Announcements

- Experiment 2 Posted
- Guest lecture feedback (Dr. Mukhopadhyay)
  - Worked problems posted to website
- Midterm Wednesday March 6th

HW4 and Quiz 2 Notes

- $v(t)$ is always a time domain expression
  - $v(t) = V_A \cos(\omega t + \varphi)$
  - $\mathbf{V} = \underline{V} = A + \text{j}\,X = V_A e^{j\varphi} = V_A \angle \varphi$
- SI prefixes

PREFIX	ABBREVIATION	MEANING
pico-	p	0.000000000001 or 10^{-12}
nano-	n	0.000000001 or 10^{-9}
micro-	µ	0.000001 or 10^{-6}
milli-	m	0.001 or 10^{-3}
kilo	k	1,000 or 10^{3}
Mega-	M	1,000,000 or 10^{6}
Giga-	G	1,000,000,000 or 10^{9}
Tera-	T	1,000,000,000,000 or 10^{12}

Midterm

- Midterm Wednesday March 6th
  - Covers coupled inductors and transformers, phasor circuit analysis and complex power
  - 3 problems, ~3x quiz length, Full class period for the exam
  - Lectures 1-16, HW 1-5, Quiz 1-2, Chapters 10,11, & 13
  - Recommended:
    - Review all lecture slides and in-class examples
    - Review solutions to HW problems where you missed points
    - Rework Quizzes
    - Create crib sheet
    - Practice complex numbers on your calculator
    - Read through Experiment 1 and 2 review/introductions

Midterm Problems:

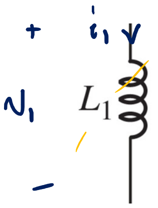
1. Phasor circuit analysis with transformer and multiple sources
   - Solve for output signal
2. Phasor circuit analysis with mutual inductances
   - Understand coupling coefficient and dot notation
   - Solve for output signal
3. Phasor power
   - Solve for and implement matched load impedance (Max Power)
   - Solve *S* and *PF*

Chapter 13

MUTUAL INDUCTANCE

Inductance: Review

We know inductor has circuit behavior $v_1 = L_1 \frac{di_1}{dt}$



More fundamentally

Ampere's law $\oint_C B \cdot dl = \mu_0 I_{encl}$

total flux

$\Phi = \alpha i_1$

↑ parameter depends on geometry & physical constants

for an inductor

$\Phi = \Phi_r$

$v_1 = \alpha \frac{di_1}{dt}$

↑

$= L$ for single turn

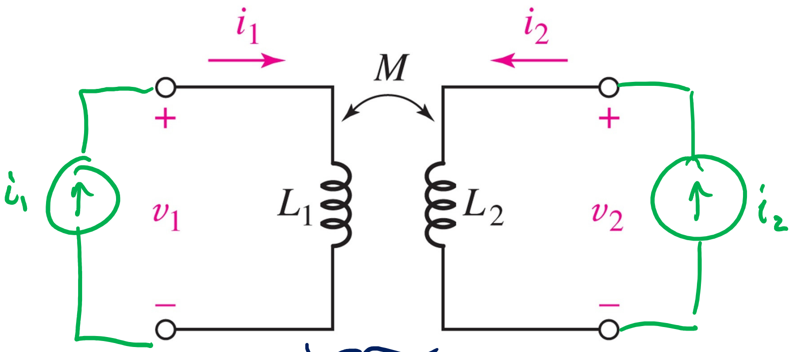
Faraday's law

$v_1 = \frac{d\Phi_r}{dt}$

For an N-turn inductor

$\Phi = N\alpha i_1$, $v_1 = N\frac{d\Phi}{dt}$, $\boxed{v_1 = N^2 \alpha \frac{di_1}{dt}}$

Mutual Inductance



New circuit symbol

Apply superposition

with $i_2 = \emptyset$

$$\begin{cases} v_1 = L_1 \dfrac{di_1}{dt} \\ v_2 = M \dfrac{di_1}{dt} \end{cases}$$

with $i_1 = \emptyset$

$$\begin{cases} v_1 = M \dfrac{di_2}{dt} \\ v_2 = L_2 \dfrac{di_2}{dt} \end{cases}$$

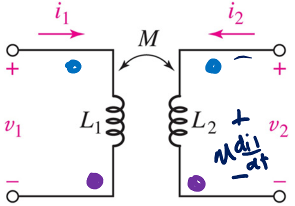
By superposition

$$\boxed{\begin{cases} v_1 = L_1 \dfrac{di_1}{dt} \pm M \dfrac{di_2}{dt} \\ v_2 = \pm M \dfrac{di_1}{dt} + L_2 \dfrac{di_2}{dt} \end{cases}}$$

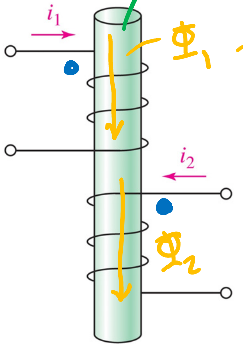
Symbols and Dot Convention

* Make sure to use passive sign convention

$$\begin{cases} v_1 = L_1 \dfrac{di_1}{dt} + M \dfrac{di_2}{dt} \\ v_2 = M \dfrac{di_1}{dt} + L_2 \dfrac{di_2}{dt} \end{cases}$$



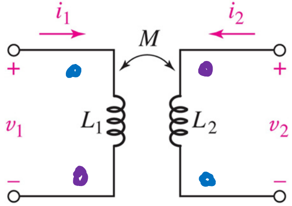
magnetic material $\mu >> \mu_0$

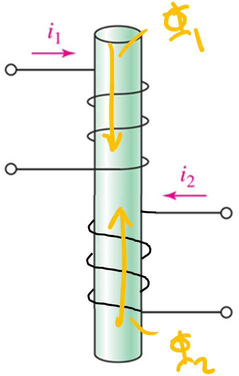


by RHR Φ_1 & Φ_2 in same direction

Physical: If both i_1 & i_2 enter the dotted terminals of L_1 & L_2, they produce additive flux

$$\begin{cases} v_1 = L_1 \dfrac{di_1}{dt} - M \dfrac{di_2}{dt} \\ v_2 = -M \dfrac{di_1}{dt} + L_2 \dfrac{di_2}{dt} \end{cases}$$

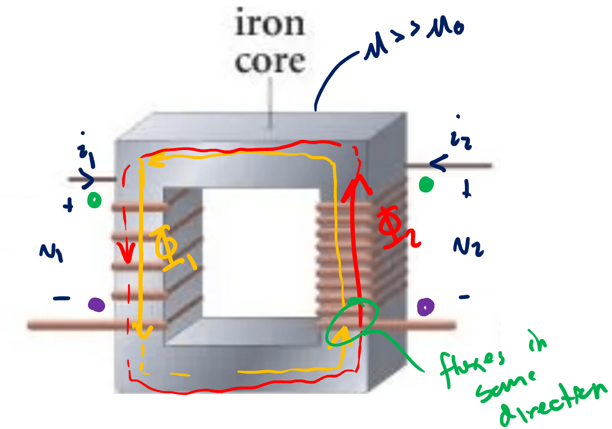




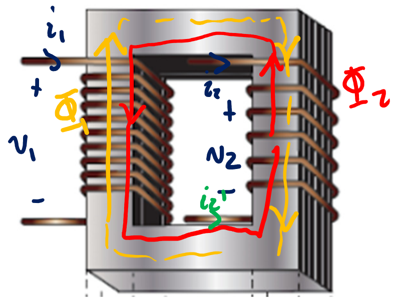
subtractive flux

Circuit: current flowing into the dotted terminal will produce a positive voltage relative to the dotted terminal of the other winding (in open circuit)

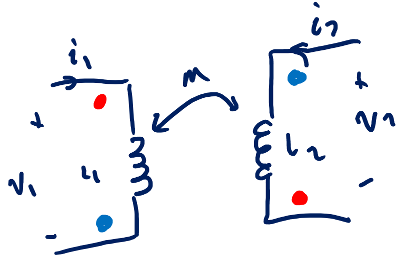
Dot Notation Example



Find which terminals should be dotted on each winding (2 possibilities)

