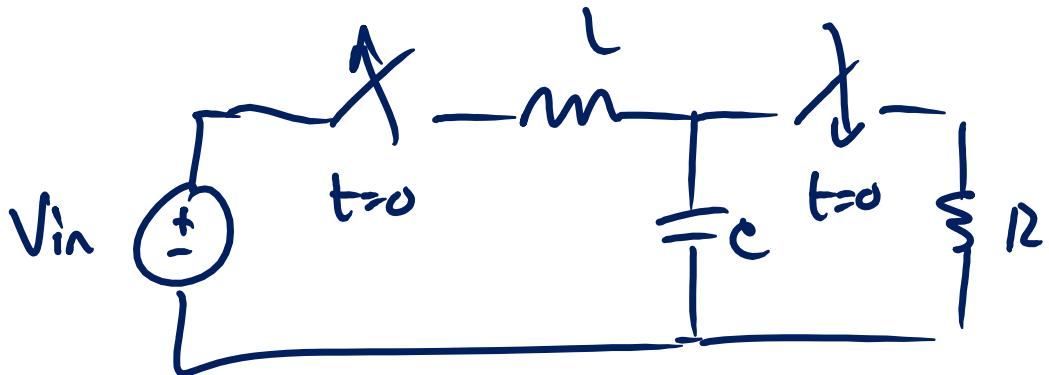
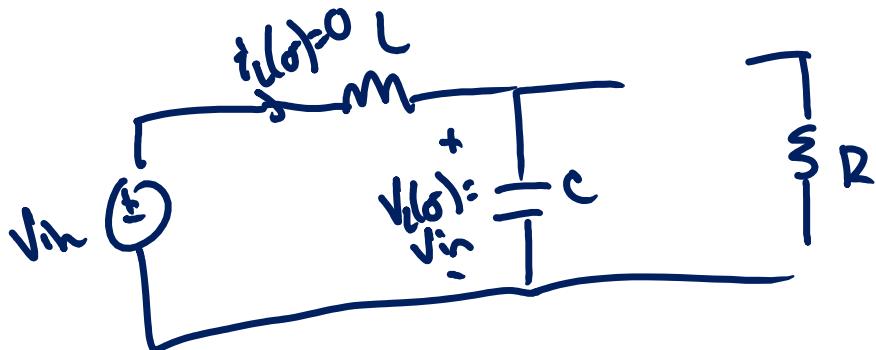


Announcements

- Midterm Exam #2 Wednesday
 - Lectures 21-36
 - Homeworks 6-9
 - Quiz 3-4
 - Chapter 17 (17.1-17.5) and Chapter 14
 - Experiment 2 (Review)
- Problems:
 1. Inverse Laplace transforms of $F(s)$
 2. Solve circuit transfer function including ICs
 3. Inverse Laplace given input and system
 4. For a system and input signal, decide if the output is bounded

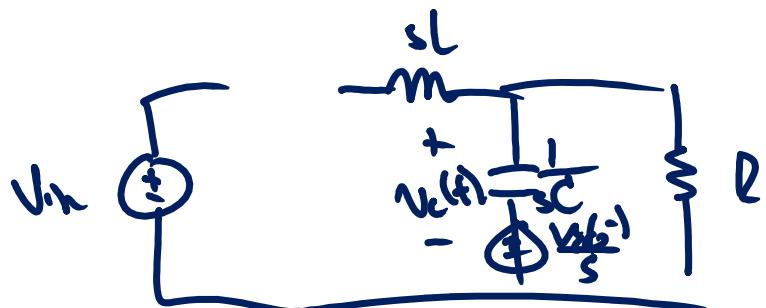


before $t=0$

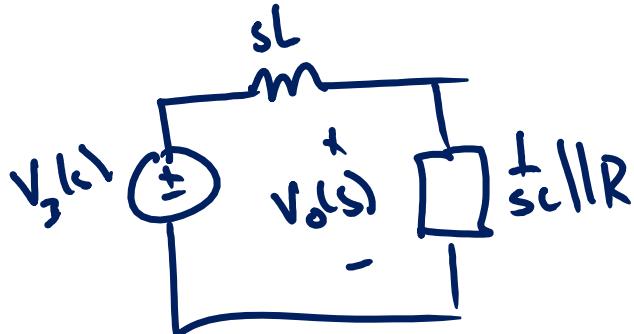
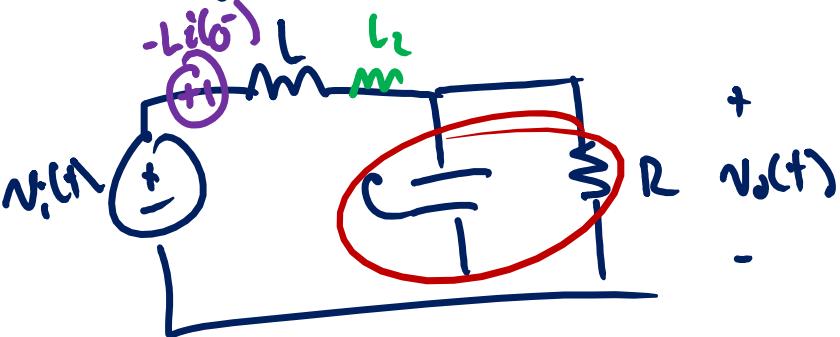


for $t > 0$

$$i_L(0^+) = 0 \\ v_C(0^+) = V_{in}$$



Quiz 4:

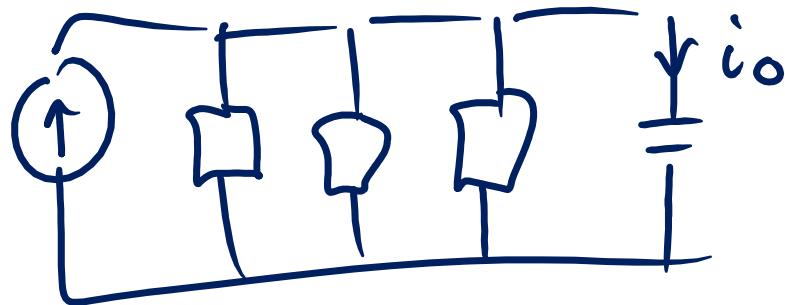
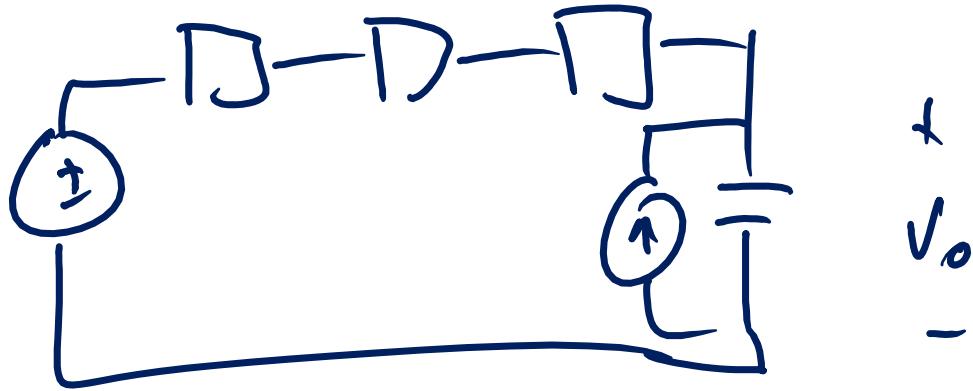


$$V_o(s) = \frac{1}{(s+2)^2} \frac{1}{s^2LC + s\frac{1}{R} + 1}$$

$$V_o(s) = \frac{1}{(s+2)^2} \frac{1}{s^2LC + s\frac{1}{R} + 1}$$

$$\begin{aligned} V_o(s) &= \frac{\frac{1}{sC} \parallel R}{sL + \frac{1}{sC} \parallel R} = \frac{\frac{1}{sC}}{sL + \frac{1}{sC} + \frac{R}{sC}} \\ &= \frac{\frac{R}{sC}}{sLR + \frac{L}{C} + \frac{R}{sC}} = \frac{1}{s^2LC + s\frac{1}{R} + 1} \\ &= \frac{k_1}{s+2} + \frac{k_2}{(s+2)^2} + \frac{k_3}{s-P_e} + \frac{k_3^*}{s-P_e^*} \end{aligned}$$

differentiation covering



MIDTERM 2 REVIEW

Frequency Response

$$\underline{V_o} = \underline{V_I} \frac{\underline{z_C}}{\underline{z_R} + \underline{z_C}} = \underline{V_I} \frac{-j/\omega C}{R - j/\omega C}$$

$$\underline{V_o} = \underline{V_I} \frac{1}{jR\omega C + 1}$$

Frequency Response

$$\frac{\underline{V_o}}{\underline{V_I}} = \frac{1}{j\omega RC + 1} = \underline{H(j\omega)}$$

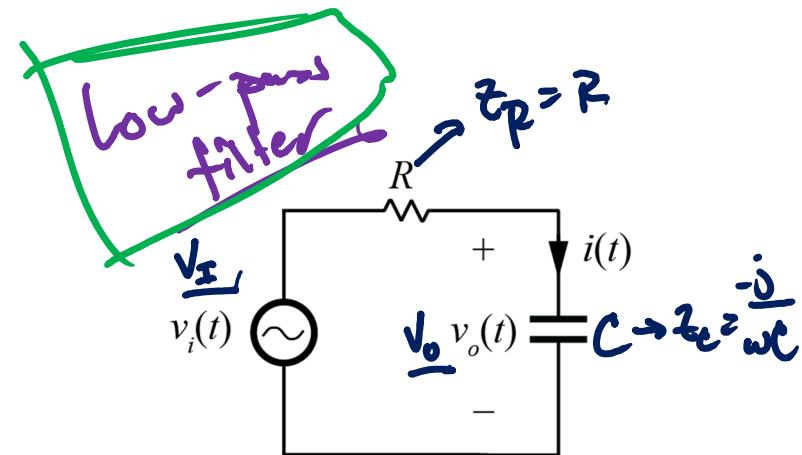
$$H(j\omega) = |H(j\omega)| \angle H(j\omega)$$

$$\underline{V_o} = \underline{V_I} \cdot H(j\omega)$$

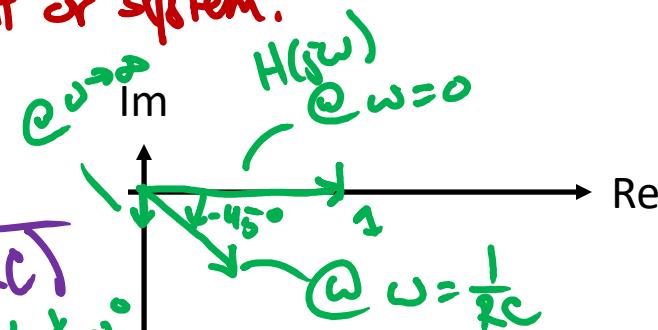
$$= |V_I| \cdot |H(j\omega)| \angle [\angle V_I + \angle H(j\omega)]$$

$$H(j\omega) = \frac{1}{j\omega RC + 1} = \frac{1}{\sqrt{1^2 + (\omega RC)^2}} \angle \tan^{-1}(-\omega RC)$$

$\begin{cases} @ \omega \rightarrow 0 \\ @ \omega = \frac{1}{RC} \\ @ \omega \rightarrow \infty \end{cases} \Rightarrow \begin{cases} H(j\omega) = 1 \angle 0^\circ \\ H(j\omega) = \frac{1}{\sqrt{2}} \angle -45^\circ \\ H(j\omega) = 0 \angle -90^\circ \end{cases}$



= Frequency Response \rightarrow some complex number that varies with ω and describes the $i - v$ relationship of the circuit or system.



