Experiment 1: Coupled Inductors and Transformers

Mutual inductance and transformers

ECE 202: Circuits II

Spring 2024

Contents

I. Introduction

Coupled inductors are used in a variety of applications for their voltage conversion, impedance conversion, and/or electrical isolation properties. The behavior of these components is dictated both by the coil inductances themselves and the coupling between them.

In this experiment, we will investigate both a tightly coupled transformer and a pair of loosely coupled coils.

The goals of this experiment are

- Measuring the circuit behavior of transformers and coupled inductors
- Establishing circuit voltage and current measurement techniques
- Investigating the impact of external elements on magnetic fields
- Characterizing coils that will be used in later experiment(s)

II. Background

a. Coupled Inductors

An example inductive coil is shown in Figure 1. The coil has 35mm diameter and 3mm height; it consists of 27 turns of wire above a high permeability magnetic material. The magnetic material (the dark grey disc in Figure 1) acts as a near short-circuit for the magnetic field which shields area below the coil from magnetic field and increases the inductance of the coil.

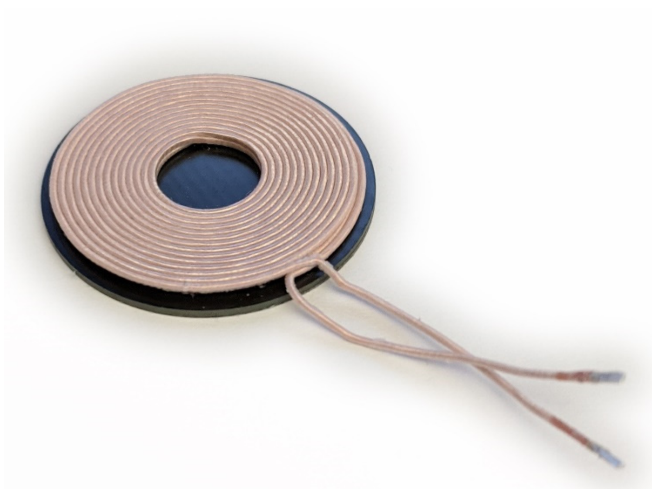


Figure 1. Abracon AWCCA-RX350300-101 WPT coil

An example finite element analysis (FEA) simulation of a single coil is shown in Figure 2. FEA simulations are beyond the scope of this course, but provide a numerical simulation of the geometry by dividing the space into small triangles (elements) and solving Maxwell's equations at the boundary of each. The diagram in Figure 2 has about 15,000 elements.

The diagram in Figure 2 is a 2D slice of the coil in Figure 1; the coil is cut through its diameter and viewed in the *r*-*z* plane of a cylindrical space. The black contours are flux lines of the magnetic field produced by the coil when current flows through the winding. The colored contours show the flux density, *B*, in units of Tesla, which is a measure of magnetic flux per unit area, $1\text{T} = 1\text{Wb/m}^2$.

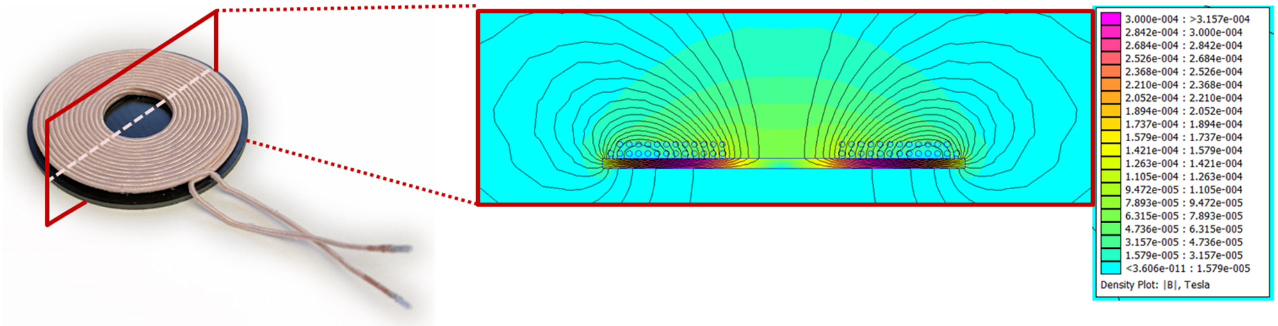


Figure 2. Finite element simulation of the magnetic field of a single WPT coil

In free space, this coil behaves as an inductor. When current $i_1(t)$ flows through the coil, it produces a flux

