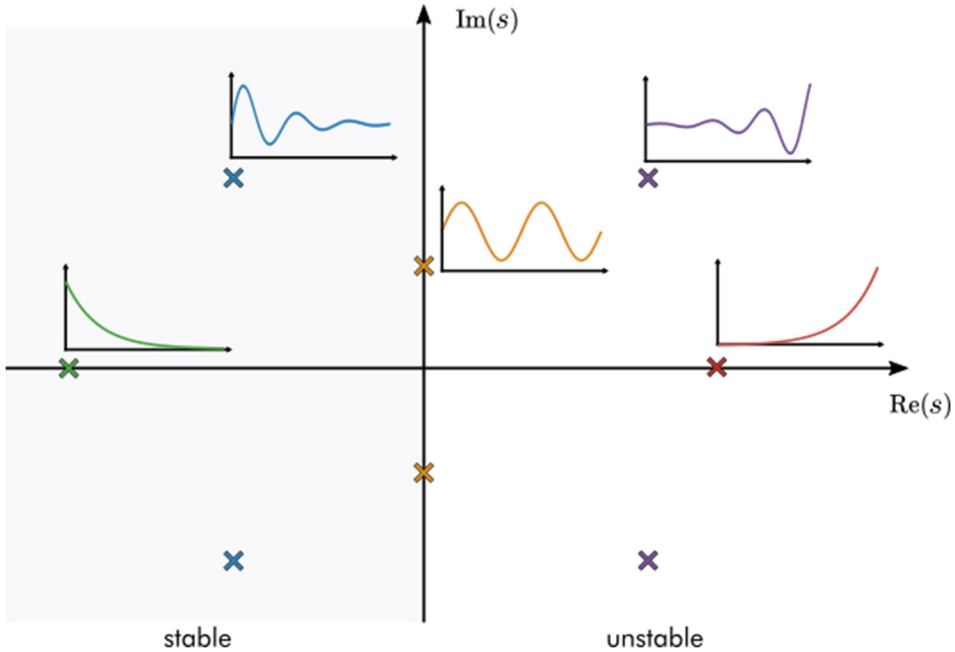


$$F(s) = -\frac{\left(s + \left(-1 + j\sqrt{2}\right)\right)\left(s + \left(-1 - j\sqrt{2}\right)\right)}{\left(s + {}^{1}/_{2}\right)(s + j2)(s - j2)}$$

<u>MATLAB:</u>

```
s = tf('s');
Fs = 2/(s^2 + 4) - 1/(s + 1/2);
pzmap(Fs);
```

Poles-Zero Plot



https://www.mathworks.com/matlabcentral/fileexchange/94635-transfer-function-analysis-of-dynamic-systems