Fourier Series

Assume we have some function $f(t)$ which is periodic with period $T_0 = \frac{2\pi}{\omega_0}$

$$f(t) = a_0 + \sum_{k=1}^{\infty} a_k \cos(k\omega_0 t) + b_k \sin(k\omega_0 t)$$

for a_0 (constant / DC term)

$$a_0 = \frac{1}{T_0} \int_0^{T_0} f(t) dt$$

for a_k : find average value of $f(t) \cdot \cos(n\omega_0 t)$

$$\frac{1}{T_0} \int_0^{T_0} f(t) \cdot \cos(n\omega_0 t) dt$$

Assuming Fourier Series is valid, this is

$$\frac{1}{T_0} \int_0^{T_0} \left[a_0 + \sum_{n=1}^{\infty} (a_n \cos(n\omega_0 t) + b_n \sin(n\omega_0 t)) \right] \cos(n\omega_0 t) dt$$

$f(t)$ can be expressed this way if

1. $f(t)$ is single-valued
2. $\int_{t_0}^{t_0+T_0} |f(t)| dt$ exists
3. $f(t)$ had finite discontinuities and max/min per period

Fourier Series Representation

Assume we have some function $f(t)$ which is periodic with period $T_0 = \frac{2\pi}{\omega_0}$

$$f(t) = a_0 + \sum_{k=1}^{\infty} a_k \cos(k\omega_0 t) + b_k \sin(k\omega_0 t)$$

$$a_k = \frac{2}{T_0} \int_{t_0}^{t_0+T_0} f(t) \cos(k\omega_0 t) dt$$

$$b_k = \frac{2}{T_0} \int_{t_0}^{t_0+T_0} f(t) \sin(k\omega_0 t) dt$$

$f(t)$ can be expressed this way if

1. $f(t)$ is single-valued
2. $\int_{t_0}^{t_0+T_0} |f(t)| dt$ exists
3. $f(t)$ had finite discontinuities and max/min per period

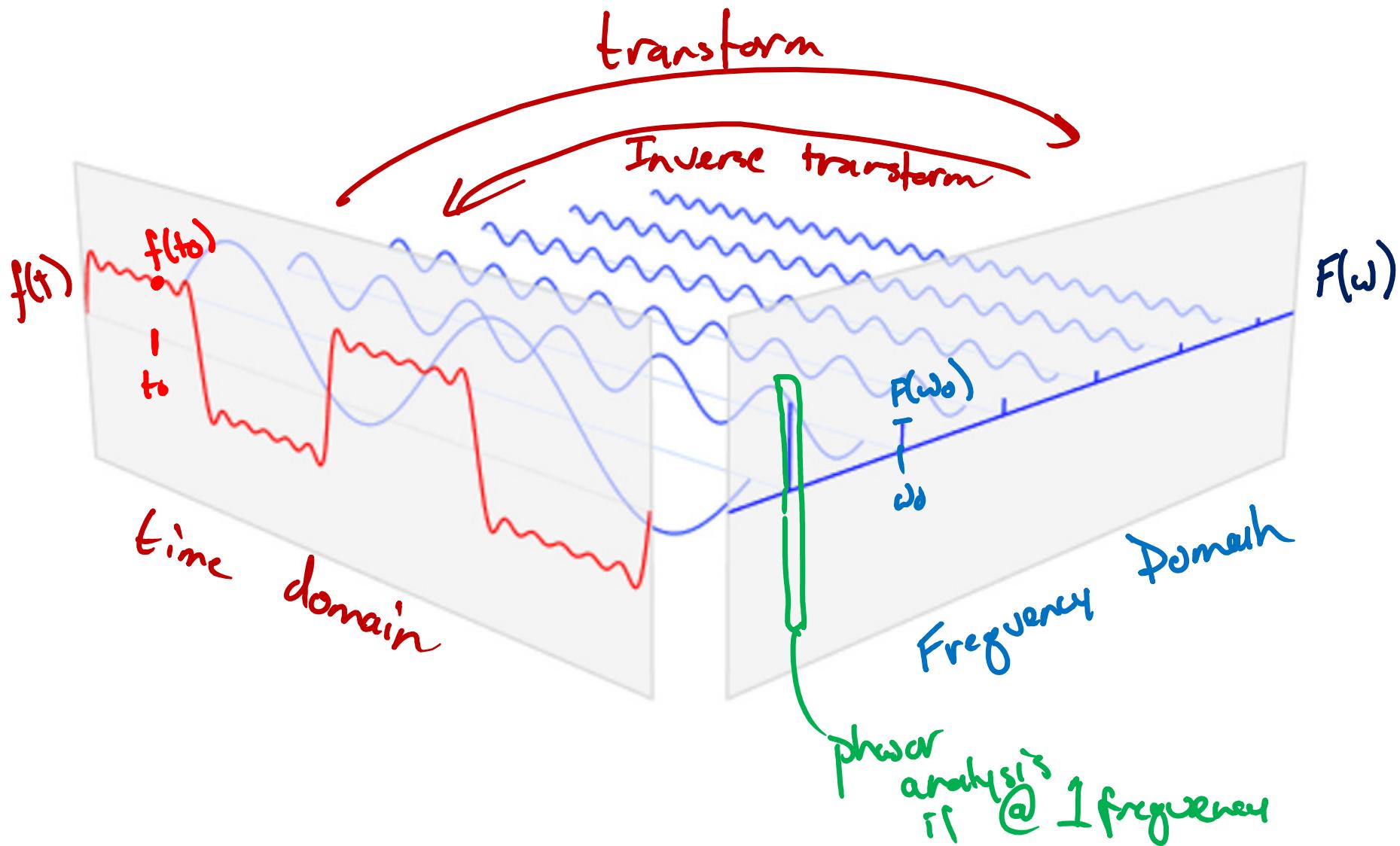
Alternate forms

$$f(t) = a_0 + \sum_{k=1}^{\infty} A_k \cos(k\omega_0 t + \varphi_k) \quad \left\{ \begin{array}{l} A_k = \sqrt{a_k^2 + b_k^2} \\ \varphi_k = \tan^{-1} \left(\frac{b_k}{a_k} \right) \end{array} \right.$$

$$f(t) = \sum_{k=-\infty}^{\infty} c_k e^{jk\omega_0 t} \quad \left\{ \begin{array}{l} c_k = \frac{1}{2} (a_k - jb_k) \\ c_{-k} = \frac{1}{2} (a_k + jb_k) \\ c_0 = a_0 \end{array} \right.$$

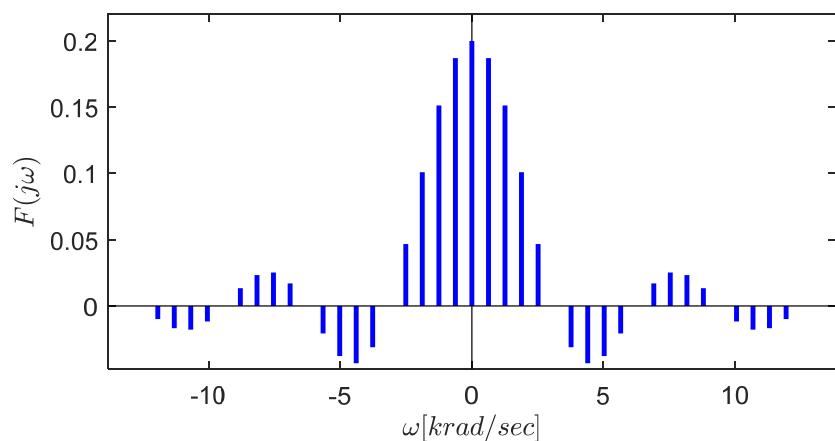
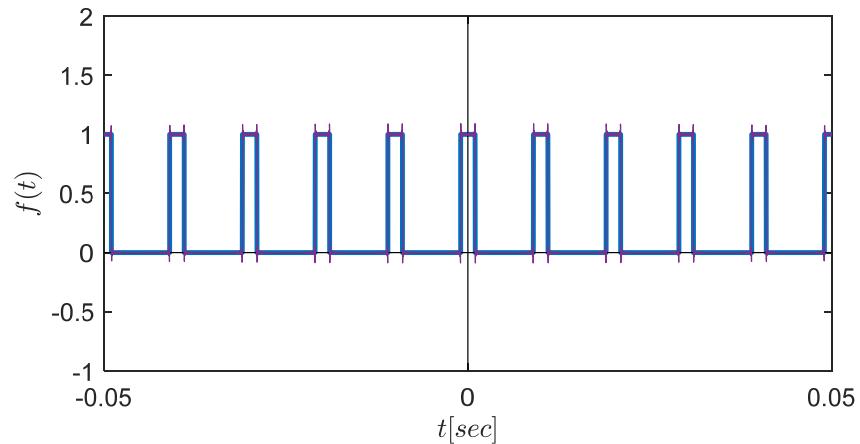
$$c_k = \frac{1}{T_0} \int_{t_0}^{t_0+T_0} f(t) e^{-jk\omega_0 t} dt$$

Input Spectrum



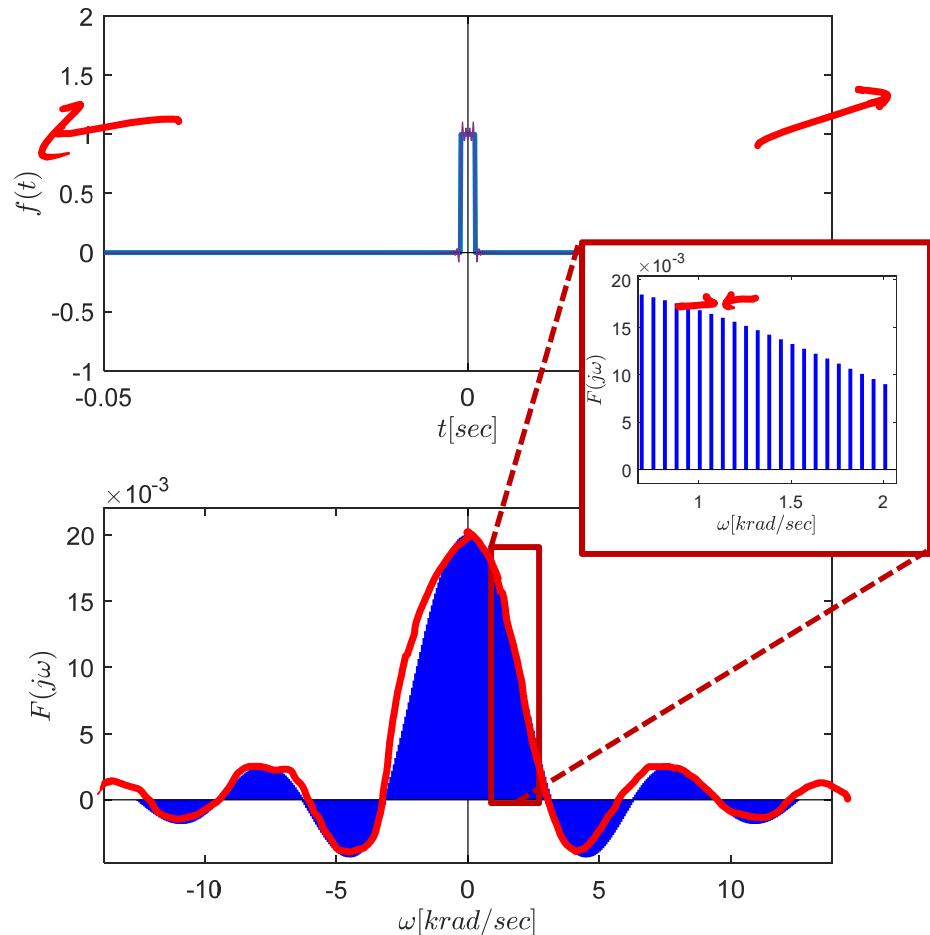
Alternate View

$$\begin{aligned}f &= 100 \text{ Hz} \\T &= 10 \text{ ms} \\&\tau = 2 \text{ ms}\end{aligned}$$



$$\begin{aligned}f &= 10 \text{ Hz} \\T &= 100 \text{ ms} \\&\tau = 2 \text{ ms}\end{aligned}$$

as $f \rightarrow 0$
 $T \rightarrow \infty$



Non-periodic Waveforms: Fourier Transform

Fourier Series → only periodic waveforms

Fourier Transform → non-periodic signals

→ treat any non-periodic signal as a periodic signal with $T \rightarrow \infty$

Fourier Series :

$$c_n = \frac{1}{T} \int_{-\pi/2}^{\pi/2} f(t) e^{-jkn\omega t} dt$$

Fourier Transform :

$$TC_k = \boxed{\int_{-\infty}^{\infty} f(t) e^{-j\omega t} dt = F(\omega)}$$

Fourier Series :

$$f(t) = c_0 + \sum_{n=-\infty}^{\infty} c_n e^{jkn\omega t}$$

Fourier Inverse Transform :

$$f(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(\omega) e^{j\omega t} dt$$