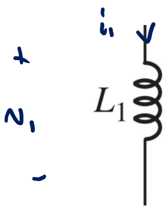


Sketch circuit symbol with same reference polarities



Energy Storage

@ $t=\phi$, $i_1=\phi$, @ $t=t_0$, $i_1=I_0$



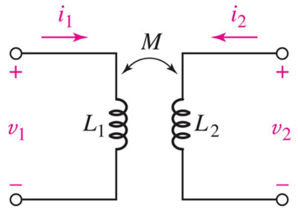
Review

$$E_L = \int_0^{t_0} v_1 \cdot i_1 \, dt = \int_0^{t_0} L_1 \left[i_1 \cdot \frac{di_1}{dt} \right] dt$$

$$i_1 \frac{di_1}{dt} = \frac{1}{2}\left[\frac{d}{dt} i_1(t)^2\right] = \frac{1}{2}\left[2 i_1(t) \frac{di_1(t)}{dt}\right]$$

$$= \frac{1}{2} L_1 \int_0^{t_0} \frac{d}{dt} i_1(t)^2 \, dt = \frac{1}{2} L_1 \left[i_1(t=t_0)^2 - i_1(t=0)^2 \right]$$

$$\boxed{E_L = \frac{1}{2} L_1 I_0^2}$$



At $t=\phi$ $i_1=\phi$ & $i_2=\phi$, at $t=t_0$ $i_1=I_1$ $i_2=I_2$

$$E = \int_0^{t_0} (v_1 i_1 + v_2 i_2) \, dt$$

$$= \int_0^{t_0} \left(L_1 i_1 \frac{di_1}{dt} \pm M i_1 \frac{di_2}{dt} + L_2 i_2 \frac{di_2}{dt} \pm M i_2 \frac{di_1}{dt} \right) dt$$

$$= \frac{1}{2} L_1 I_1^2 + \frac{1}{2} L_2 I_2^2 \pm \int_0^{t_0} \underbrace{\left(M i_1 \frac{di_2}{dt} + M i_2 \frac{di_1}{dt} \right)}_{= M \frac{d}{dt}(i_1 \cdot i_2)} dt$$

$$\boxed{E = \frac{1}{2} L_1 I_1^2 + \frac{1}{2} L_2 I_2^2 \pm M I_1 I_2}$$

starting from zero currents & ramping up to $i_1 = I_1$ & $i_2 = I_2$, must be true that

$$E = \frac{1}{2} L_1 I_1^2 + \frac{1}{2} L_2 I_2^2 \pm M I_1 I_2 > \phi$$

$$M < \frac{\frac{1}{2} L_1 I_1^2 + \frac{1}{2} L_2 I_2^2}{I_1 I_2} = \frac{1}{2} L_1 \frac{I_1}{I_2} + \frac{1}{2} L_2 \frac{I_2}{I_1}$$

find minimum of $\frac{1}{2} L_1 x + \frac{1}{2} L_2 \frac{1}{x}$, $x = \frac{I_1}{I_2}$

$$\frac{\partial M}{\partial x} = \frac{1}{2} L_1 - \frac{1}{2} L_2 \frac{1}{x^2} = \phi \qquad x = \sqrt{\frac{L_2}{L_1}}$$

$$\frac{\partial^2 M}{\partial x^2} = L_2 \frac{1}{x^3} \quad \checkmark$$

so,

$$M < \frac{1}{2} L_1 \sqrt{\frac{L_2}{L_1}} + \frac{1}{2} L_2 \sqrt{\frac{L_1}{L_2}} = \frac{1}{2}\sqrt{L_1 L_2} + \frac{1}{2}\sqrt{L_1 L_2}$$

$$\boxed{M \leq \sqrt{L_1 L_2}}$$

Coupling Coefficient

define coupling coefficient $k = \frac{M}{\sqrt{L_1 L_2}}$

$0 \le k \le 1$

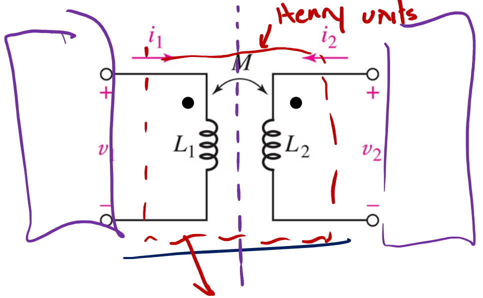
$k = \phi \rightarrow$ two separate inductors

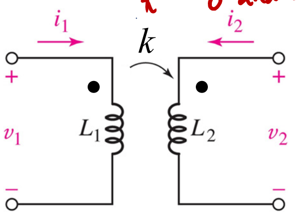
$k = 1 \rightarrow$ perfect coupling between them

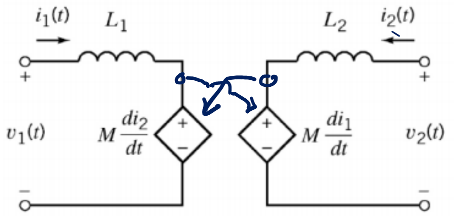
all of Φ_1 flows through L_2 & vice-versa

known as a "transformer"

Equivalent Circuits

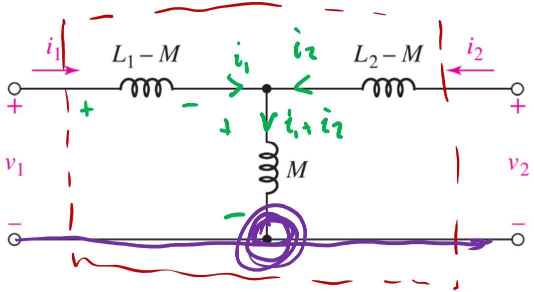






$$\begin{cases} v_1 = L_1\frac{di_1}{dt} \pm M\frac{di_2}{dt} \\ v_2 = \pm M\frac{di_1}{dt} + L_2\frac{di_2}{dt} \end{cases}$$

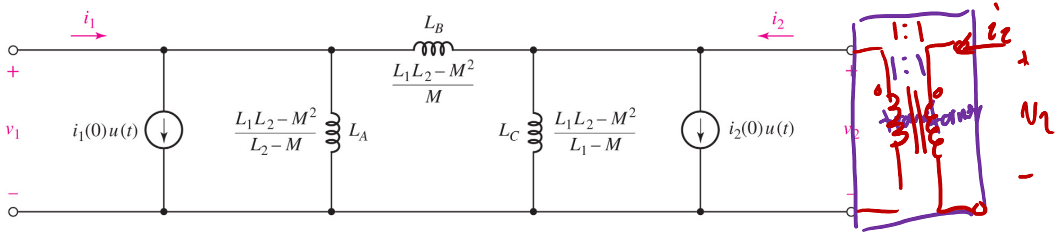
T-network



$$\begin{cases} v_1 = (L_1 - \cancel{M})\frac{di_1}{dt} + M\frac{d}{dt}(\cancel{i_1} + i_2) \\ v_2 = M\frac{d}{dt}(i_1 + \cancel{i_2}) + (L_2 - \cancel{M})\frac{di_2}{dt} \end{cases}$$

for positive polarity

Π-network



Transformers

Special case of coupled inductors with $k = 1 \longleftrightarrow$ perfect coupling $\longleftrightarrow M = \sqrt{L_1 L_2}$

$$\begin{cases} v_1 = L_1 \dfrac{di_1}{dt} \pm M \dfrac{di_2}{dt} \\ v_2 = \pm M \dfrac{di_1}{dt} + L_2 \dfrac{di_2}{dt} \end{cases} \xleftrightarrow{M = \sqrt{L_1 L_2}} \begin{cases} v_1 = L_1 \dfrac{di_1}{dt} \pm \sqrt{L_1 L_2} \dfrac{di_2}{dt} \\ v_2 = \pm \sqrt{L_1 L_2} \dfrac{di_1}{dt} + L_2 \dfrac{di_2}{dt} \end{cases}$$

$$v_1 = \sqrt{\frac{L_1}{L_2}}\, v_2$$

$$\downarrow$$

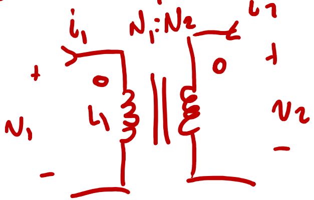
$$v_1 = \sqrt{\frac{\cancel{\alpha_1} N_1^2}{\cancel{\alpha_2} N_2^2}}\, v_2$$

if $k=1$, it must be true that $\alpha_1 = \alpha_2$

$$\boxed{v_1 = \frac{N_1}{N_2} v_2}$$

$\frac{N_1}{N_2}$ is the "turns ratio"