$$\Phi_1(t) = \alpha_{11} N_1 i_1(t) \tag{1}$$

where k_{11} is a constant determined by the geometry of the coil and N_1 is the number of turns in the coil. The direction of the flux is in accordance with the right-hand rule. All of the flux $\Phi_1(t)$ flows through the coil. By Farady's Law of Induction, if this flux is time varying, it will produce a voltage at the terminals of the coil

$$|v_1(t)| = N_1 \frac{d\Phi_1(t)}{dt} \tag{2}$$

The polarity of $v_1(t)$ is given by Lentz's Law, and is in the direction that would *oppose* the flux $\Phi_1(t)$ if allowed to generate current through some resistor. Combining (1) and (2) to eliminate $\Phi_1(t)$,

$$v_1(t) = \alpha_{11} {N_1}^2 \frac{di_1(t)}{dt} \tag{3}$$

or, in more familiar terms

$$v_1(t) = L_1 \frac{di_1(t)}{dt} \tag{4}$$

where $L_1 = \alpha_{11} N_1^2$ is the coil inductance. Thus, this coil is modeled by the schematic of Figure 3.

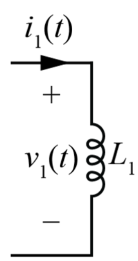


Figure 3. Schematic diagram of a single inductor

A diagram of the magnetic field of a pair of these coils in close proximity is shown in Figure 4. The diagram shows a 2D slice, of two vertically-aligned coils, through the center of each coil. The two coils have 35mm diameter, and are separated by 9mm. Many of the flux lines in Figure 4 enclose (at least in portion) both coils. This indicates that the two inductors are coupled: flux produced by current through one coil will result in a time-varying $d\Phi/dt$ in the other coil, thereby inducing a voltage on the second coil.

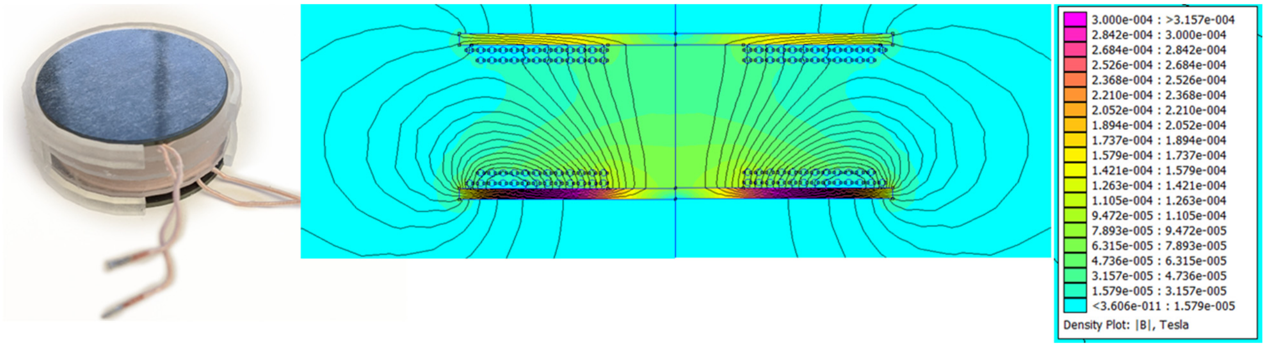


Figure 4. Finite element simulation of the magnetic field of two WPT coils with a distance of 9mm between them.