The Laplace Transform

Take Fourier Transform & replace $(j\omega)$ w/ $s = \sigma + j\omega$

$$F(s) = \int_{-\infty}^{\infty} e^{-st} f(t) dt$$

$$f(t) = \frac{1}{2\pi j} \int_{\sigma_0 - j\infty}^{\sigma_0 + j\infty} F(s) e^{st} ds$$

Usually (always in ECE 202)
Laplace transform

$$F(s) = \int_{0^-}^{\infty} e^{-st} f(t) dt$$

we'll use the unilateral

$$f(t) = \frac{1}{2\pi j} \int_{\sigma_0 - j\infty}^{\sigma_0 + j\infty} F(s) e^{st} ds$$

Short-hand:

$$F(s) = \mathcal{L}\{f(t)\} = \mathcal{L}\{f(t)\}$$

$$f(t) = \mathcal{L}^{-1}\{F(s)\} = \mathcal{L}^{-1}\{F(s)\}$$

$$f(t) \rightarrow F(s)$$

time-domain

$f(t)$
signals

ODEs

systems

Frequency / Fourier Domain

$F(j\omega)$

signals

$H(j\omega)$

systems

Laplace / s / complex freq
Domain

$F(s)$

signals

$H(s)$

systems

Complex Frequency

$$F(s) = \mathcal{L}\{f(t)\}$$

$s = \sigma + j\omega$ is a "complex frequency"

Laplace makes our signal $f(t)$ look like a superposition of terms $K_i e^{(\sigma+j\omega)t} = k_i e^{\sigma t} e^{j\omega t}$

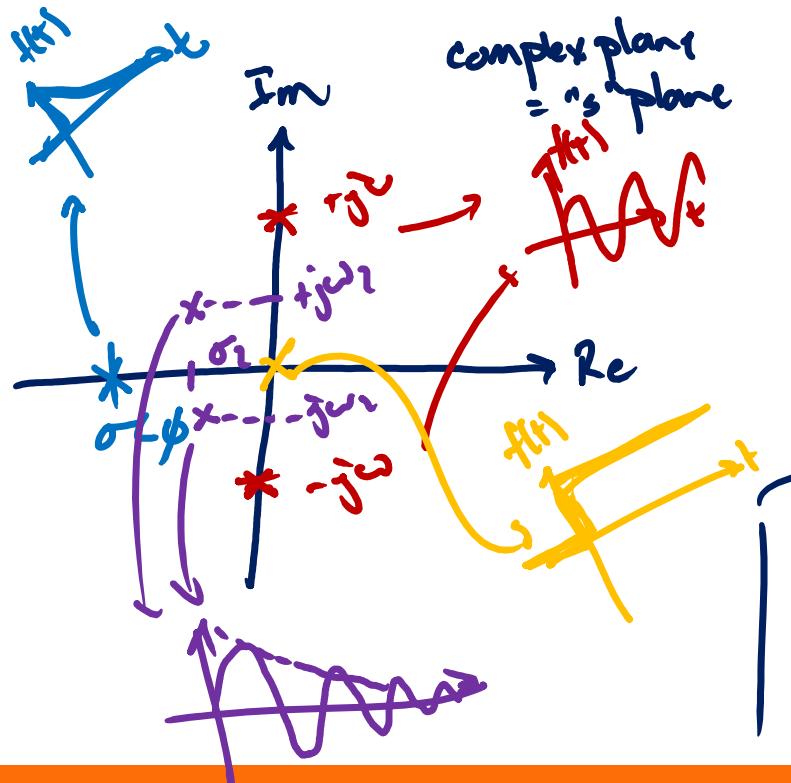
if $\sigma=0$, $s=\sigma+j\omega \rightarrow$ sinusoids
(ω / $-\omega$ included for Euler)
 $A \cos(\omega t + \phi)$

$\omega=\phi$, $s=\sigma+j\phi \rightarrow$ exponentials $e^{\sigma t}$
converging if $\sigma < \phi$ Be

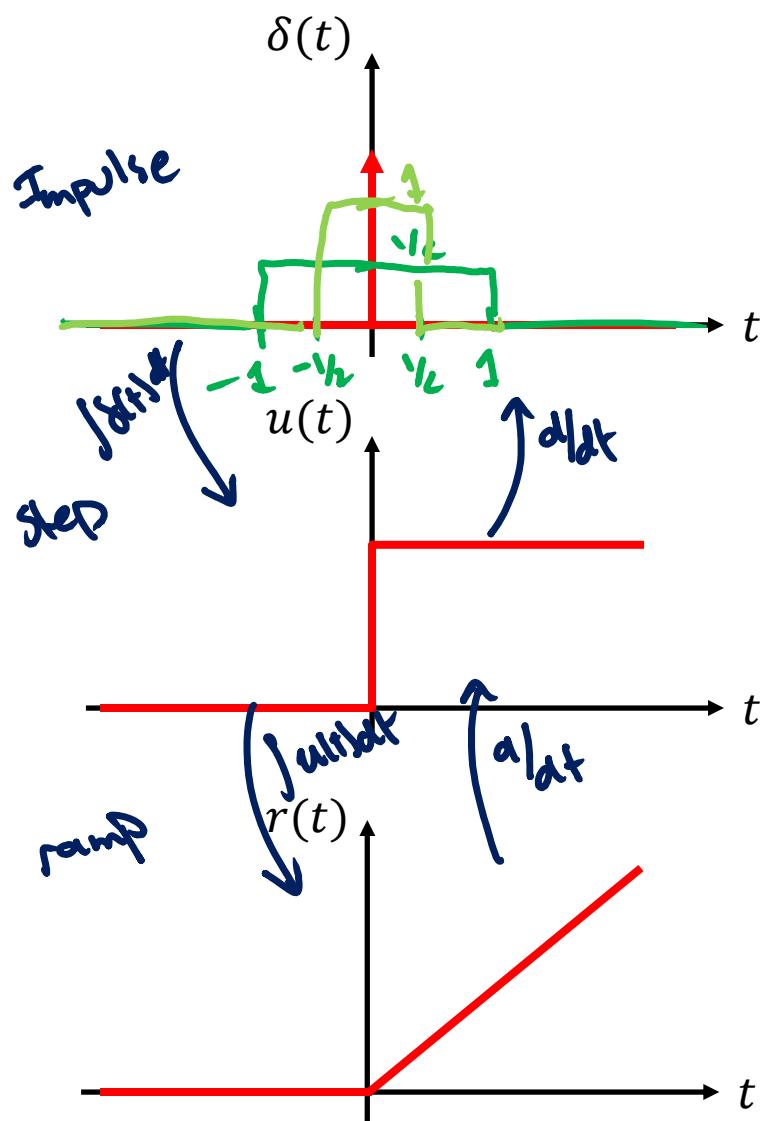
if $\omega=\phi$ $\& \sigma=0$, $s=\phi \rightarrow$ constants C

nothing, $s=\sigma+j\omega \rightarrow$ exponentials $\&$ sinusoids
 $D e^{\sigma t} \cos(\omega t + \phi)$

- Conditions for $F(s)$ to exist
1. $f(t)$ must be integrable over any finite time range
 2. $\lim_{t \rightarrow \infty} e^{-\sigma_0 t} |f(t)|$ at exists for some real σ_0



Impulse, Step, and Ramp Functions



$$\delta(t) \begin{cases} 0 & t \neq 0 \\ \infty & t = 0 \end{cases}$$

$$\int_{-\infty}^{+\infty} \delta(t) dt = \int_{0^-}^{0^+} \delta(t) dt = 1$$

$$u(t) \begin{cases} 0 & t < 0 \\ 1 & t > 0 \end{cases}$$

$$r(t) = tu(t) = \begin{cases} 0 & t \leq 0 \\ t & t \geq 0 \end{cases}$$

Example Signal Laplace Transforms

$$\underline{f(t) = u(t)}$$

$$\begin{aligned} \mathcal{L}\{f(t)\} = F(s) &= \int_0^{\infty} e^{-st} u(t) dt = \int_0^{\infty} e^{-st} dt \\ &= \left[-\frac{1}{s} e^{-st} \right] \Big|_{t=0}^{t \rightarrow \infty} = \left[0 - \left(-\frac{1}{s} \right) \right] = \frac{1}{s} \end{aligned}$$

$$\boxed{\mathcal{L}\{u(t)\} = \frac{1}{s}, \text{ if } \operatorname{Re}\{s\} > 0}$$

Region of convergence (R.O.C)

$$f(t) = e^{-at} u(t)$$

$a \in \mathbb{R}^+$

$$\mathcal{L}\{f(t)\} = \int_0^{\infty} e^{-st} e^{-at} u(t) dt = \int_0^{\infty} e^{-(s+a)t} dt$$

using previous

$$\boxed{\mathcal{L}\{e^{-at} u(t)\} = \frac{1}{s+a} \text{ if } \operatorname{Re}\{s+a\} > 0}$$

Generalize: $\mathcal{L}\{f(t)e^{-at}\} = F(s+a)$ where $\mathcal{L}\{f(t)\} = P(s)$

$a \in \mathbb{R}^+$

TABLE 14.1 Laplace Transform Pairs

$f(t) = \mathcal{L}^{-1}\{\mathbf{F}(s)\}$	$\mathbf{F}(s) = \mathcal{L}\{f(t)\}$	$f(t) = \mathcal{L}^{-1}\{\mathbf{F}(s)\}$	$\mathbf{F}(s) = \mathcal{L}\{f(t)\}$
$\delta(t)$	1	$\frac{1}{\beta - \alpha} (e^{-\alpha t} - e^{-\beta t}) u(t)$	$\frac{1}{(s + \alpha)(s + \beta)}$
$u(t)$	$\frac{1}{s}$	$\sin \omega t u(t)$	$\frac{\omega}{s^2 + \omega^2}$
$t u(t)$	$\frac{1}{s^2}$	$\cos \omega t u(t)$	$\frac{s}{s^2 + \omega^2}$
$\frac{t^{n-1}}{(n-1)!} u(t), n = 1, 2, \dots$	$\frac{1}{s^n}$	$\sin(\omega t + \theta) u(t)$	$\frac{s \sin \theta + \omega \cos \theta}{s^2 + \omega^2}$
$e^{-\alpha t} u(t)$	$\frac{1}{s + \alpha}$	$\cos(\omega t + \theta) u(t)$	$\frac{s \cos \theta - \omega \sin \theta}{s^2 + \omega^2}$
$t e^{-\alpha t} u(t)$	$\frac{1}{(s + \alpha)^2}$	$e^{-\alpha t} \sin \omega t u(t)$	$\frac{\omega}{(s + \alpha)^2 + \omega^2}$
$\frac{t^{n-1}}{(n-1)!} e^{-\alpha t} u(t), n = 1, 2, \dots$	$\frac{1}{(s + \alpha)^n}$	$e^{-\alpha t} \cos \omega t u(t)$	$\frac{s + \alpha}{(s + \alpha)^2 + \omega^2}$

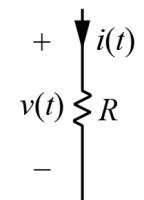
$2|k|e^{\alpha t} \cos(\omega t - \Delta h) u(t)$