Circuit Symbol



$N_1:N_2$ is the same as $1:N$, $N = \frac{N_2}{N_1}$

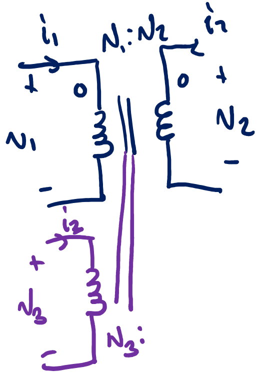
Ideal Transformer

"Ideal" transformer has very large L_1 & L_2

Recall: Inductors (& transformers) cannot have DC voltage applied to their windings

- $v_1 = L_1 \frac{di_1}{dt} \rightarrow i_1 = \frac{1}{L_1}\int_0^t v_1\, dt \rightarrow$ DC voltage causes current to go to ∞ & will saturate & stop
- Materials $\rightarrow$ core materials needed for $K \approx 1$ working at some finite current
- $v = N \frac{d\Phi}{dt} \rightarrow$ Faraday's Law, need time-varying waveforms

when L_1 & L_2 are large enough $\rightarrow$ No energy storage



$$\boxed{v_1 = \frac{N_1}{N_2} v_2}$$

No energy storage, so $v_1 i_1 + v_2 i_2 = \phi$

$$\left(\frac{N_1}{N_2} v_2\right) i_1 + v_2 i_2 = \phi$$

$$\boxed{N_1 i_1 + N_2 i_2 = \phi}$$

$$i_1 = -\frac{N_2}{N_1} i_2$$

for more than two turns

$$\frac{v_1}{N_1} = \frac{v_2}{N_2} = \frac{v_3}{N_3} = \ldots$$

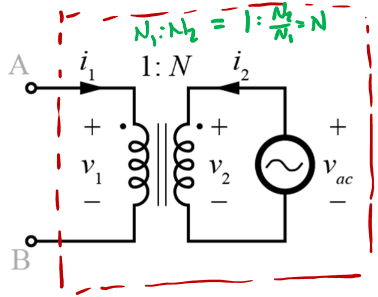
$$N_1 i_1 + N_2 i_2 + N_3 i_3 + \ldots = \phi$$

Transformer Reflection

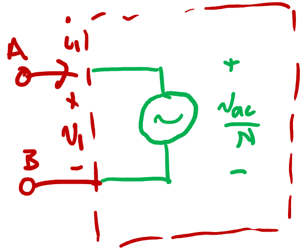
$$\boxed{\begin{array}{c} \frac{v_1}{1} = \frac{v_2}{N} \\ 1 i_1 + N i_2 = \phi \end{array}}$$