Using the same procedure as in (1)-(4), this pair of coils exhibits terminal characteristics

$$\begin{aligned} v_1(t) &= \alpha_{11} {N_1}^2 \frac{di_1}{dt} \pm \alpha_{12} N_1 N_2 \frac{di_2}{dt} \\ v_2(t) &= \pm \alpha_{21} N_2 N_1 \frac{di_1}{dt} + \alpha_{22} {N_2}^2 \frac{di_2}{dt} \end{aligned} \tag{5}$$

where α_{ij} is the geometrical constant modeling how much flux generated from coil i impinges on windings of coil j. In general, this relationship is symmetric, $\alpha_{ij} = \alpha_{ji}$, so (5) can be written

$$\begin{aligned} v_1(t) &= L_1 \frac{di_1}{dt} \pm M \frac{di_2}{dt} \\ v_2(t) &= \pm M \frac{di_1}{dt} + L_2 \frac{di_2}{dt} \end{aligned} \tag{6}$$

where L_1 and L_2 are the self-inductances of each coil; that is, the inductance that you would measure at the terminals of one coil with the other coil open-circuited. The term M is the mutual inductance, which models the effect of the flux from one coil on the other coil. The "±" operator is dictated by the relative winding direction of each coil, which is denoted in the symbol of Figure 5 by the dot convention. Positive currents flowing into the dotted terminals always produces additive flux (and therefore voltage) on each winding. If one reference current is flowing into the non-dotted terminal, then positive currents will produce subtractive flux and voltages.

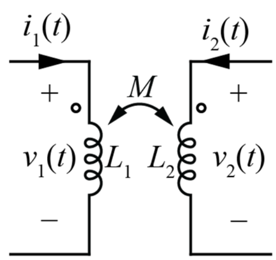


Figure 5. Schematic of two coupled inductors

The coupling coefficient of a pair of inductors is

$$k = \frac{M}{\sqrt{L_1 L_2}} \tag{7}$$

which is a value in the range $0 \le k \le 1$. The coupling coefficient models how tightly coupled the two inductors are: $k = 0$ indicates that there is no coupling and the two inductors are completely independent, while $k = 1$ indicates perfect coupling. In the latter case, (7) indicates that there is a maximum value of mutual inductance M that can be achieved. When *all* flux generated by current in one coil induces voltage in the other, $k = 1$ and $M = \sqrt{L_1 L_2}$.

Individual inductors, such as those that make up WPT coils, are often characterized by their quality factor, Q. Quality factor is defined at a specified frequency f, as

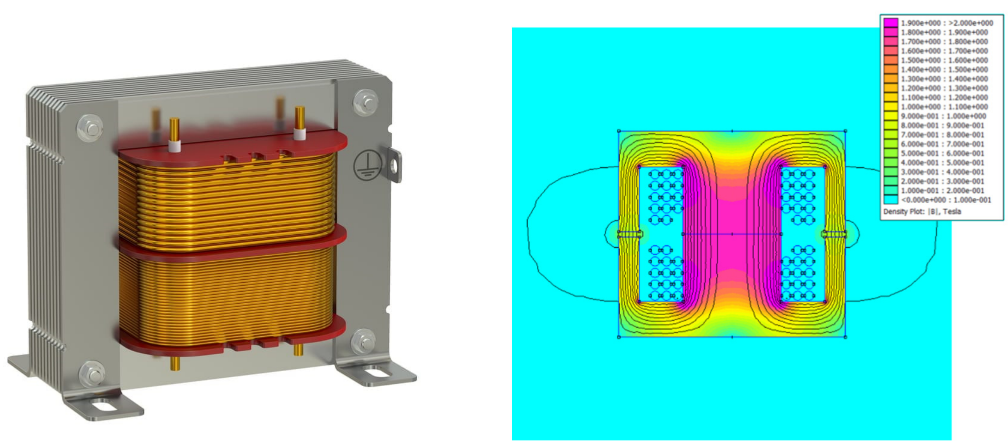
$$Q = \frac{\omega L}{R} \tag{8}$$

where R is the parasitic series resistance of the (non-ideal) inductor. Higher quality factor inductors lose a smaller percentage of their energy to heat per period of sinusoidal current flowing through them. When used in WPT applications, it is critical to operate with high quality factor to prevent the coils from overheating. One way to do this is to increase the operating frequency, which increases the numerator in (8). However, for reasons explained in latter courses, the resistance R increases with frequency. At high enough frequency, this increase is quite rapid, and outpaces the increase in the numerator of (9), establishing a limited range over which increased frequency can benefit Q. In this lab, we will be working at low power levels, such that heating and field strength are not a concern. However, the main limitation for current commercial wireless phone chargers is the low power levels necessary to prevent the onboard coils from overheating.