$s \frac{k}{s - (\omega + j\Delta)} + s \frac{t^*}{s - (\omega - j\Delta)}$

Operation	$f(t)$	$\mathbf{F}(s)$
Addition	$f_1(t) \pm f_2(t)$	$\mathbf{F}_1(s) \pm \mathbf{F}_2(s)$
Scalar multiplication	$kf(t)$	$k\mathbf{F}(s)$
Time differentiation	$\frac{df}{dt}$	$s\mathbf{F}(s) - f(0^-)$
	$\frac{d^2f}{dt^2}$	$s^2\mathbf{F}(s) - sf(0^-) - f'(0^-)$
	$\frac{d^3f}{dt^3}$	$s^3\mathbf{F}(s) - s^2f(0^-) - sf'(0^-) - f''(0^-)$
Time integration	$\int_{0^-}^t f(t) dt$	$\frac{1}{s}\mathbf{F}(s)$
	$\int_{-\infty}^t f(t) dt$	$\frac{1}{s}F(s) + \frac{1}{s} \int_{-\infty}^{0^-} f(t) dt$
Convolution	$f_1(t) * f_2(t)$	$\mathbf{F}_1(s)\mathbf{F}_2(s)$
Time shift	$f(t-a)u(t-a), a \geq 0$	$e^{-as}\mathbf{F}(s)$
Frequency shift	$f(t)e^{-at}$	$\mathbf{F}(s+a)$
Frequency differentiation	$tf(t)$	$-\frac{d\mathbf{F}(s)}{ds}$
Frequency integration	$\frac{f(t)}{t}$	$\int_s^{\infty} \mathbf{F}(s) ds$
Scaling	$f(at), a \geq 0$	$\frac{1}{a}\mathbf{F}\left(\frac{s}{a}\right)$
Initial value	$f(0^+)$	$\lim_{s \rightarrow \infty} s\mathbf{F}(s)$
Final value	$f(\infty)$	$\lim_{s \rightarrow 0} s\mathbf{F}(s)$, all poles of $s\mathbf{F}(s)$ in LHP
Time periodicity	$f(t) = f(t+nT), n = 1, 2, \dots$	$\frac{1}{1-e^{-Ts}}\mathbf{F}_1(s),$ where $\mathbf{F}_1(s) = \int_{0^-}^T f(t) e^{-st} dt$

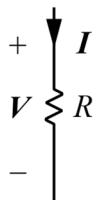
Circuit Laplace Transform

Time Domain



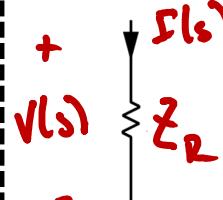
$$v(t) = i(t)R$$

Phasor Domain



$$V = IR$$

s-Domain

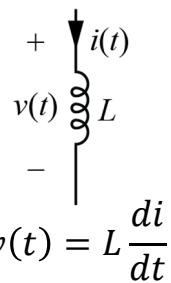


$$\mathcal{L}\{v(t)\} = \mathcal{L}\{i(t)R\}$$

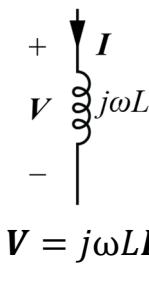
$$V(s) = I(s) \cdot R$$

$$Z_R = R$$

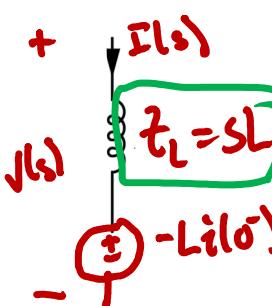
still called "impedance"



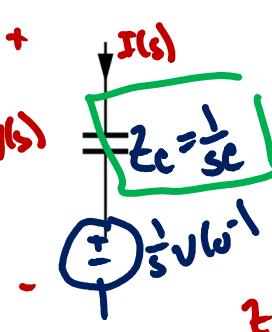
$$v(t) = L \frac{di}{dt}$$



$$V = j\omega LI$$



$$Z_L = sL$$

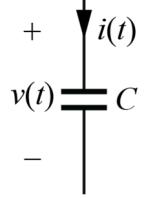


$$Z_C = \frac{1}{sC}$$

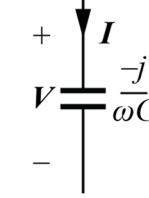
$$\mathcal{L}\{v(t)\} = \mathcal{L}\left\{L \frac{di}{dt}\right\}$$

$$V(s) = sL I(s) - L i(0^-)$$

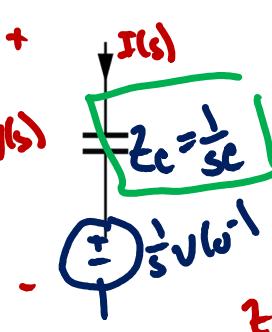
$$I(s) = \frac{V(s)}{sL} + \frac{1}{s} i(0^-)$$



$$i(t) = C \frac{dv}{dt}$$



$$V = -\frac{j}{\omega C} I$$



$$Z_C = \frac{1}{sC}$$

$$\mathcal{L}\{i(t)\} = \mathcal{L}\left\{C \frac{dv}{dt}\right\}$$

$$I(s) = sC V(s) - C v(0^-)$$

$$V(s) = \frac{1}{sC} I(s) + \frac{1}{s} v(0^-)$$

Laplace Transform of Diff EQs

N^{th} order circuit with sinusoidal input described by ($M \leq N$ for causality)

$$b_N \frac{d^N}{dt^N} v_o(t) + \cdots + b_1 \frac{d}{dt} v_o(t) + b_0 v_o(t) = a_M \frac{d^M}{dt^M} v_i(t) + \cdots + a_1 \frac{d}{dt} v_i(t) + a_0 v_i(t)$$

$$\sum_{i=0}^N b_i \frac{d^i}{dt^i} v_o(t) = \sum_{i=0}^M a_i \frac{d^i}{dt^i} v_i(t)$$

Then the Laplace transform of the circuit, neglecting initial conditions, is

$$\mathcal{L} \left\{ \sum_{i=0}^N b_i \frac{d^i}{dt^i} v_o(t) \right\} = \mathcal{L} \left\{ \sum_{i=0}^M a_i \frac{d^i}{dt^i} v_i(t) \right\}$$

$$\sum_{i=0}^N b_i s^i V_o(s) = \sum_{i=0}^M a_i s^i V_i(s)$$

Rearranging:

Solver will
be a will
polynomials of
 s

Transfer
function

$$\frac{V_o(s)}{V_i(s)} = H(s) = \frac{\sum_{i=0}^M a_i s^i}{\sum_{i=0}^N b_i s^i}$$

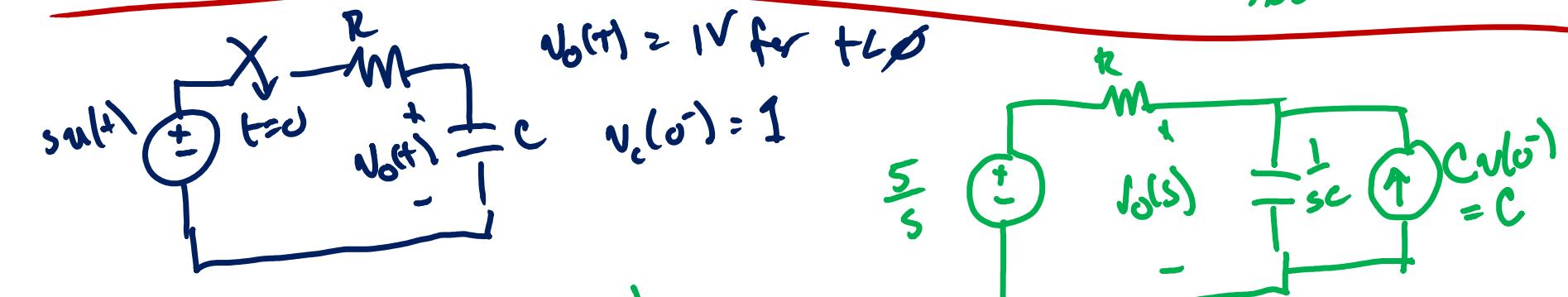
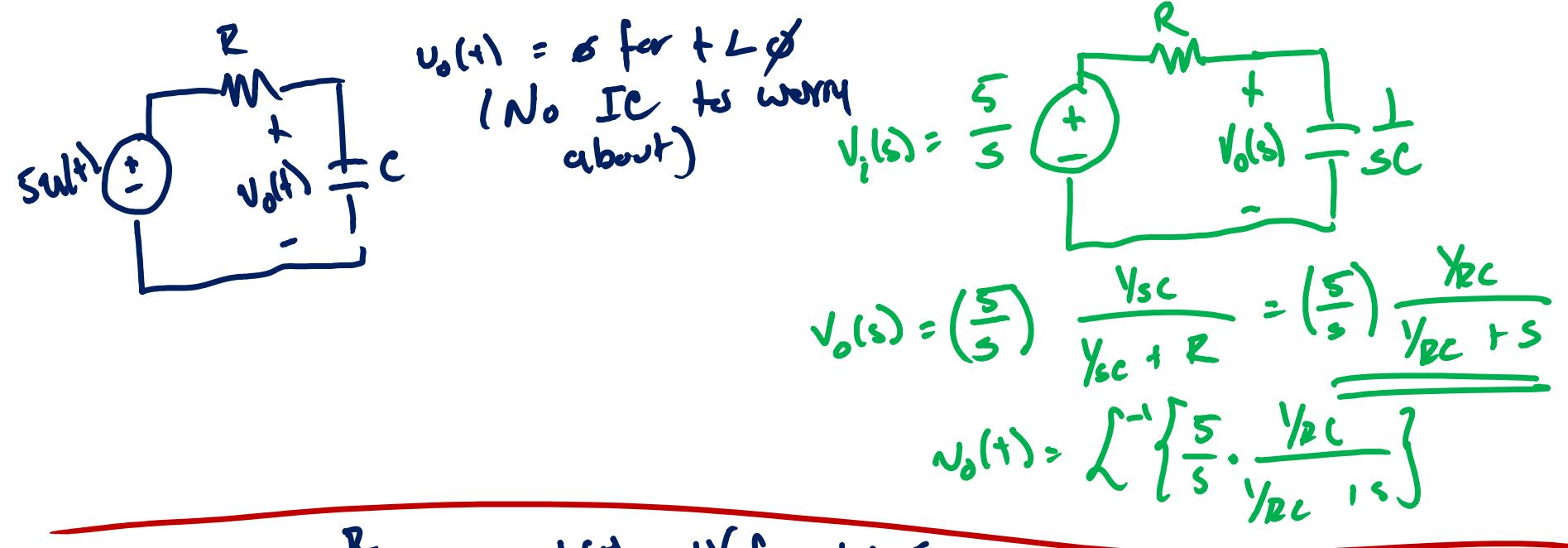
if we replace
 $s \rightarrow j\omega$
this is the
frequency response

Laplace Circuit Solution Algorithm

1. Transform all sources, signals into Laplace Domain
2. Transform circuit components (including initial conditions) into Laplace Domain
3. Solve the circuit using 201 techniques

$$V_o(s) = H(s)V_i(s) = \frac{\sum_{i=0}^M a_i s^i}{\sum_{i=0}^N b_i s^i}$$

4. Inverse Laplace Transform to get back to time domain



Apply superposition

$$V_o(s) = \frac{5}{s} \frac{\frac{1}{RC}}{s + \frac{1}{RC}}$$

$$V_o(s) = \frac{5}{s} \frac{\frac{1}{RC}}{s + \frac{1}{RC}} + \frac{1}{s + \frac{1}{RC}}$$

$$V_o(t) = \mathcal{L}^{-1}\left\{\frac{5}{s} \frac{\frac{1}{RC}}{s + \frac{1}{RC}}\right\} + e^{-\frac{1}{RC}t} u(t)$$

Inverse Transforms

solve circuit to get

$$V_o(s) = H(s)V_{in}(s)$$

\uparrow

s-domain
circuit solution

laplace transform of
input

$$V_o(t) = L^{-1}\{V_o(s)\} + L^{-1}\{H(s)V_{in}(s)\}$$

Generally, we'll need to factor
the polynomials

will usually look like

$$V_o(t) = L^{-1}\left\{\frac{\sum a_i s^i}{\sum b_i s^i}\right\}$$

ex/

$$F(s) = \frac{10}{s^2 + 4s + 4} = \frac{10}{(s+2)(s+2)} = \frac{10}{(s+2)^2}$$

find roots by quadratic formula

from tables

$$r = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$f(t) = L^{-1}\{P(s)\} \Rightarrow 10t e^{-2t} u(t)$$

ex/

$$F(s) = \frac{5s+1}{s+1} \rightarrow \text{if } M > N, \text{ use long division}$$

$$= 5 + \frac{-4}{s+1}$$

$$f(t) = 5\delta(t) - 4e^{-t} u(t)$$

$$\begin{array}{r} 5 \\ s+1 \longdiv{5s+1} \\ \underline{-5s-5} \\ 4 \end{array}$$