$$v_1 = \frac{v_2}{N}$$

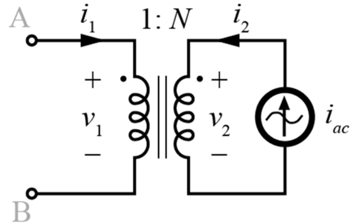
$$i_1 = -N i_2$$

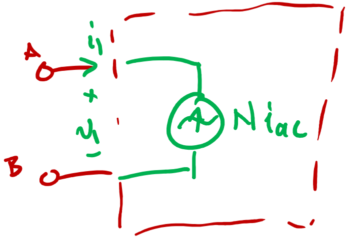


equivalent circuit @ primary

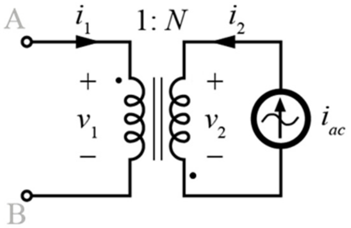


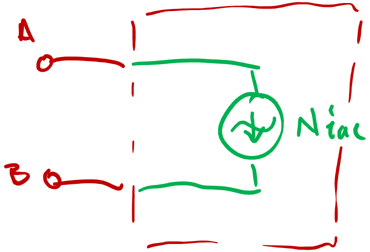
$$v_1 = \frac{v_2}{N}$$





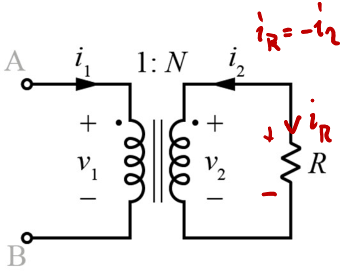
$$i_1 = -N i_2$$

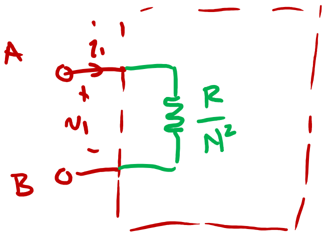




$$v_1 = \frac{v_2}{N}$$

$$i_1 = -N i_2$$

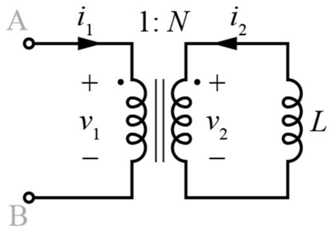


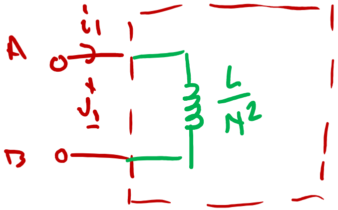


$$v_2 = (-i_2)R$$

$$N v_1 = \left(\frac{1}{N}\right) i_1 R$$

$$v_1 = \left(\frac{1}{N^2}R\right) i_1$$

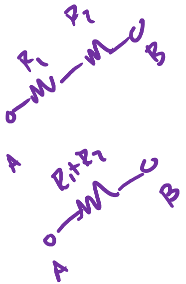


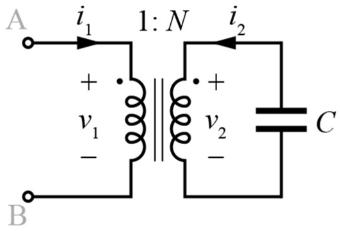


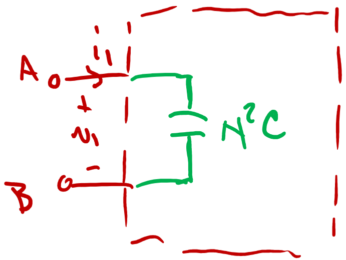
$$v_2 = L\frac{d(-i_2)}{dt}$$

$$N v_1 = L\frac{d}{dt}\left(\frac{1}{N} i_1\right)$$

$$v_1 = \frac{1}{N^2} L \frac{di_1}{dt}$$







$$-i_2 = C\frac{dv_2}{dt}$$

$$\frac{1}{N} i_1 = C\frac{d}{dt} N v_1$$

$$i_1 = N^2 C \frac{dv_1}{dt}$$

Chapter 10

PHASOR MODELING

Sinusoids as Complex Numbers

$$v(t) = A\cos(\omega t + \varphi)$$

$$= \mathrm{Re}\{A e^{j(\omega t + \varphi)}\}$$

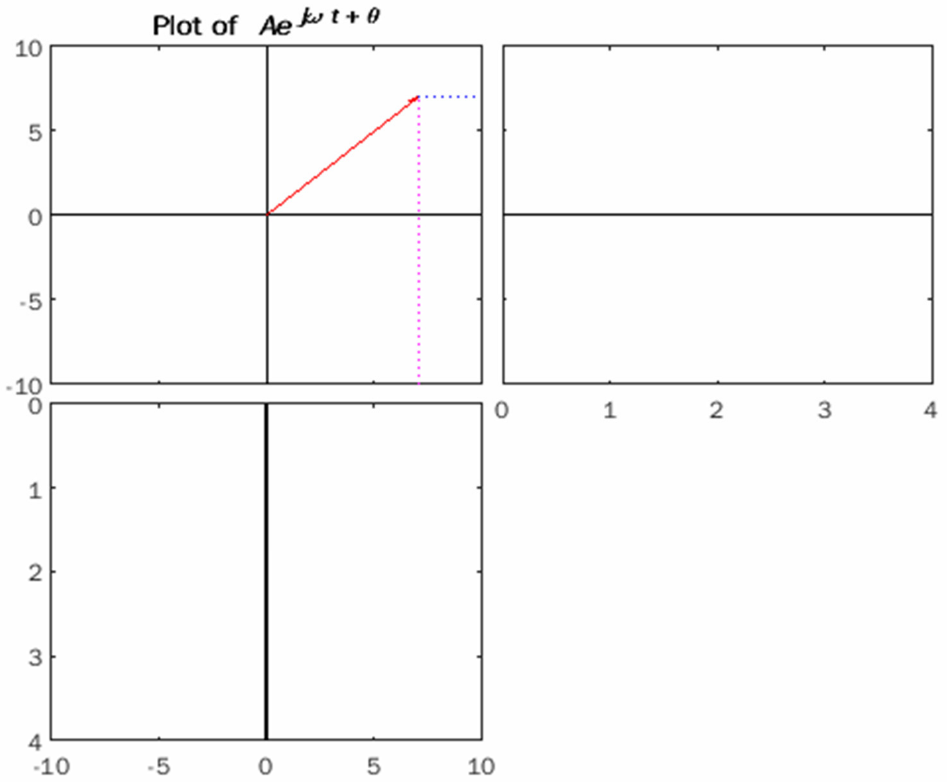
$$= \mathrm{Re}\{A \; e^{j\omega t} \; e^{j\varphi}\}$$

cosine magnitude sinusoid phase

$$\frac{d}{dt} v(t) = -A\omega \sin(\omega t + \varphi) = A\omega \cos(\omega t + \varphi + 90^\circ)$$

$$= \mathrm{Re}\{A\omega \; e^{j\omega t} \; e^{j\varphi} \; e^{j\frac{\pi}{2}}\}$$

$$= \mathrm{Re}\{A(j\omega) e^{j\omega t} e^{j\varphi}\}$$



Plot of $Ae^{j\omega t + \theta}$
10
5
0
-5
-10
0
1
2
3
4
-10
-5
0
5
10
0
1
2
3
4

Phasor Transformation

"Phasor" is a complex number that represents a sinusoid in steady-state

→ useful in analyzing single-frequency sinusoidal circuits in steady-state

$A\cos(\omega t+\varphi)$

- A amplitude
- φ phase
- ~~ω frequency~~ → for single-frequency circuits
- ~~cos sinusoidal~~ function → by convention, always use cos
- ~~t time~~ → steady-state, single-frequency

$$A\cancel{\cos}(\cancel{\omega t}+\varphi) = \cancel{Re}\{A\cancel{e^{j\omega t}}\,e^{j\varphi}\}$$

$$\xrightarrow{\text{phasor transform}} Ae^{j\varphi}$$

Phasor Notation

$v(t) = A\cos(\omega t + \varphi) = Re\{A \cdot e^{j\omega t} e^{j\varphi}\}$

transform @ ω

$\underline{V} = Ae^{j\varphi} \longleftrightarrow A \angle \varphi$ (short hand)

Bold in book
underline in lecture

$i(t) = B\sin(\omega t + \theta)$
$= B\cos(\omega t + \theta - 90°)$

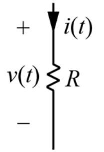
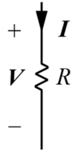
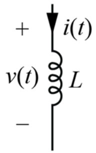
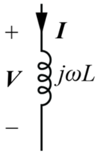
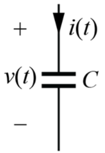
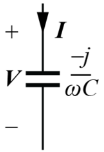
transform

$\underline{I} = Be^{j(\theta - \frac{\pi}{2})} \longleftrightarrow B \angle (\theta - \frac{\pi}{2})$

Comments:

- Phasor transform works for all voltage / current sources & signals
- Everything must be at same frequency ω
- Using complex numbers in transformation
  - No "t" in any phasor expression
  - No complex # in time domain

Phasor Circuit Elements

Time Domain	Phasor Domain
	
$v(t) = i(t)R$ $\longrightarrow$	$\underline{V} = \underline{I}R$
	
$v(t) = L\dfrac{di}{dt}$ $\longrightarrow$	$\underline{V} = \underbrace{j\omega L}_{Z_L}\underline{I}$
	
$i(t) = C\dfrac{dv}{dt}$ $\longrightarrow$	$\underline{V} = \underbrace{\dfrac{-j}{\omega C}}_{Z_C}\underline{I}$

$v(t) = A\cos(\omega t + \varphi) \longrightarrow \underline{V} = Ae^{j\varphi}$

$i(t) = \dfrac{A}{R}\cos(\omega t + \varphi) \longrightarrow \underline{I} = \dfrac{A}{R}e^{j\varphi}$

$v(t) = i(t)R \longrightarrow \boxed{\underline{V} = \underline{I}R}$

$i(t) = A\cos(\omega t + \varphi) \longrightarrow \underline{I} = Ae^{j\varphi}$

$v(t) = AL\omega\cos(\omega t + \varphi + 90^\circ) \longrightarrow \underline{V} = A\omega L e^{j(\varphi + \frac{\pi}{2})}$

$\underline{V} = Aj\omega L e^{j\varphi}$

$\boxed{\underline{V} = (j\omega L)\underline{I}}$

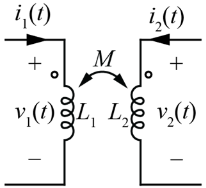
$v(t) = A\cos(\omega t + \varphi) \longrightarrow \underline{V} = Ae^{j\varphi}$

$i(t) = AC\omega\cos(\omega t + \varphi + 90^\circ) \longrightarrow \underline{I} = AC\omega e^{j\varphi}e^{j\frac{\pi}{2}}$

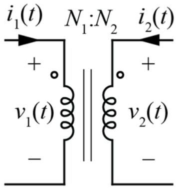
$\underline{I} = Aj\omega C e^{j\varphi}$

$\boxed{\underline{V} = \dfrac{1}{j\omega C}\underline{I} = \dfrac{-j}{\omega C}\underline{I}}$

Time Domain

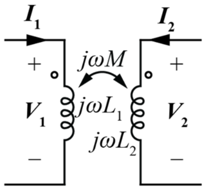


$$v_1(t) = L_1\frac{di_1}{dt} + M\frac{di_2}{dt}$$
$$v_2(t) = M\frac{di_1}{dt} + L_2\frac{di_2}{dt}$$

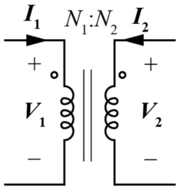


$$\frac{v_1(t)}{N_1} = \frac{v_2(t)}{N_2}$$
$$N_1 i_1(t) + N_2 i_2(t) = 0$$

Phasor Domain



$$\underline{V_1} = j\omega L_1 \underline{I_1} + j\omega M \underline{I_2}$$
$$\underline{V_2} = j\omega M \underline{I_1} + j\omega L_2 \underline{I_2}$$



$$\frac{\underline{V_1}}{N_1} = \frac{\underline{V_2}}{N_2}$$
$$N_1 \underline{I_1} + N_2 \underline{I_2} = \emptyset$$

Impedance

Phasor equivalent of ohm's law

$$\underline{V} = \underline{I}\, Z$$

$Z \rightarrow$ "impedance" , $Z = R + jX$

"resistance" $\mathrm{Re}\{Z\}$

"reactance" $\mathrm{Im}\{Z\}$

$Z_R = R$ for resistor

$Z_L = j\omega L$ for inductor

$Z_C = \dfrac{-j}{\omega C}$ for capacitor

All Z have units of Ohms

Y = "Admittance" $= \dfrac{1}{Z} = G + jB$ $\longrightarrow$ units of siemens (or mhos)