The changing Q with frequency is also one (of many) reasons why it is desirable to operate WPT systems with single-frequency, sinusoidal currents and voltages. If the system operates at a single frequency, it can be tuned to operate with high efficiency. If nonsinusoidal waveforms are present, the coils may exhibit a suboptimal Q-factor.

b. Ideal Transformer

An ideal transformer is a circuit element that approximates the behavior of two (or more) tightly coupled coils. Most commonly, this is achieved by using high-permeability magnetic material to "guide" the flux between two coils. An example is given in Fig. 6, where two windings share a magnetic "E" core. In the simulated fields of Fig. 6(b), a small air gap is included in the outer arms. Note that the flux is predominately contained within he magnetic material but "fringes" out at the air gaps. Regardless, nearly all flux loops flow through the center leg of the core, and therefore all flux from one coil flows through the other generating a coupling coefficient close to unity.



(a) (b)

Figure 6. Transformer (a) and cross-section of solved fields

Taking a coupling coefficient k=1 in (7) and plugging into (6),

$$v_1(t) = L_1 \frac{di_1}{dt} \pm \sqrt{L_1 L_2} \frac{di_2}{dt}$$
$$v_2(t) = \pm\sqrt{L_1 L_2} \frac{di_1}{dt} + L_2 \frac{di_2}{dt} \tag{9}$$

which is of the form

$$v_1(t) = \pm\sqrt{\frac{L_1}{L_2}} v_2(t) \tag{10}$$

Replacing inductances with geometric parameters from (3), this relationship is

$$v_1(t) = \pm\sqrt{\frac{\alpha_{11}(N_1)^2}{\alpha_{22}(N_2)^2}} v_2(t) \tag{11}$$

For perfect coupling, k=1, the two coils must be virtually identical such that all flux from one winding flows through all turns of the other. For this to be the case, the geometric parameters must be $\alpha_{11} = \alpha_{22}$. Then, (11) simplifies to

$$v_1(t) = \pm \frac{N_1}{N_2} v_2(t) \tag{12}$$

Like previously, the "±" operator is dictated by the relative winding direction of each coil. In a schematic, the polarity is again defined by the dot notation, as shown in the ideal transformer symbol of Fig. 7.

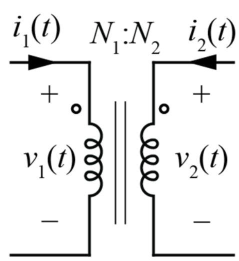


Figure 7. Schematic of an ideal transformer

A transformer is merely a special case of coupled inductors, but its behavior still conforms to (9). If, however, we make an additional approximation that the inductances L_1 and L_2 are very large such that we can approximate both as infinite, we can additionally state that there is no internal energy storage. This is because an infinite inductance would require infinite voltage to induce a non-zero current. If this is true, then because there is no energy storage the instantaneous power in each winding must be balanced,

$$v_1(t)i_1(t) + v_2(t)i_2(t) = 0 \tag{13}$$

Then, combining (13) and (12),

$$\pm N_1 i_1(t) + N_2 i_2(t) = 0 \tag{14}$$

As long as both currents are defined the same with respect to the dotted terminals (i.e. either both into or both out-of the dotted terminal) the "+" sign is used, and the defining equation for the ideal transformer are

$$\frac{v_1(t)}{N_1} = \frac{v_2(t)}{N_2}$$
$$N_1 i_1(t) + N_2 i_2(t) = 0 \qquad (15)$$

both the winding currents and voltages are simply scaled versions of one another with a ratio determined by the number of turns in each winding. The behavior of the ideal transformer is then entirely defined by these two equations; the pair of equations in (6) is no longer needed.