Transfer Functions

$$H(s) = \frac{\sum_{i=0}^M a_i s^i}{\sum_{i=0}^N b_i s^i} \stackrel{\text{factoring}}{=} \frac{(s - z_1)(s - z_2) \cdots (s - z_M)}{(s - p_1)(s - p_2) \cdots (s - p_N)}$$

polynomial form factored pole-zero form

roots of numerator, z_i , are called "zeros"
 - values of s where $H(s) = \emptyset$

- values of s where $H(s) \rightarrow \infty$

if all a_i are real, all z_i are either real or complex-conjugate pairs

and all π 's are " "

"bits" in "all pi" are "poles define" form of $f(t)$, zeros will define coefficients of terms \sum constant terms from $H(s)$

$$V(s) = H(s) V_F(s) \quad \{ \text{ } \beta \text{ terms from } V_F(s)$$

Partial Fraction Expansion / Decomposition

$$\frac{\sum_{i=0}^M a_i s^i}{\sum_{i=0}^N b_i s^i} = \frac{(s - z_1)(s - z_2) \cdots (s - z_M)}{(s - p_1)(s - p_2) \cdots (s - p_N)} = \frac{k_1}{(s - p_1)} + \frac{k_2}{(s - p_2)} + \cdots + \frac{k_N}{(s - p_N)}$$

factored pole/zeros form

Partial Fraction Expansion

k_i are "readily"

if all poles p_i are real & unique $\nvdash n \geq m$

"cover-up" method: find k_i by multiplying both sides by $(s - p_i)$ & evaluating at $s = p_i$

$$\frac{(s - z_1)(s - z_2) \cdots (s - z_M)}{(s - p_1)(s - p_2) \cdots (s - p_N)} \Big|_{s=p_1} = \frac{k_1(s-p_1)}{(s-p_1)} + \frac{(k_2)(s-p_1)}{(s-p_2)} + \cdots + \frac{k_N(s-p_1)}{(s-p_N)} \Big|_{s=p_2}$$

ex

$$f(s) = \frac{4(s+2)}{s^2 + 4s + 3} = \frac{4(s+2)}{(s+1)(s+3)} = \frac{k_1}{s+1} + \frac{k_2}{s+3} = \frac{2}{s+1} + \frac{2}{s+3}$$

$$k_1 = \frac{4(s+2)}{s+3} \Big|_{s=-1} = 2$$

$$k_2 = \frac{4(s+2)}{s+1} \Big|_{s=-3} = 2$$

$$f(t) = L^{-1}[F(s)]$$

$$f(t) = [2e^{-t} + 2e^{-3t}]u(t)$$

PFE: Repeated Roots

$$F(s) = \frac{N(s)}{(s-p_1)(s-p_2)^2} = \frac{k_1}{s-p_1} + \frac{k_2}{s-p_2} + \frac{k_3}{(s-p_2)^2}$$
$$= \frac{k_1}{s-p_1} + \frac{k_2 s + (k_3 - p_2 k_2)}{(s-p_2)^2} \rightarrow \frac{A s + B}{(s-p_2)^2}$$

for $k_1 \rightarrow$ no change (find with coverup method)
 $k_2 \pm k_3 \rightarrow$ two options { equating coefficients method
differentiating method

$$P(s) = \frac{N_2(s)}{(s-p_1)(s-p_2)^3} = \frac{k_1}{s-p_1} + \frac{k_2}{s-p_2} + \frac{k_3}{(s-p_2)^2} + \frac{k_4}{(s-p_2)^3}$$

and so on for higher-order repeated roots

Repeated Roots: Equating Coefficients

~~CX~~ $F(s) = \frac{32s(s+1)}{(s+2)(s+10)^2} = \frac{k_1}{s+2} + \frac{As+B}{(s+10)^2} = \frac{1}{s+2} + \frac{31s - 50}{(s+10)^2}$

find k_1 by normal coverup method

$$k_1 = \left. \frac{32s(s+1)}{(s+10)^2} \right|_{s=-2} = 1$$
$$f(t) = [e^{-2t} + 31e^{-10t} - 360te^{-10t}]$$

to find k_2 : Multiply both sides by full original denominator
then solve to match coefficients on powers of s

$$\begin{aligned} 32s^2 + 32s &= 1(s+10)^2 + (As+B)(s+2) \\ &= s^2 + 20s + 100 + As^2 + 2As + Bs + 2B \end{aligned}$$

$$\left\{ \begin{array}{l} s^2: 32 = 1 + A \rightarrow A = 31 \\ s^1: 32 = 20 + 2A + B \\ s^0: 0 = 100 + 2B \end{array} \right. \rightarrow B = -50$$

Repeated Roots: Differentiation

$$\cancel{e^s} \quad F(s) = \frac{32s(s+1)}{(s+2)(s+10)^2} = \frac{k_1}{s+2} + \frac{k_2}{s+10} + \frac{k_3}{(s+10)^2}$$

find $k_1 \neq k_3$ by coverup method

$$k_1 = \frac{32s(s+1)}{(s+10)^2} \Big|_{s=-2} = \underline{\underline{1}}$$

$$k_3 = \frac{32s(s+1)}{s+2} \Big|_{s=-10} = \underline{\underline{-360}}$$

Differentiating method: Multiply both sides by $(s+10)^2$, then
differentiate w.r.t. s before evaluating at $s = -10$

$$\frac{d}{ds} \left[\frac{k_1(s+10)^2}{s+2} + k_2(s+10) + k_3 \right] \Big|_{s=-10} = \frac{d}{ds} \left[\frac{32s(s+1)}{s+2} \right] \Big|_{s=-10}$$
$$k_2 = \left[\frac{(64s+32)(s+2) - (32s^2+32)}{(s+2)^2} \right] \Big|_{s=-10} = \underline{\underline{31}}$$

$$F(s) = \frac{1}{s+2} + \frac{31}{s+10} + \frac{-360}{(s+10)^2}$$

Complex Roots: Complex Math

$$F(s) = \frac{1}{s^2 - 2s + 2} = \frac{1}{(s-(1+j))(s-(1-j))} \text{ (roots } e^{\frac{2 \pm \sqrt{4-4 \cdot 1 \cdot 2}}{2}} = 1 \pm j)$$

Complex poles (and their residues) will always show up in complex conjugate pairs for real signals & systems

$$F(s) = \frac{1}{s^2 - 2s + 2} = \frac{k_1}{s-(1+j)} + \frac{k_2}{s-(1-j)} \quad k_2 = k_1^*$$

Option 1: Do nothing different

By cover-up method.

$$k_1 = \left. \frac{1}{s-(1-j)} \right|_{s=1+j} = \frac{1}{2j} = \frac{-j}{2}$$

$$k_2 = \left. \frac{1}{s-(1+j)} \right|_{s=1-j} = \frac{1}{-2j} = \frac{j}{2} = k_1^* \quad \checkmark$$

$$f(t) = \mathcal{L}^{-1} \left\{ \frac{\frac{1}{2}j}{s-(1+j)} + \frac{-\frac{1}{2}j}{s-(1-j)} \right\} = \frac{1}{2j} e^{(1+j)t} u(t) + \frac{-1}{2j} e^{(1-j)t} u(t)$$

Complex Roots: General Case

$$\begin{aligned} & \mathcal{L}^{-1} \left\{ \frac{k}{s - (\sigma + j\omega)} + \frac{k^*}{s - (\sigma - j\omega)} \right\} \\ &= \left[k e^{(\sigma+j\omega)t} + k^* e^{(\sigma-j\omega)t} \right] u(t) \\ &= e^{\sigma t} \left[k e^{j\omega t} + k^* e^{-j\omega t} \right] u(t) \\ &= e^{\sigma t} \left[\operatorname{Re}\{k\} \left(e^{j\omega t} + e^{-j\omega t} \right) + j \operatorname{Im}\{k\} \left(e^{j\omega t} - e^{-j\omega t} \right) \right] u(t) \quad \begin{matrix} k = \operatorname{Re}\{k\} + j \operatorname{Im}\{k\} \\ k^* = \operatorname{Re}\{k\} - j \operatorname{Im}\{k\} \end{matrix} \\ &= e^{\sigma t} \left[2 \operatorname{Re}\{k\} \cos(\omega t) - 2 \operatorname{Im}\{k\} \sin(\omega t) \right] u(t) \quad 2j \sin \theta \\ &= e^{\sigma t} \left[2 \sqrt{\operatorname{Re}\{k\}^2 + \operatorname{Im}\{k\}^2} \cos\left(\omega t + \tan^{-1}\left(\frac{-\operatorname{Im}\{k\}}{\operatorname{Re}\{k\}}\right)\right) \right] u(t) \\ &= \boxed{e^{\sigma t} 2|k| \cos(\omega t - \angle k) u(t)} \end{aligned}$$

Complex Roots: Table Lookup

$$F(s) = \frac{1}{s^2 - 2s + 2} = \frac{1}{(s-1)^2 + 1}$$

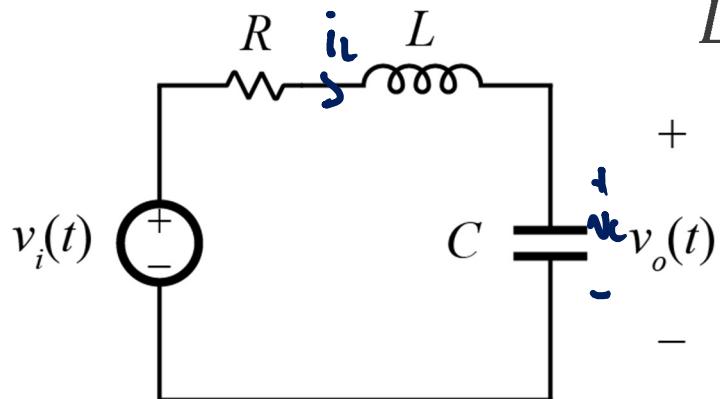
By tables $\omega/ \alpha = -1$, $\omega = 1$

$$\mathcal{L}^{-1} \left\{ \frac{1}{(s-1)^2 + 1} \right\} \rightarrow \cancel{e^t \sin(t) u(t)}$$