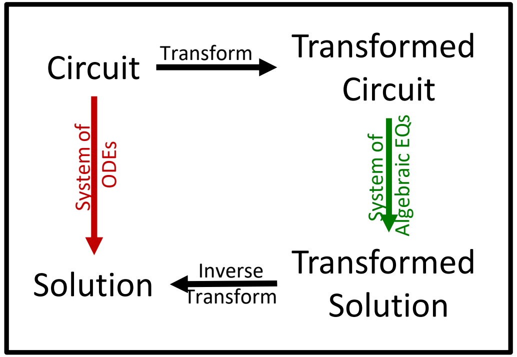
"conductance"

"susceptance"

$$\frac{1}{Z} \neq \frac{1}{R} + j\frac{1}{X}$$

$$\frac{1}{Z} = \frac{1}{R+jX}\,\frac{(R-jX)}{(R-jX)} = \frac{R-jX}{R^2+X^2} = \frac{R}{R^2+X^2} - j\frac{X}{R^2+X^2}$$

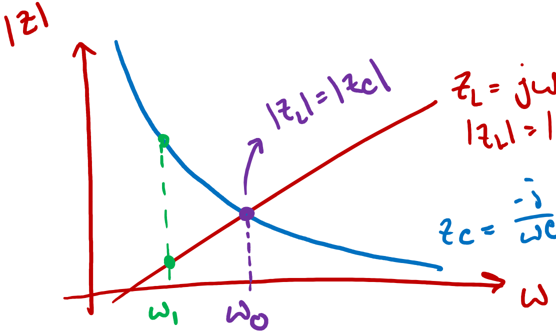
Phasor Circuit Analysis



Goal: Analyze a LTI circuit to find steady-state solution with only single-frequency sinusoidal source(s)

1. Transform all sources & signals into their phasor equivalents
2. Transform all passives into impedances
3. Solve the circuit
   - Use 201 techniques for DC resistor-only circuits
4. Transform solution back into the time domain

Reactance and Resonance



$|Z_L| = |Z_C|$

$Z_L = j\omega L$

$|Z_L| = |j\omega L| = |\omega L|$

$Z_C = \frac{-j}{\omega C} \rightarrow |Z_C| = \frac{1}{\omega C}$

@ $\omega_0 \rightarrow |Z_L| = |Z_C|$

$\omega_0 L = \frac{1}{\omega_0 C}$

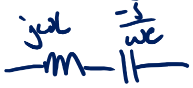
$$\boxed{\omega_0 = \frac{1}{\sqrt{LC}}}$$

resonant frequency

At Low frequency $\omega \rightarrow \emptyset$ ($\approx$ DC) $Z_C \rightarrow \infty$ (open) $Z_L \rightarrow \emptyset$ (short)

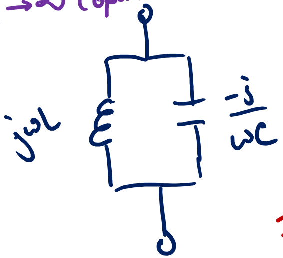
At high frequency $\omega \rightarrow \infty$ $Z_C \rightarrow \emptyset$ (short) $Z_L \rightarrow \infty$ (open)

At resonance



$\Rightarrow$ short @ resonance (ω_0)

$Z_{eq} = j\omega L + \frac{-j}{\omega C}$



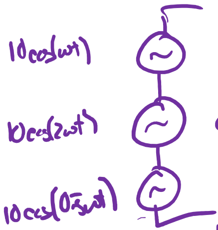
for only 2 in parallel

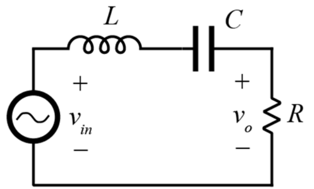
$Z_{eq} = \frac{1}{\frac{1}{j\omega L} + \frac{\omega C}{-j}} = \frac{j\omega L \cdot \left(\frac{-j}{\omega C}\right)}{j\omega L + \frac{-j}{\omega C}}$

$\Rightarrow$ open @ resonance

Phasor Superposition

3 frequencies → apply superposition in time domain





Find $v_o(t)$ for $v_{in}(t) = 10\cos(\omega t) + 10\cos(2\omega t) + 10\cos(0.5\omega t)$ and $\omega = 2\pi 100$ kHz, $R = 10\ \Omega$, $L = 10\ \mu$H, and $C = 253$ nF

$\omega_{LF} = \frac{1}{2}\omega = 2\pi 50\text{kHz}$

$V_{in} = 10\angle 0^0$

$Z_L = j\omega_{LF}L = j\pi$

$Z_C = \frac{-j}{\omega_{LF}C} = -j4\pi$

$Z_R = R = 10\Omega$

same

$V_o = V_{in}\frac{Z_R}{Z_R + Z_L + Z_C} = 0.73\angle 43^0$

$v_{oLF}(t) = 0.73\cos(0.5\omega t + 43^0)$

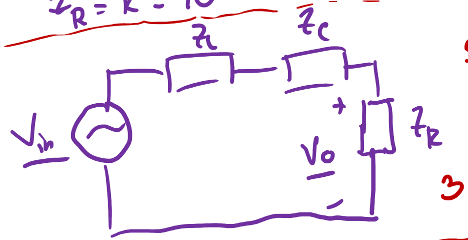
$\omega = 2\pi 100\text{kHz}$

$V_{in} = 10\angle 0^0$ → V_o

$Z_L = j\omega L = j2\pi$

$Z_C = \frac{-j}{\omega C} = -j2\pi$

$Z_R = R = 10$



same for all 3 frequencies

$V_o = V_{in}\frac{Z_R}{Z_R + Z_L + Z_C} = 10\angle 0^0$

$v_o(t) = 10\cos(\omega t)$

$\omega_{HF} = 2\omega = 2\pi 200\text{kHz}$

$V_{in} = 10\angle 0^0$

$Z_L = j\omega_{HF}L = j4\pi$

$Z_C = \frac{-j}{\omega_{HF}C} = -j\pi$

$V_o = 0.73\angle -43^0$

$v_{oHF}(t) = 0.73\cos(2\omega t - 43^0)$

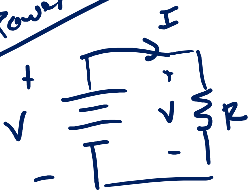
$$\boxed{v_o(t) = 10\cos(\omega t) + 0.73\cos(0.5\omega t + 43^0) + 0.73\cos(2\omega t - 43^0)}$$

Chapter 11

POWER IN SINUSOIDAL STEADY STATE

Average Power

DC Power

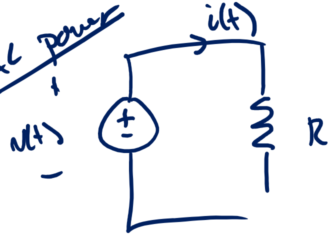


$P_R = V \cdot I \rightarrow$ Generally true for any 2-terminal element

for a resistor $V = IR$

$$P_R = I^2 R = \frac{V^2}{R}$$

AC power



$P_R(t) = v(t) \cdot i(t) \rightarrow$ Generally true of all 2-terminal elements

for a resistor $P_R(t) = i(t)^2 R = \dfrac{v(t)^2}{R}$

In any case, power calculation is not LTI

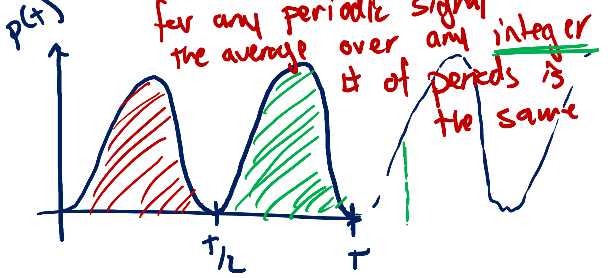
Average Power

Average power defined over some time interval $t \in [t_1, t_1 + T]$

capital "P" denotes average $\rightarrow$ $P = \dfrac{1}{T}\displaystyle\int_{t_1}^{t_1+T} p(t)\,dt$