III. Prelab Exercises

Complete the following exercises prior starting the lab. In this prelab, you will complete calculations and simulations to design related to a pair of coupled coils driven by a function generator, as will be tested in the lab. We will assume that the power source for our system is modeled by a sinusoidal Thévenin source with 2 V amplitude and 50 Ω Thévenin resistance.

PE1 Prelab Exercise 1: Source Impedance Impact

Consider the circuit of Fig. 8. The source and resistor inside the grey, dashed box represent the circuit model of a function generator. Coil L_2 is open-circuited.

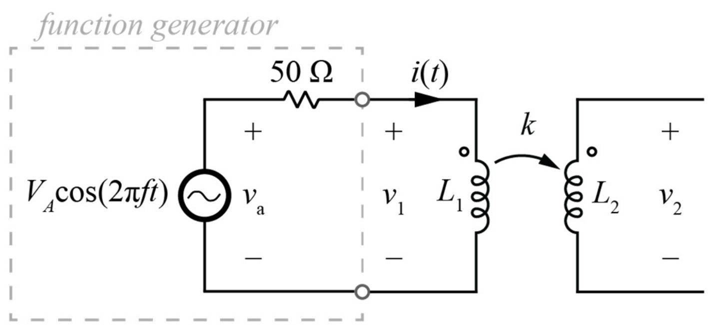


Figure 8. Schematic of a function generator supplying a pair of coupled coils

Solve for a differential equation relating $i(t)$ and $v_a(t)$. Assume that $i(t)=I_A\cos(\omega t+\varphi)$ with $\omega=2\pi f$. Find the value of $v_1(t)$ if

a. $\omega L \gg 50\ \Omega$
b. $\omega L \ll 50\ \Omega$

If L_1 and L_2 are identical coils, find $v_2(t)$ as a function of the source voltage and coupling coefficient (only) for the two cases above.

PE2 Prelab Exercise 2: LTSpice Simulation

Using LTspice, simulate the circuit of Fig. 8 with the values given in Table I.

Table I: Starting Values for Simulation

f	V_A	k	L_1	L_2
50 kHz	2 V	0.5	100 μH	100 μH

Include a plot of $v_1(t)$ and $v_2(t)$ over two periods after the circuit has reached steady-state

PE3 Prelab Exercise 3: LTSpice Simulation Sweep

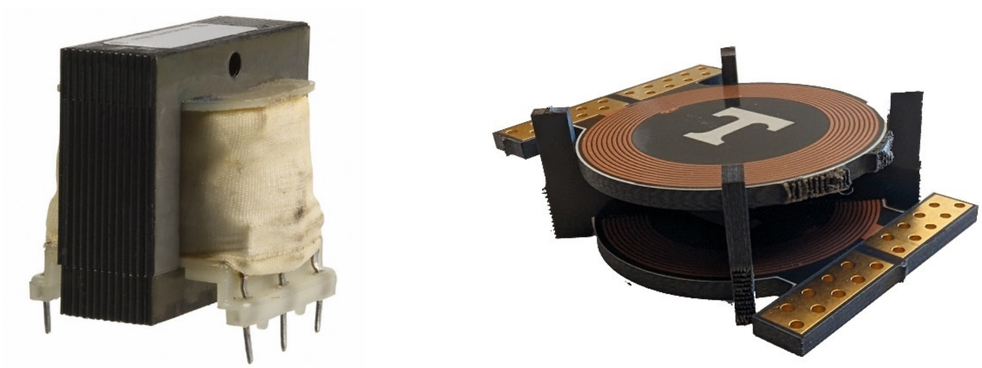
Simulate the circuit from PE2 for k = [0.25, 0.5, 0.75, 1] and f = [1 kHz, 50 kHz, 1 MHz] (12 total points).