Example

Find $v_o(t)$

$$v_i(t) = \sin(2t) u(t)$$



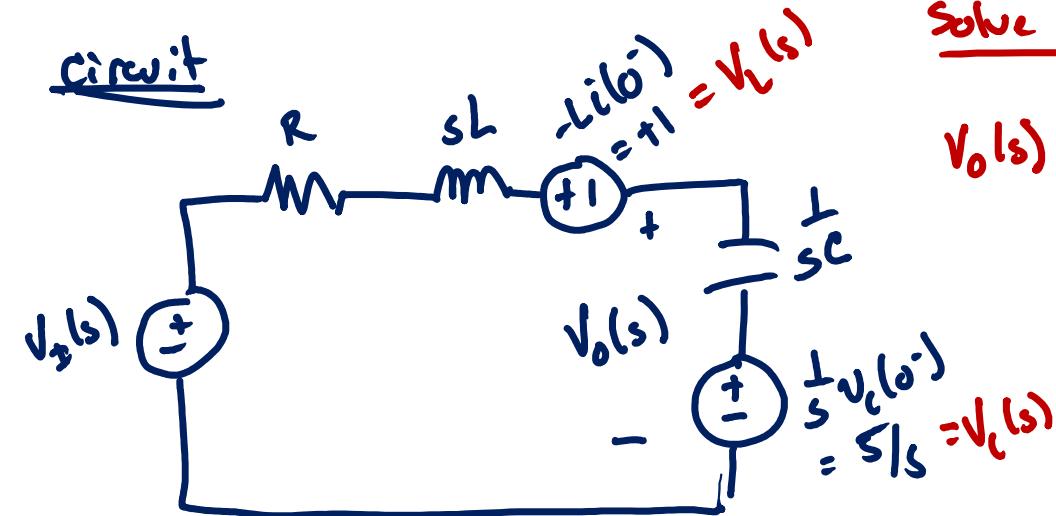
$$L = 500\text{mH}, \quad C = 500\text{nF}, \quad R = 2\Omega$$

$$v_c(0) = 5V, \quad i_L(0) = -2A$$

Input

$$V_I(s) = \mathcal{L}\{v_i(t)\} = \frac{2}{s^2 + 4}$$

Circuit



Solve in s-domain

$$V_o(s) = A_I(s)V_I(s) + H_L(s)V_I(s) + A_C(s)V_C(s)$$

$$H_I(s) = \frac{\frac{1}{sC}}{R + sL + \frac{1}{sC}} = \frac{1}{s^2LC + sRC + 1}$$

$$H_L(s) = (-1)H_I(s)$$

$$A_C(s) = (1 - H_I(s))$$

Pole Locations

$$x = N_H + N_I$$

$$\begin{aligned}
 V_o(s) &= \cancel{V_I(s)} \cancel{H(s)} = \left(\frac{\sum_{i=0}^{M_I} c_i s^i}{\sum_{i=0}^{N_I} d_i s^i} \right) \left(\frac{\sum_{i=0}^{M_H} a_i s^i}{\sum_{i=0}^{N_H} b_i s^i} \right) = \frac{(s - z_1)(s - z_2) \cdots (s - z_{M_H+M_I})}{(s - p_1)(s - p_2) \cdots (s - p_{N_H+N_I})} \\
 &\quad \text{Input} \qquad \text{Transfer function} \qquad \text{further} \\
 &= \frac{k_1}{(s - p_1)} + \frac{k_2}{(s - p_2)} + \cdots + \frac{k_x}{(s - p_x)} \\
 V_o(t) &= \underline{k_1 e^{p_1 t}} + k_2 e^{p_2 t} + \dots + k_x e^{p_x t}
 \end{aligned}$$

PFG

poles define the terms in time domain

real poles \rightarrow exponentially

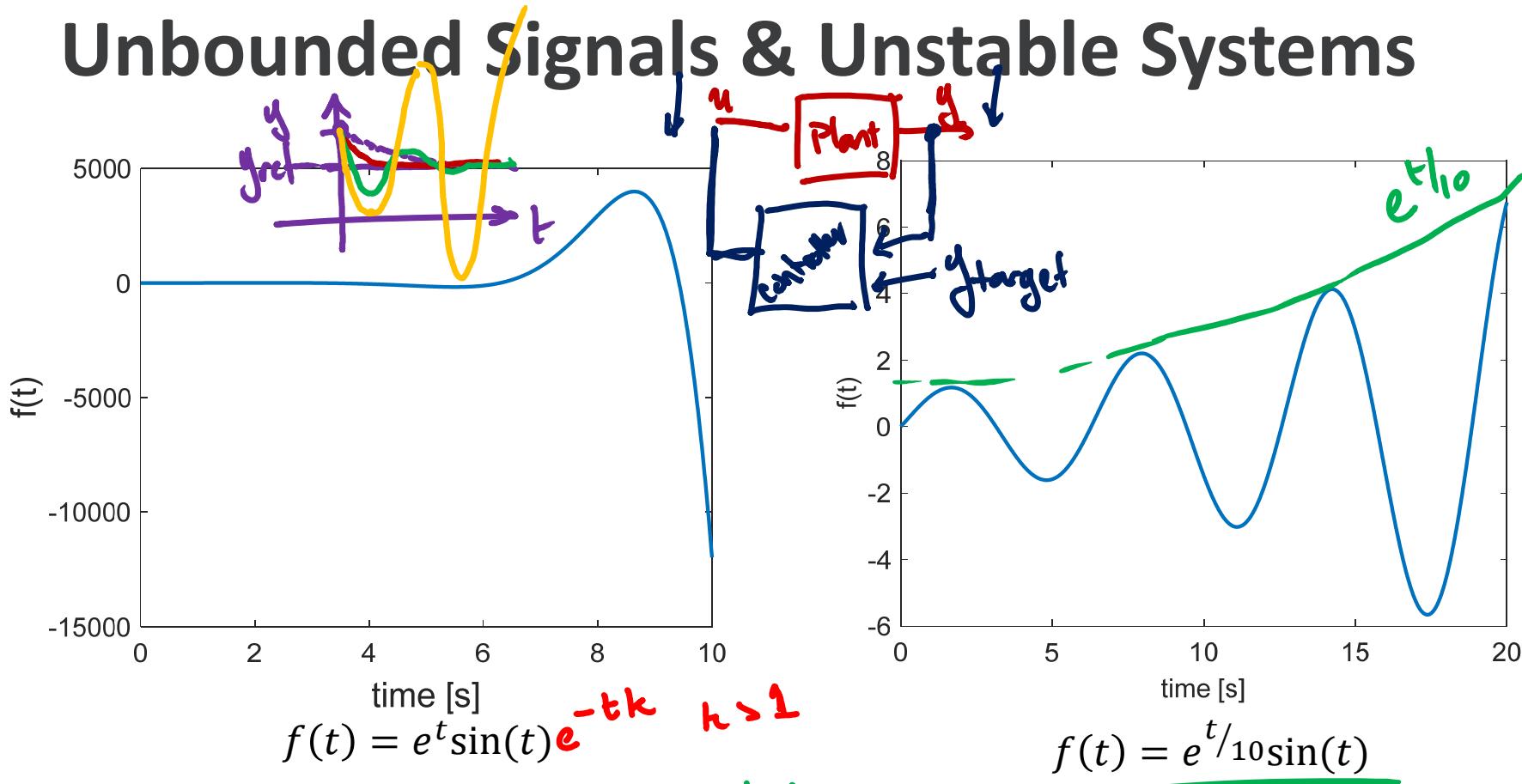
repeated poles \rightarrow $t \cdot e^{ot}$

complex poles (pairs) \rightarrow $z|k| e^{ot} \cos(\omega t + \angle k)$

some terms in output come from $H(s)$ independent of $V_I(s)$
 $V_I(s)$ \leftrightarrow $H(s)$

Unilateral laplace always has some transient @ $t=0$

Unbounded Signals & Unstable Systems



Bounded signal $\exists B$ s.t. $|f(t)| \leq B \forall t$
 BIBO stable system: "BIBO" = bounded input, bounded output

Always want any hardware circuit to be bounded & stable

Laplace Explanation

$$F(s) = \int_{0^-}^{+\infty} e^{-st} f(t) dt = \int_{0^-}^{+\infty} e^{-\sigma t} e^{-j\omega t} f(t) dt$$

L21

Fourier Series

Assume we have some function $f(t)$ which is periodic with period $T_0 = \frac{2\pi}{\omega_0}$.
 $f(t) = a_0 + \sum_{k=1}^{\infty} a_k \cos(k\omega_0 t) + b_k \sin(k\omega_0 t)$
 $f(t)$ can be expressed this way if
 1. $f(t)$ is single-valued
 2. $\int_{T_0}^{t+T_0} |f(t)| dt$ exists
 3. $f(t)$ had finite discontinuities and max/min per period

for a_0 (constant / DC term): $a_0 = \frac{1}{T_0} \int_0^{T_0} f(t) dt$

for a_k : find average value of $f(t) \cdot \cos(n\omega_0 t)$
 $\frac{1}{T_0} \int_0^{T_0} f(t) \cos(n\omega_0 t) dt$
 Assuming Fourier Series is valid, this is
 $\frac{1}{T_0} \int_0^{T_0} \left[a_0 + \sum_{k=1}^{\infty} a_k \cos(k\omega_0 t) + b_k \sin(k\omega_0 t) \right] \cos(n\omega_0 t) dt$

L24

Non-periodic Waveforms: Fourier Transform

Fourier Series \rightarrow only periodic waveforms

Fourier Transform \rightarrow non-periodic signals

\rightarrow treat any non-periodic signal as a periodic signal with $T \rightarrow \infty$

$$\text{Fourier Series: } c_k = \frac{1}{T} \int_{T_0}^{T_0+T} f(t) e^{-jk\omega_0 t} dt$$

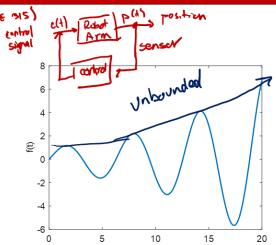
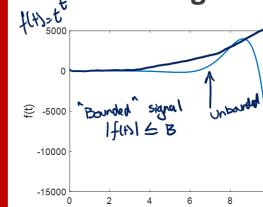
$$\text{Fourier transform: } T c_k = \int_{-\infty}^{\infty} f(t) e^{-j\omega t} dt = F(\omega)$$

$$\text{Fourier Series: } f(t) = c_0 + \sum_{k=-\infty}^{\infty} c_k e^{jk\omega_0 t}$$

$$\text{Fourier Inverse Transform: } f(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(\omega) e^{j\omega t} d\omega$$

L30

Unstable Signals



$f(t) = e^t \sin(t)$
 $|f(t)|$ explodes \rightarrow bad in any real circuit
 BIBO stability = "Bounded input bounded output"
 Always want signals with $\sigma \leq 0$ in any $e^{\sigma t}$ term or factor

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L23

Complex Form of Fourier Series

Euler: $e^{j\omega t} = \cos(\omega t) + j\sin(\omega t)$

Fourier Series: $f(t) = a_0 + \sum_{n=1}^{\infty} a_n \frac{1}{2} (e^{jn\omega_0 t} + e^{-jn\omega_0 t}) + b_n \frac{1}{2j} (e^{jn\omega_0 t} - e^{-jn\omega_0 t})$

$f(t) = a_0 + \sum_{n=-\infty}^{\infty} c_n e^{jn\omega_0 t}$

c_n for $k < 0$
 c_n for $k > 0$

L25

Example Signal Laplace Transforms

$$f(t) = u(t)$$

$$\mathcal{L}\{f(t)\} = F(s) = \int_0^{\infty} e^{-st} u(t) dt = \int_0^{\infty} e^{-st} dt$$

$$= \left[\frac{-1}{s} e^{-st} \right]_0^{\infty} = \left[\frac{-1}{s} (-1) \right] = \frac{1}{s}$$

$$\mathcal{L}\{u(t)\} = \frac{1}{s}$$

if $\operatorname{Re}\{s\} > 0$

Region of convergence

$$f(t) = e^{at} u(t)$$

$$a \in \mathbb{R}^+$$

$$\mathcal{L}\{f(t)\} = \int_0^{\infty} e^{-st} e^{at} u(t) dt = \int_0^{\infty} e^{-(s-a)t} dt$$