Average power over all time

$$P = \lim_{T\to\infty} \frac{1}{T}\int_{-T/2}^{T/2} p(t)\,dt$$



for any periodic signal the average over any integer # of periods is the same

Power in a Resistor

Average power in a resistor with periodic (e.g. sinusoidal) sources

$$P_R = \frac{1}{T}\int_0^T P_R(t)\,dt = \frac{1}{T}\int_0^T i(t)^2 R\,dt$$

$$= R\,\frac{1}{T}\int_0^T i(t)^2\,dt$$

$$P_R = R\,\underbrace{\sqrt{\frac{1}{T}\int_0^T i(t)^2\,dt}^{\,2}} \longleftrightarrow P_R = I_{rms}^2 R = \frac{V_{rms}^2}{R}$$

rms is 'root mean squared'

Define rms $\quad X_{rms} = \sqrt{\frac{1}{T}\int_0^T x(t)^2\,dt}$

Note: Book calls this "effective" instead of rms

$I_{eff} = I_{rms} \qquad V_{eff} = V_{rms}$

RMS of a sinusoid

$\omega = 2\pi f \qquad T = \frac{1}{f} = \frac{2\pi}{\omega}$

$i(t) = I_A \cos(\omega t + \varphi)$

$$I_{rms} = \sqrt{\frac{1}{T}\int_0^T \left(I_A \cos(\omega t + \varphi)\right)^2 dt}$$

trig identity

$$\cos^2(\theta) = \frac{1}{2}(1 + \cos 2\theta)$$

$$I_{rms}^2 = \frac{1}{T}\int_0^T I_A^2 \cos^2(\omega t + \varphi)\, dt$$

$$I_{rms}^2 = \frac{1}{T} I_A^2 \frac{1}{2}\int_0^T 1 + \cos(2\omega t + 2\varphi)\, dt$$

$$I_{rms}^2 = \frac{1}{T} I_A^2 \frac{1}{2}\left[t + \sin(2\omega t + 2\varphi)\frac{1}{2\omega}\right]\Bigg|_0^{T = \frac{2\pi}{\omega}}$$

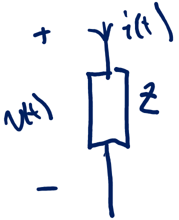
$$I_{rms}^2 = \frac{1}{T} I_A^2 \frac{1}{2}\left[\left(\frac{2\pi}{\omega} - 0\right) + \frac{1}{2\omega}\left(\cancel{\sin(4\pi + 2\varphi) - \sin(2\varphi)}\right)\right] \quad \phi$$

$$I_{rms}^2 = \frac{\cancel{\omega}}{\cancel{2\pi}} I_A^2 \frac{1}{2}\frac{\cancel{2\pi}}{\cancel{\omega}}$$

$$I_{rms}^2 = \frac{I_A^2}{2} \rightarrow \boxed{I_{rms} = \frac{I_A}{\sqrt{2}}}$$

for any sinusoidal signal

Power with Sinusoidal Sources



In steady-state with single-frequency sinusoidal sources @ ω

$$v(t) \rightarrow \underline{V} = V_A e^{j\varphi_V}$$

$$i(t) = \underline{I} = I_A e^{j\varphi_I}$$

$$\underline{V} = \underline{I}\, Z \quad \rightarrow \quad Z = \frac{\underline{V}}{\underline{I}} = \frac{V_A e^{j\varphi_V}}{I_A e^{j\varphi_I}} = \frac{V_A}{I_A} e^{j(\varphi_V - \varphi_I)}$$

power: since power is <u>not</u> LTI, we don't know how to calculate power w/ phasors

<u>Not</u> $\underline{V} \cdot \underline{I}$ → go back to time domain to calculate

<u>Trig identity</u>: $2\cos\theta \cos\varphi = \cos(\theta+\varphi) + \cos(\theta-\varphi)$

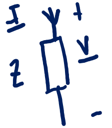
$$p(t) = i(t) \cdot v(t) = I_A \cos(\omega t + \varphi_I) \cdot V_A \cos(\omega t + \varphi_V)$$

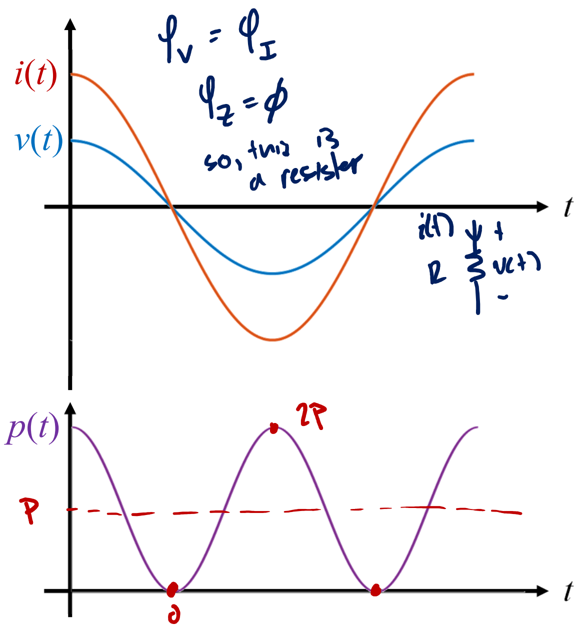
$$= \frac{I_A V_A}{2} \left(\underbrace{\cos(2\omega t + \varphi_I + \varphi_V)}_{\text{sinusoid @ } 2\omega} + \underbrace{\cos(\varphi_I - \varphi_V)}_{\text{constant (not time-varying)}} \right)$$

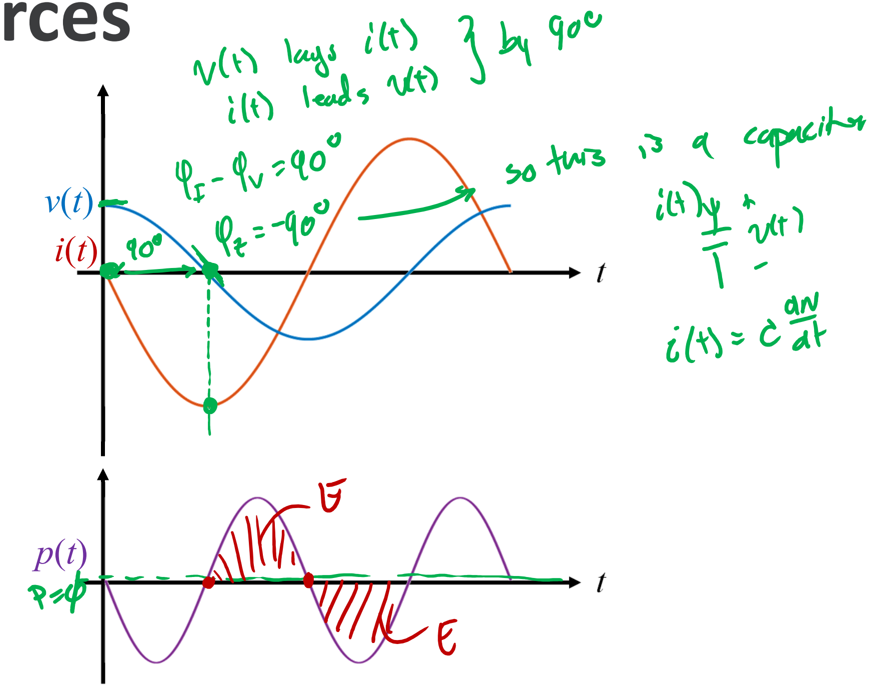
<u>Average power</u>

$$\boxed{P_Z = \frac{I_A V_A}{2} \cos(\varphi_I - \varphi_V)} = \frac{I_A V_A}{2} \cos(\varphi_Z) = V_{rms} I_{rms} \cos(\varphi_Z)$$