In each case, record the amplitude of $v_1(t)$ and $v_2(t)$. In MatLab, create a plot of each signal where the x-axis is frequency, the y-axis is voltage amplitude, and there are four curves corresponding to each value of k. Use `semilogx()` instead of `plot()` so that all three frequencies are visible.

Comment on your results. Do these plots match your results predicted in PE1?

IV. Laboratory Experiment

In this experiment, we will be testing both a 3-winding transformer and a pair of WPT coils, both of which are shown in Fig. 9



(a) (b)

Figure 9. Transformer (a) and coupled coils (b) from parts kit

LE1 Lab Exercise 1: Transformer Turns Ratio

Your lab kit contains a PC-34-125 transformer. This transformer is rated for 50-500 Hz. Note the winding connections of the transformer as given in the datasheet and repeated in Fig. 10. The dot on the top of the transformer (labeled P1) indicates the location of pin 1.

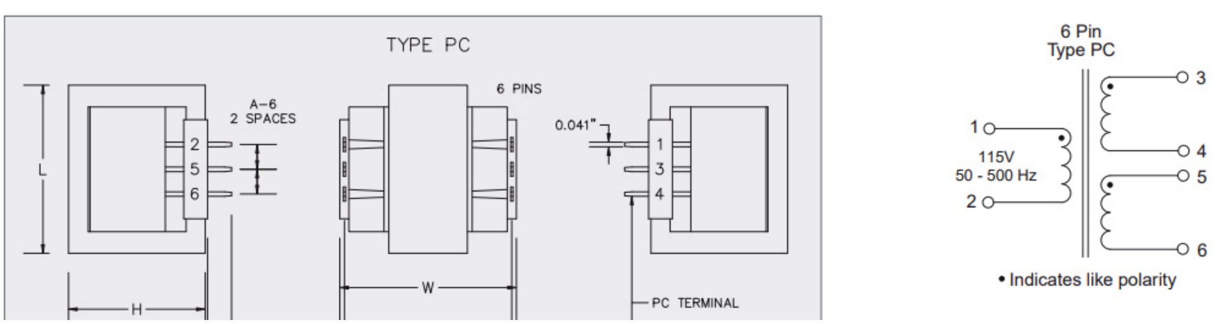


Figure 10. Transformer winding connections from PC-34-125 datasheet.

The transformer has three windings including a primary (pins 1-2) and two identical secondaries (pins 3-4 and 5-6).

Configure the function generator output to a 2V amplitude, 500 Hz sinusoid. Connect the output of the function generator to the primary winding. Using both oscilloscope channels, measure the primary winding voltage, and one of the secondary winding voltages. Using the measurement function of the oscilloscope, measure the amplitude of both signals.

Assuming the transformer behaves as an ideal transformer, the ratio of these amplitudes is determined by the turns ratio. From your measurement, compute the turns ratio, and express it 1:N, with $N \geq 1$. Repeat your test until you have a turns ratio for all three windings, and give the overall turns ratio as 1:N_1:N_2 with $N_1 \geq 1$ and $N_2 \geq 1$. In your report, draw the schematic symbol of this transformer with the turns labeled on each winding.

LE2 Lab Exercise 2: Transformer Inductances

An ideal transformer has infinite inductances, but a real component will always have finite values for all inductances. In this exercise you'll measure the actual winding and mutual inductances of the transformer. For simplicity, in this exercise and all remaining exercises, you can ignore the third winding (pins 5&6) and treat the component as a two-winding transformer.

In order to measure the impedances (resistance, inductance, and/or capacitance) we can simultaneously measure the voltage and current on an element, then use the elementary equations to solve the impedance. In this experiment, we'll use the following circuit for this purpose