using previous

$$\mathcal{L}\{e^{at} u(t)\} = \frac{1}{s-a}$$

$$\text{if } \operatorname{Re}\{s\} > a$$

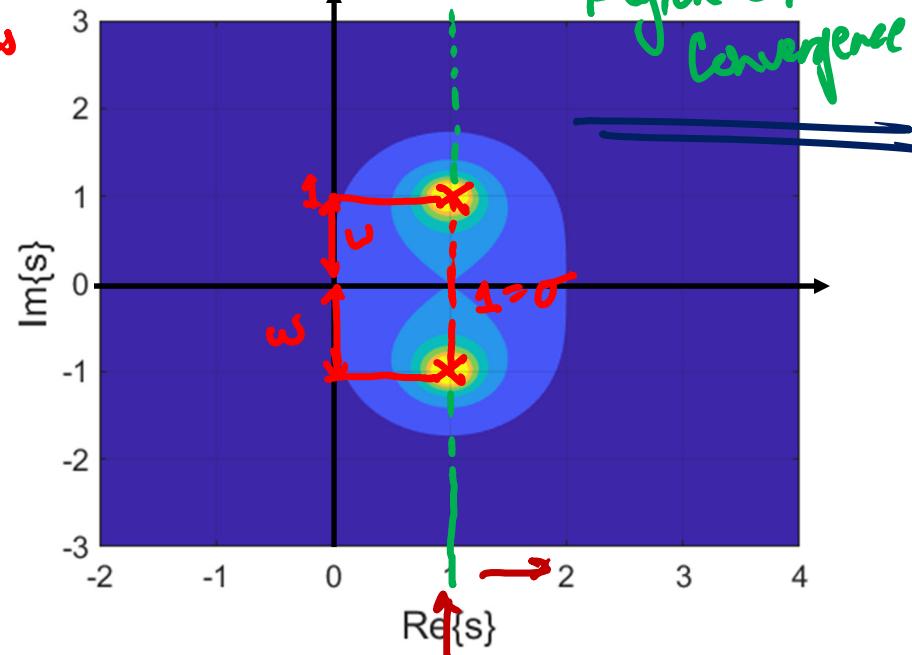
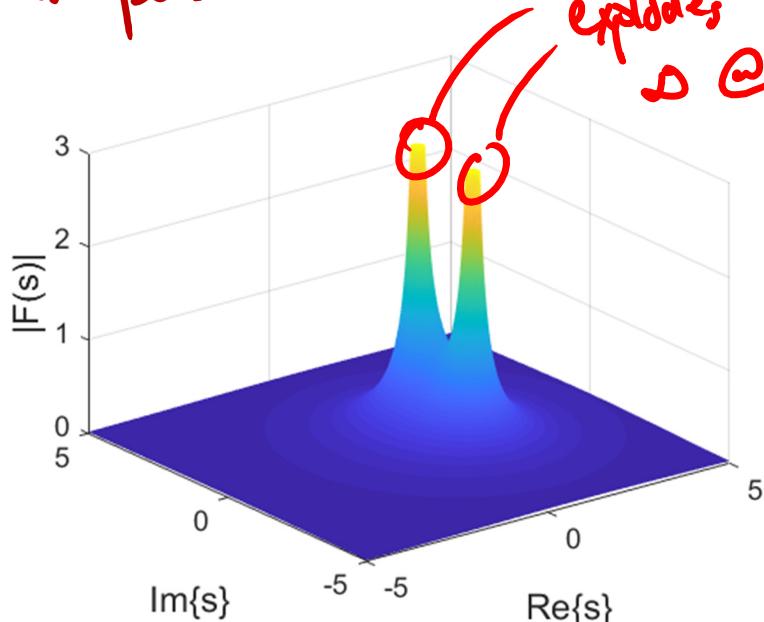
Generalize: $\mathcal{L}\{f(t)e^{-at}\} = F(s+a)$ where $\mathcal{L}\{f(t)\} = F(s)$

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The s-plane

The R.O.C. of the Laplace Transform is the complex plane to the right of all poles

If all poles are in the open left half-plane signal/system is Bounded
explodes to infinity \rightarrow poles

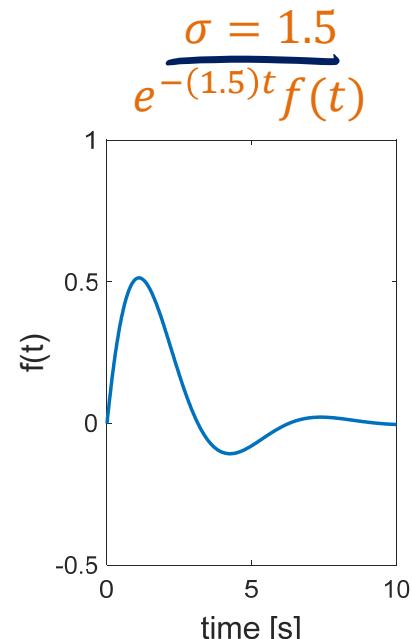
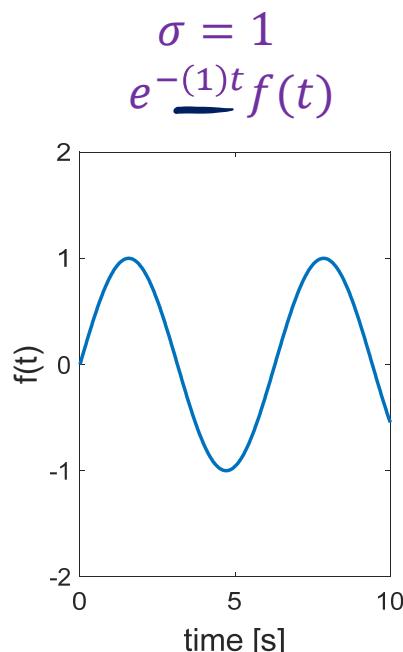
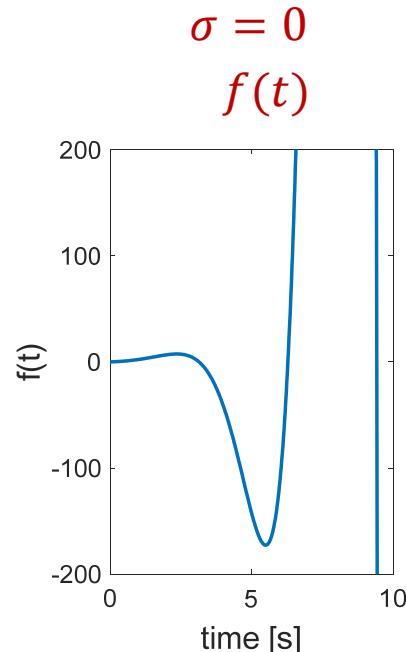
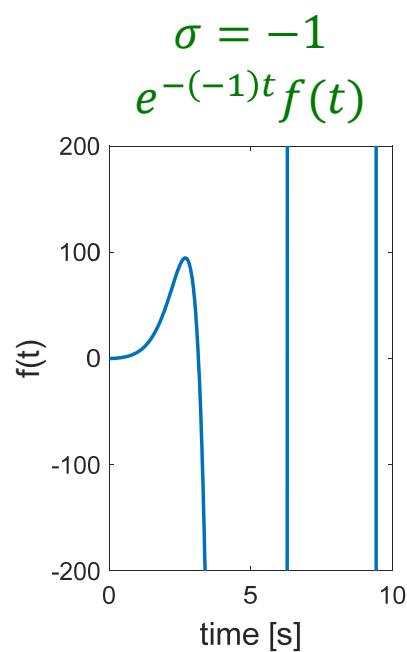
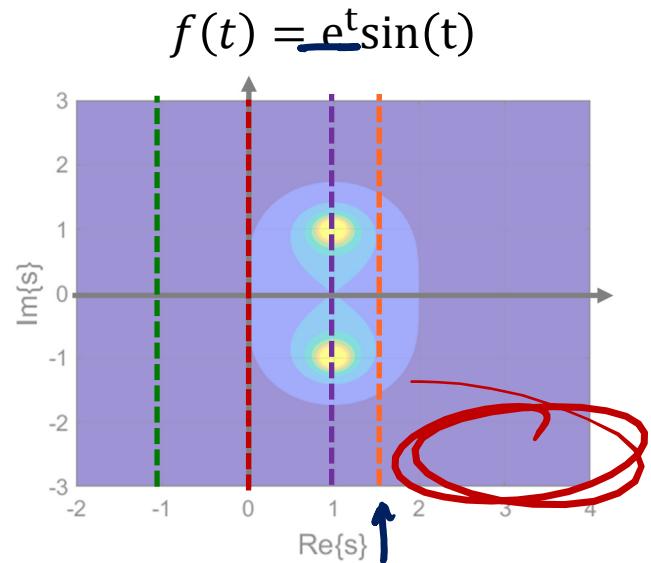


$$F(s) = \frac{1}{s^2 - 2s + 2} = \frac{1}{(s - (1 + j))(s - (1 - j))} \rightarrow \text{poles } s = 1 \pm j$$

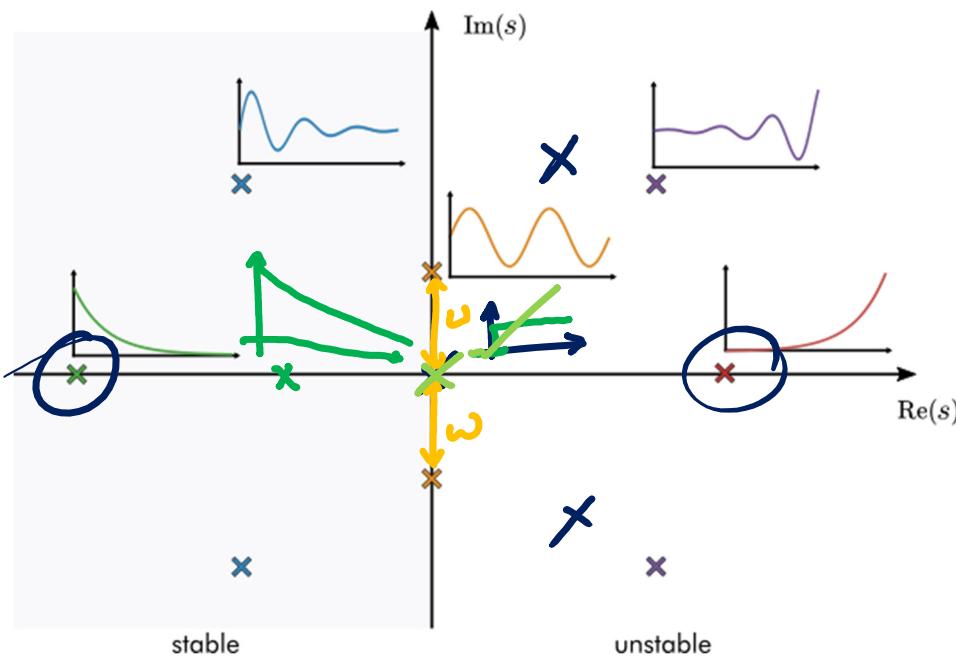
$f(t) = e^t \sin(t) u(t)$ → in Laplace transform must multiply by $e^{-\sigma t}$ $\sigma \geq 1$ for convergence

Example R.O.C.

$$F(s) = \int_{0^-}^{+\infty} e^{-st} f(t) dt = \int_{0^-}^{+\infty} e^{-j\omega t} e^{-\sigma t} f(t) dt$$



Poles-Zero Plot



If $F(s)$ has poles at $-\sigma \pm j\omega$

$$F(s) = \frac{N(s)}{\dots (s - (-\sigma + j\omega))(s - (-\sigma - j\omega)) \dots}$$

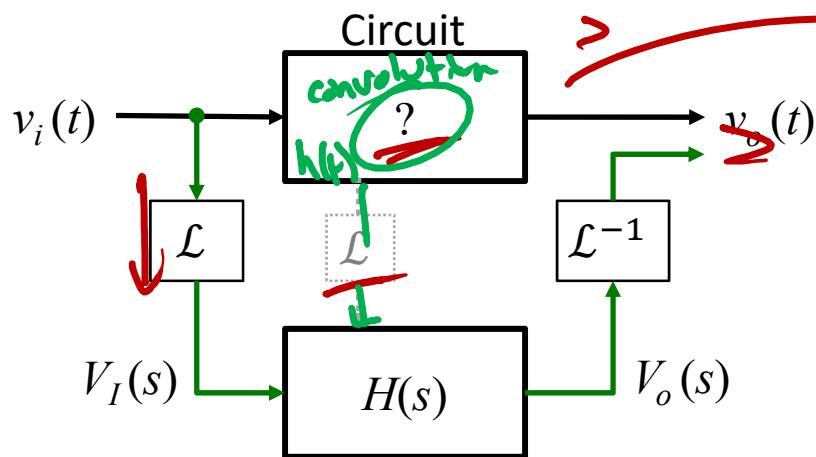
$$F(s) = \dots + \frac{h}{s - (-\sigma + j\omega)} + \frac{h^*}{s - (-\sigma - j\omega)} + \dots$$

$$f(t) = \dots + 2|h| e^{-\sigma t} \cos(\omega t - \Delta t) + \dots$$

Takeaways

- complex poles/ zeros always show up in conjugate pairs (required for real-valued time-domain function)
- If all poles in open LHP system is stable / signal is bounded
- If all poles in open LHP $j\omega$ -axis is in region of convergence
(ch 15 - frequency response)