Power with Sinusoidal Sources







for a resistor

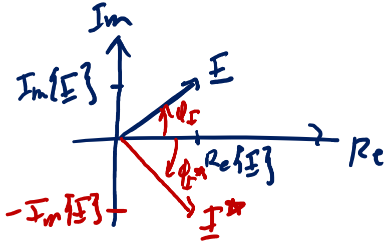
$$P = \frac{V_A I_A}{2}\cos(\varphi_V - \varphi_I) = \frac{V_A I_A}{2}\cos(\varphi_Z) = \frac{V_A I_A}{2}\cos(\phi) = \frac{V_A I_A}{2}$$

for a capacitor

$$P = \frac{V_A I_A}{2}\cos(-90°) = \phi$$

Capacitors & inductors must have $P = \phi$ in steady-state

Complex Power



$$P = \frac{V_A I_A}{2}\cos(\varphi_V - \varphi_I) = \boxed{\frac{1}{2}\mathrm{Re}\{\underline{V}\,\underline{I}^*\}}$$

$$= \frac{1}{2}\mathrm{Re}\{V_A e^{j\varphi_V} \cdot I_A e^{j(-\varphi_I)}\}$$

$$= \frac{1}{2}\mathrm{Re}\{V_A I_A e^{j(\varphi_V - \varphi_I)}\}$$

$$= \frac{1}{2} V_A I_A \cos(\varphi_V - \varphi_I)$$

what about the imaginary part of $\underline{V}\,\underline{I}^*$

$$S = \frac{1}{2}\underline{V}\,\underline{I}^* = P + jQ$$

complex power [VA] Volt-amps

Average power "Real power" [W] watts

Reactive power [VAR] "volt-amp reactive"

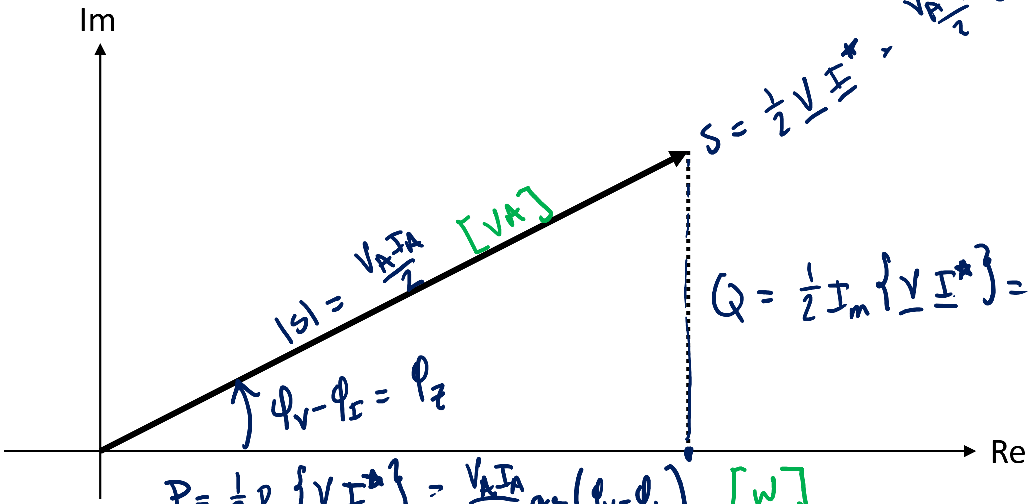
Power Triangle

$V_A = \sqrt{2}\, V_{rms}$

$I_A = \sqrt{2}\, I_{rms}$

$\varphi_V - \varphi_I = \varphi_Z$

$\underline{V} = \underline{I}\,\underline{Z}$



$S = \frac{1}{2}\underline{V}\,\underline{I}^* = \frac{V_A I_A}{2} e^{j(\varphi_V - \varphi_I)}$ [VA]

$|S| = \frac{V_A I_A}{2}$ [VA]

$\varphi_V - \varphi_I = \varphi_Z$

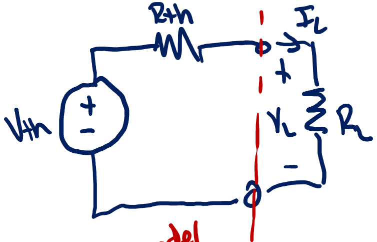
$Q = \frac{1}{2} I_m\{\underline{V}\,\underline{I}^*\} = \frac{V_A I_A}{2} \sin(\varphi_V - \varphi_I)$ [VAR]

$P = \frac{1}{2} R_e\{\underline{V}\,\underline{I}^*\} = \frac{V_A I_A}{2} \cos(\varphi_V - \varphi_I)$ [W]

$= \frac{1}{2} Re\left\{\underline{V}\left(\frac{\underline{V}}{\underline{Z}}\right)^*\right\}$

$PF = \frac{P}{|S|} = \cos(\varphi_V - \varphi_I)$ leading or lagging

Maximum Power Transfer



Thevenin model of source

model of load

what value of R_L will yield maximum power $\quad P_L = V_L I_L$

Answer: $R_L = R_{th}$

$$P_L = V_L I_L = \left(V_{th}\frac{R_L}{R_L+R_{th}}\right)\left(\frac{V_{th}}{R_L+R_{th}}\right) = V_{th}^2 \frac{R_L}{(R_L+R_{th})^2}$$

to find extrema

$$\frac{\partial P_L}{\partial R_L} = \cancel{V_{th}^2}\left[\frac{1\cdot(R_L+R_{th})^{\cancel{2}} - R_L(2)\cancel{(R_L+R_{th})}}{(R_L+R_{th})^{\cancel{4}\,3}}\right] = \emptyset$$

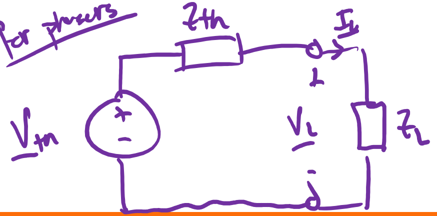
$$0 = \frac{R_L+R_{th}-2R_L}{(R_L+R_{th})^3} \to \text{numerator} \quad R_{th}-R_L = \emptyset$$

what value of R_{th} will give maximum power to a fixed R_L?

$\boxed{R_{th} = \emptyset}$

---

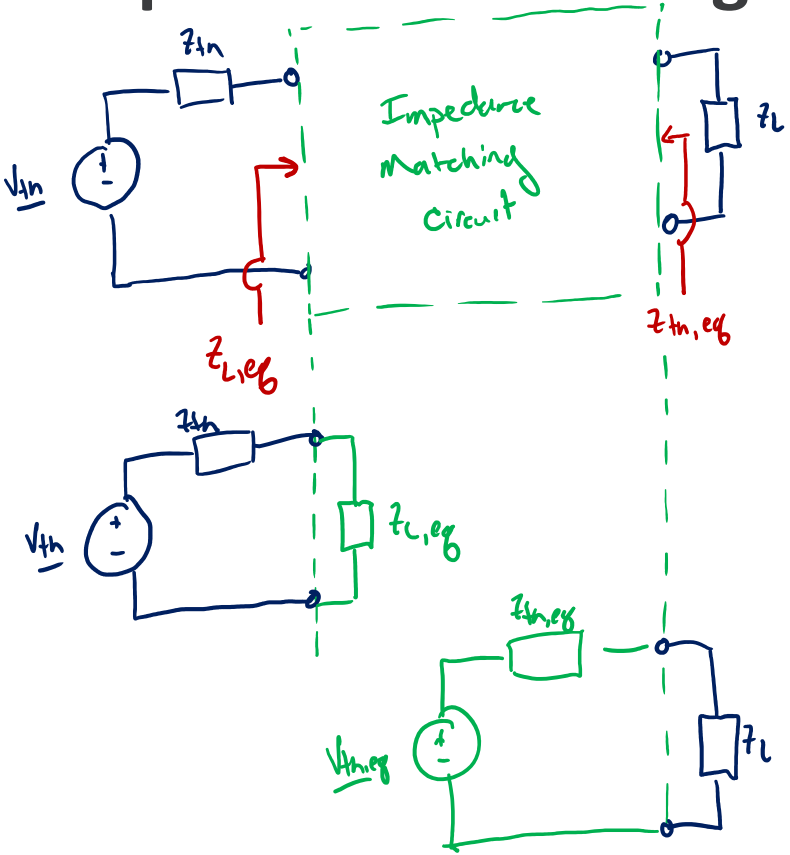
for phasors



what Z_L will maximize power transfer to the load?