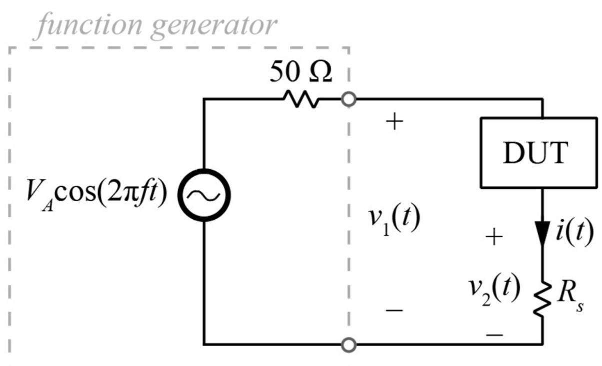


Figure 11. Circuit for impedance measurement

DUT stands for Device Under Test, which is the component that we are measuring. The dashed box gives an equivalent model of the function generator in sinusoidal mode. Note that you can set V_A and f, but the 50 Ω resistor cannot be altered or removed and is internal to the function generator itself. In this circuit, the function generator output $v_1(t)$ is applied across the series connection of the DUT and a sensing resistor Rs which is a known value. The sensing resistor allows measurement of current by noting that $v_2(t) = i(t)R_s$. Measuring $v_1(t)$ and $v_2(t)$ with the oscilloscope, the voltage and current of the DUT are then

$$v_{\text{DUT}}(t) = v_1(t) - v_2(t)$$

$$i_{\text{DUT}}(t) = \frac{1}{R_s} v_2(t)$$

Again using a 2V amplitude, 500 Hz sinusoid from the function generator and R_s = 1kΩ construct the circuit of Fig. 11 with the DUT being the primary winding of the transformer. Measure and compute the winding voltage and current, then compute the inductance of the winding based on these measurements. Do the same for the secondary winding. Include the measured voltages, currents, and inductances for both cases in your report.

Next, connect both the primary and secondary windings in series and measure/compute the inductance. Finally, flip *one* of the windings from the prior measurement (i.e. reverse the connection of the pins) so that the windings are still in series but a flux addition that is opposite what it was previously. Measure/compute the inductance in this. Use these two measured series inductances to compute the mutual inductance of the two windings. Report your measurements, computed mutual inductance M, and the value of the winding coupling coefficient k in your lab report. Comment on the value of k and whether it matches your prediction for this component.

> *Hint: using the equation for mutual inductance when neither of the ideal transformer assumptions are applied, write the equation for the total voltage when they are connected in either series or anti-series and compare.*

LE3 Lab Exercise 3: Transformer Reflection

Take the 1μF capacitor from your parts kit (yellow, with print "105" on the disc). Using the circuit of Fig. 11 with this capacitor as the DUT, measure/compute the capacitance of this component.

Next, using the same measurement circuit, instead connect the capacitor through the transformer as shown in Fig. 12. Test both when the primary is connected to the capacitor and when the secondary is connected to it. In each case, measure the capacitance seen from the winding on the left.