System I/O Relationship



201 approach: solve ODEs

$$\begin{aligned} 202: \quad V_{\text{I}}(s) &= \mathcal{L}\{v_i(t)\} \\ V_{\text{O}}(s) &= H(s)V_{\text{I}}(s) \\ v_o(t) &= \mathcal{L}^{-1}\{V_{\text{O}}(s)\} = \mathcal{L}^{-1}\{H(s)V_{\text{I}}(s)\} \end{aligned}$$

$$v_o(t) = \mathcal{L}^{-1}\{H(s)V_{\text{I}}(s)\}$$

$\rightarrow \int_0^{\infty} h(t-\tau)W(s)dt$ property for Laplace transform of a product of two s-domain functions?

$$\mathcal{L}^{-1}\{V_{\text{I}}(s)\} = v_i(t)$$

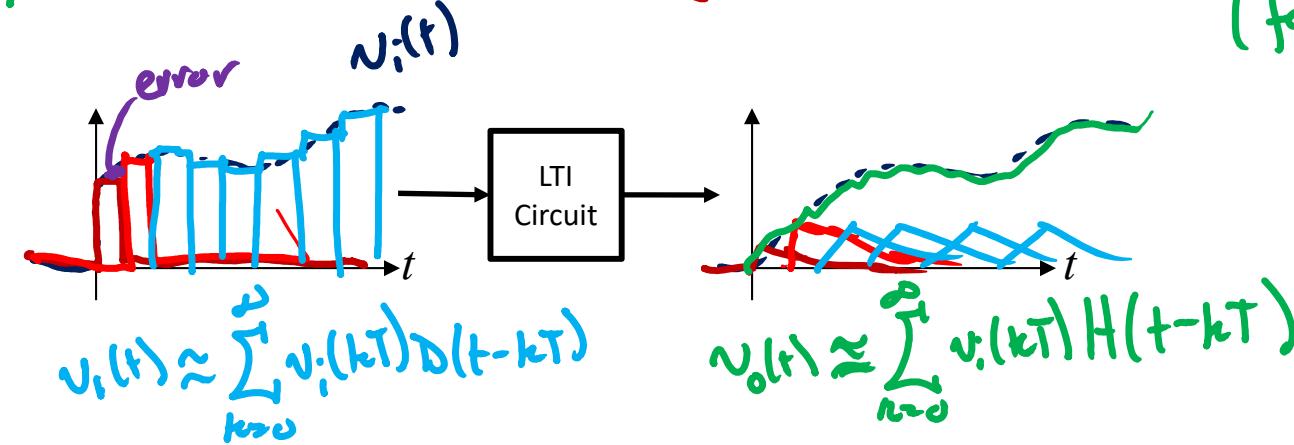
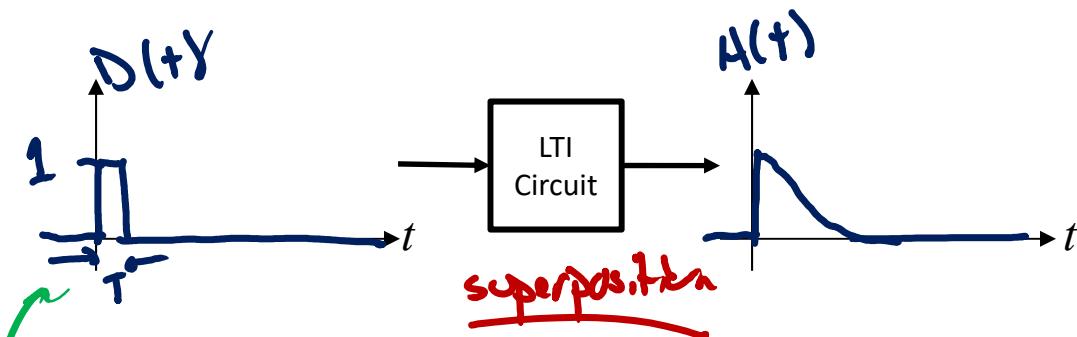
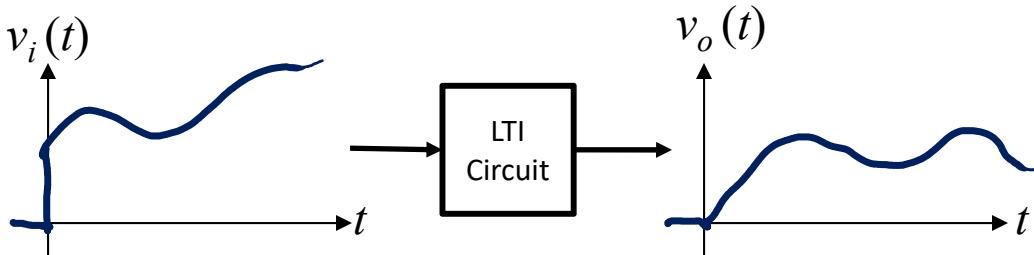
$$\mathcal{L}^{-1}\{H(s)\} = h(t)$$

$$\text{if } V_{\text{I}}(s) = 1 \iff v_i(t) = \delta(t)$$

$$\text{then } v_o(t) = \mathcal{L}^{-1}\{H(s) \cdot 1\} = \mathcal{L}^{-1}\{H(s)\}$$

$\mathcal{L}^{-1}\{H(s)\} = h(t) = \text{"impulse response"}$

Convolution



as $T \rightarrow 0$

$$v_i(t) = \int_0^\infty v_i(\tau) \delta(t-\tau) d\tau$$

sifting property of $\delta(t)$

$$\begin{aligned} v_o(t) &= \int_0^\infty v_i(\tau) h(t-\tau) d\tau \\ &= \int_0^\infty v_i(t-\tau) h(\tau) d\tau \end{aligned}$$

Convolution integral

(formally $\int_{-\infty}^\infty v_i(\tau)h(t-\tau) d\tau$
but with unilateral Laplace
zero for $t < 0$)

The Convolution Integral

Laplace Transform property:

$$V_o(s) = V_i(s) H(s)$$



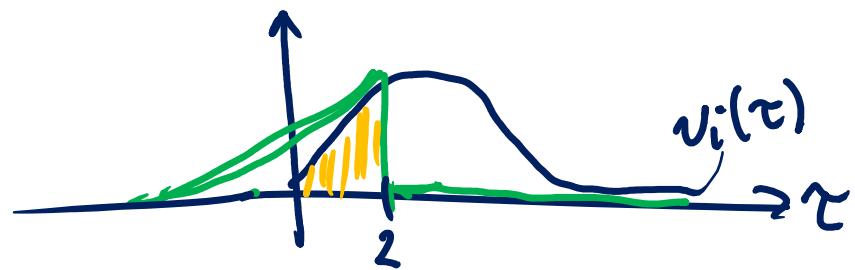
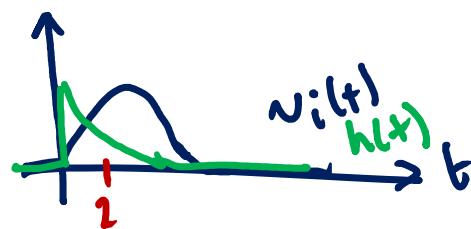
$$V_o(t) = \int_0^{\infty} h(t-\tau) v_i(\tau) d\tau = h(t) * v_i(t)$$

short-hand
↓

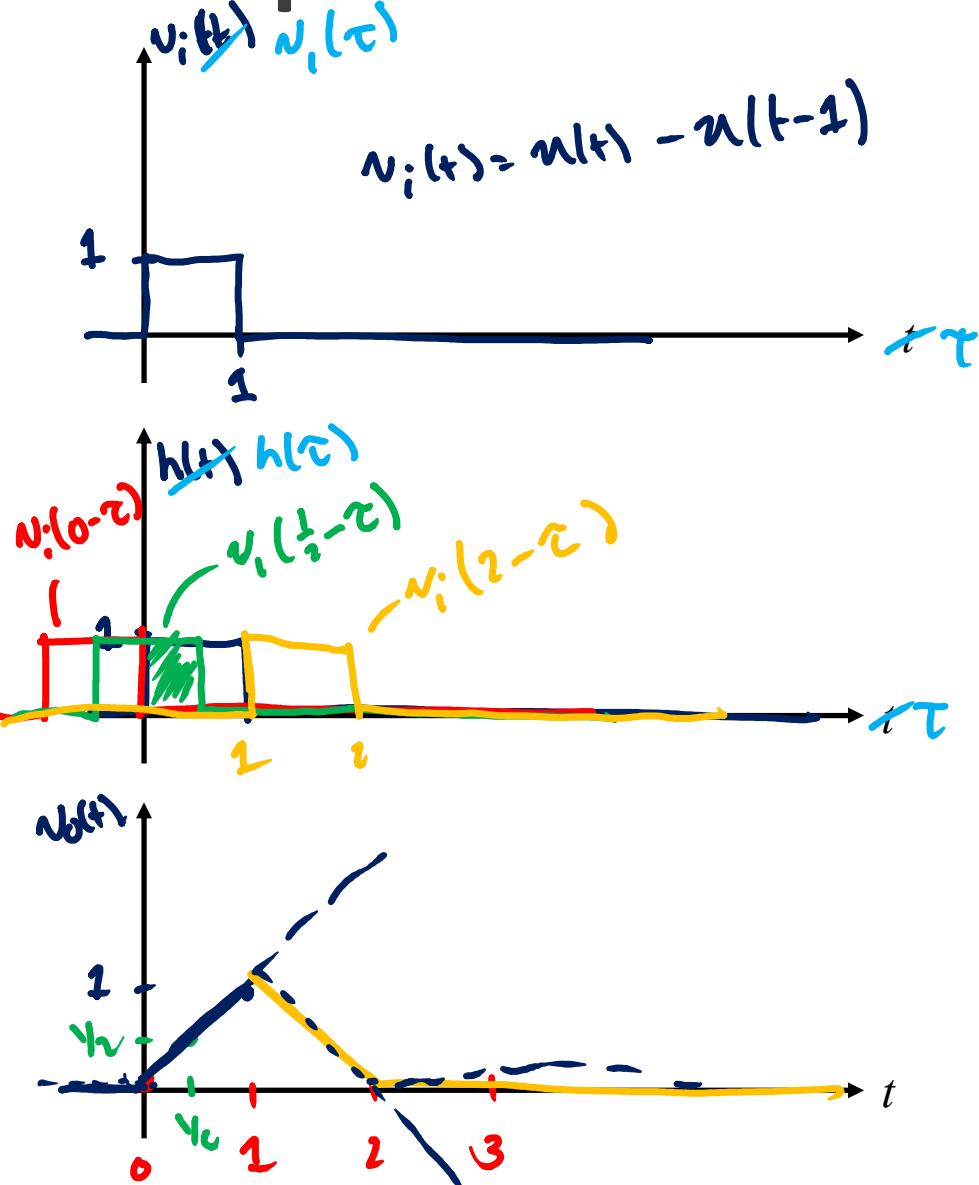
$$V_o(t) = \int_0^{\infty} h(-\tau+t) v_i(\tau) d\tau$$

↑ flip ↑ shift
at $t=2$

$$V_o(t=2) = \int_0^{\infty} h(-\tau+2) v_i(\tau) d\tau$$



Graphical Convolution



$$v_o(t) = \int_{-\infty}^{\infty} u_i(t-\tau) h(\tau) d\tau$$

$$v_o(t) = \begin{cases} 0, & t \leq 0 \\ 0, & t \geq 2 \\ \int_0^t 1 dt, & 0 < t \leq 1 \\ \int_{t-1}^1 1 dt, & 1 < t \leq 2 \end{cases}$$

$$v_o(t) = r(t) - 2r(t-1) + r(t-2)$$

$$V_I(s) = \frac{1}{s} - \frac{1}{s}e^{-s} = H(s)$$

$$V_o(s) = V_I(s) H(s) = \frac{1}{s^2} + \frac{1}{s^2}e^{-2s} - 2\frac{1}{s^2}e^{-s}$$

$$v_o(t) = r(t) - 2r(t-1) + r(t-2)$$