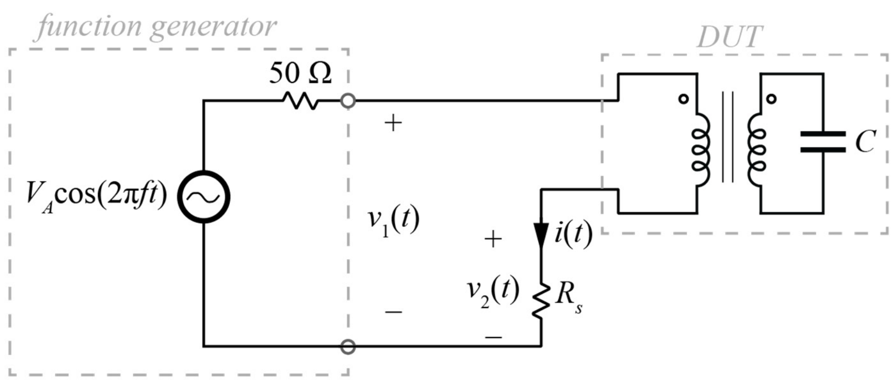


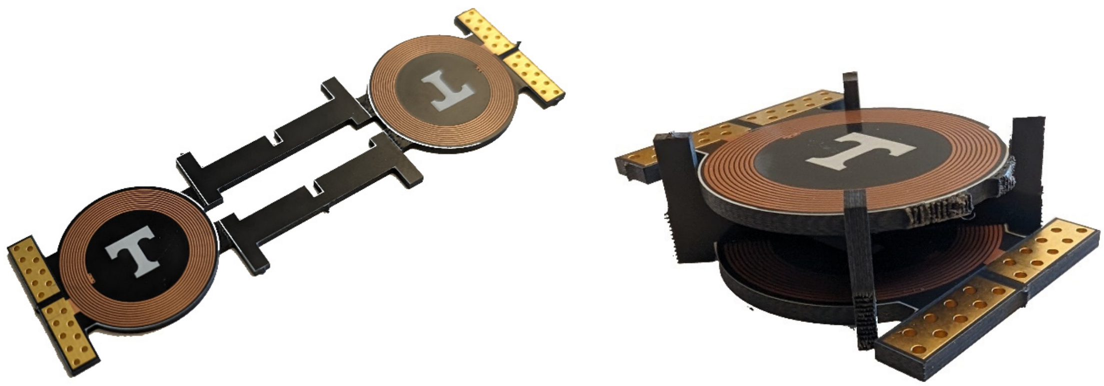
Figure 12. Impedance measurement with transformer scaling

Report all three measured capacitance, and compare them to your expectations given the results of LE1.

LE4 Lab Exercise 4: Air Core Coil Coupling

The remaining exercises will use the air core coils in the ECE 202 supplement kit. These coils come as a single printed circuit board (PCB) containing two identical coils and a mechanical stand that fixes the distance and alignment between them. See Fig. 13. Break apart the coils and holder and assemble them as shown in Fig. 13(b). Note that the coil holder is asymmetric; use it with the longer legs upwards in this experiment, as shown in the photograph.

> *Note: If you have access to electrical tape, you can tape this whole structure together. Do not use tapes with conductive material or with strong adhesive, though, as we'll be using the other stand arrangement in future experiments.*



(a) (b)

Figure 13. Coil printed circuit board (a) and assembled coils with holder (b)

Apply a 2V sinusoidal 500 kHz (note kilohertz in this and future tests) signal from the function generator to one coil and measure the voltage on both coils. Knowing that the two coils are identical, compute the coupling coefficient from this measurement, alone.

LE5 Lab Exercise 5: Air Core Inductances

Using a similar procedure to that in LE2, measure the inductance of each coil and the mutual inductance of the pair. Keep the source at 500 kHz, and use a resistor R_s of 20 Ω instead of the 1 kΩ used previously. Compare the coupling coefficient from these measurements to the value found in LE4.

Additionally, compare the coupling coefficient of these coils to that of the transformer in LE2. Comment on why you think they are different, given the physical construction of the two.

LE6 Lab Exercise 6: Magnetic shielding

Magnetic fields flow more easily though materials with high permeability μ but are blocked by materials with high conductivity. Connect the coils as in LE4, with the function generator on one coil and both coil voltages measured by the oscilloscope. Make sure that the coil connected to the function generator is on top. Again note the ratio of the voltages and the coupling coefficient.

Next, place the transformer from LE1-LE3, upside-down, on top of the top coil. This should place its magnetic core directly in contact with the top coil. Measure the voltages and the coupling coefficient in this setup.

Remove the transformer, and replace it with a sheet of metal. You can use aluminum foil, a coin, a foil-lined gum wrapper, a metal cup or can, or any continuous sheet of conductive material that covers the top coil. Take a picture of your coils with this metal on top, and submit it along with your measurements of voltages and the coupling coefficient in this setup.

Finally, remove the conductor and place your finger on top of the coil. Measure the voltages and the coupling coefficient in this setup.

In your lab report, comment on the difference between *k* values in the base setup, magnetic material, conductor, and finger measurements. Explain how any why these measurements differ (or don't).

V. Conclusions

In your report, make sure to include the results of *all* measurements and calculations requested in the lab. Whenever the exercise asks for a "comparison", the comparison should be discussed explicitly in the report. When measurements are asked for, you should show the results of the measurements with oscilloscope waveforms (preferred) or tabular data.