$\boxed{Z_L = Z_{th}^*}$

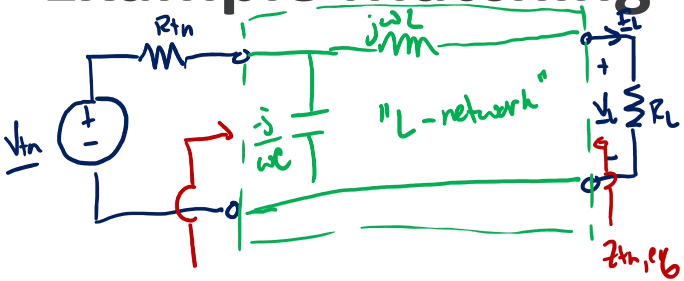
Impedance Matching



Impedance matching goals

- Maximize power transfer $\rightarrow Z_L = Z_{th,eq}^*$
- Minimize distortion $\rightarrow Z_L = Z_{th,eq}$
- Maximize efficiency $\rightarrow Re\{Z_{th,eq}\} \ll Re\{R_L\}$
- Maximize Quality Factor $\rightarrow Im\{Z_{L,eq}\} = 0$

Example Matching Circuits



$R_{th} > R_L$, but want maximum possible power to R_L

Using "L-network" set $Z_L = Z_{th,eq}^*$ or $Z_{L,eq} = Z_{th}^*$

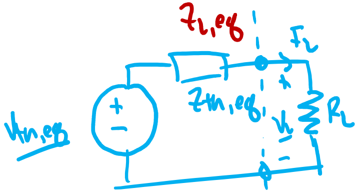
$Z_L = j\omega L = jX_L$, $X_L = \omega L$ | $Z_C = \frac{-j}{\omega C} = -jX_C$, $X_C = \frac{1}{\omega C}$

$$Z_{th,eq} = (R_{th} \| -jX_C) + jX_L$$

$$= \frac{-jX_C R_{th}}{R_{th} - jX_C}\frac{(R_{th}+jX_C)}{(R_{th}+jX_C)} + jX_L$$

$$= \frac{-jX_C R_{th}^2 + X_C^2 R_{th}}{R_{th}^2 + X_C^2} + jX_L$$

$$Z_{th,eq} = \underbrace{\frac{X_C^2 R_{th}}{R_{th}^2 + X_C^2}}_{Re\{Z_{th,eq}\} = R_L} + \underbrace{j\left[X_L - \frac{X_C^2 R_{th}}{R_{th}^2 + X_C^2}\right]}_{Im\{Z_{th,eq}\} = \emptyset}$$

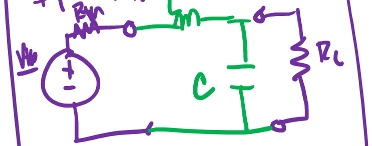


$$V_{th,eq} = \frac{V_{th}}{R_{th}}(R_{th} \| -jX_C)$$

$$X_C = \sqrt{\frac{R_L R_{th}^2}{R_{th} - R_L}}$$

$$X_L = \frac{X_C R_{th}^2}{R_{th}^2 + X_C^2}$$

If $R_{th} < R_L$?

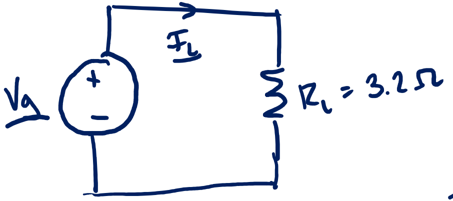


Just flip the two ports

transformer can be used for scaling

$X_C \to \infty$ $Re\{Z_{th,eq}\} \to R_{th}$
$X_C \to \emptyset$ $Re\{Z_{th,eq}\} \to \emptyset$
} this circuit reduces R_{th}

Matching Example



$\omega = 2\pi 60\,\text{Hz}$

$V_g = 170\,\angle 0^\circ\ \text{V}$

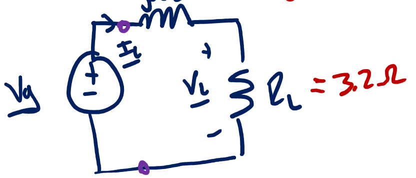
residential wall outlet is $120\,V_{rms} = 170\,V_{pk}$ @ 60 Hz

with $R_L = 3.2\,\Omega \approx$ equivalent model of plugged-in load

$$I_L = \frac{V_g}{R_L} = \frac{170\angle 0^\circ}{3.2} = 53\angle 0^\circ\ \text{A}$$

$$S_g = \tfrac{1}{2} V_g I_L = \tfrac{1}{2}(170\angle 0^\circ)(53\angle 0^\circ) = 4.5\text{kW} + j0\ \text{VAR} = 4.5\angle 0^\circ\ \text{kVA}$$

$j\omega L \rightarrow L = 5\,\text{mH}$ $\quad Z_L = j(2\pi 60)(5e{-3}) = j1.89\,\Omega$



$$I_L = \frac{V_g}{j\omega L + R_L} = \frac{170\angle 0^\circ}{3.2 + j1.89} = 45\angle -30^\circ\ \text{A}$$

$$V_L = R_L I_L = (3.2\,\Omega)(45\angle -30^\circ\ \text{A}) = 146.5\angle -30^\circ\ \text{V}$$

$$P_L = \tfrac{1}{2}\text{Re}\{V_L I_L^*\} = \frac{|V_L||I_L|}{2}\cos(\phi_{V_L} - \phi_{I_L}) = \tfrac{1}{2}(146.5)(45) = 3.3\text{kW}$$

$$S_g = \tfrac{1}{2} V_g I_L^* = \tfrac{1}{2}(170\angle 0^\circ)(45\angle +30^\circ) = 3.8\angle 30^\circ\ \text{kVA} = \underbrace{3.3\text{kW}}_{\text{to } R_L} + j\,\underbrace{1.9\text{kVAR}}_{\text{to } L}$$

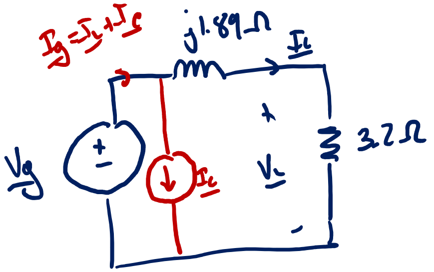
$$PF_g = \frac{P_g}{|S_g|} = \frac{3.3\text{kW}}{3.8\text{kVA}} = 0.87 \text{ lagging}$$

↳ current w.r.t. voltage

$PF = \cos(\phi_V - \phi_I) = \cos(\phi_Z)$

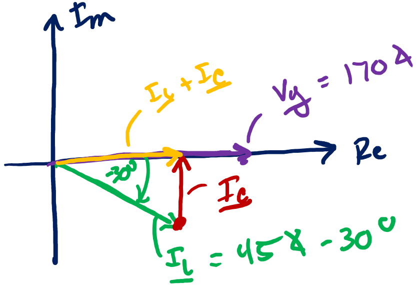
$\phi_Z = \angle(j\omega L + R_L)$

Generally, "lagging" corresponds to inductive loads
"leading" to capacitive



Can we return grid PF to unity PF = 1, S_g is all real

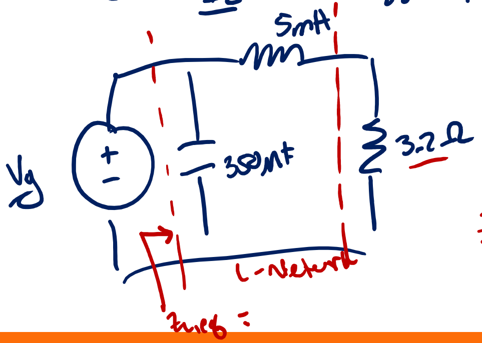
$PF = \cos(\varphi_v - \varphi_I)$



$$I_c = j\,\mathrm{Im}\{I_L^*\} = j45\sin(30^\circ)$$
$$= j\,22.5\,A = 22.5\angle +90^\circ$$

Can this be a passive impedance?

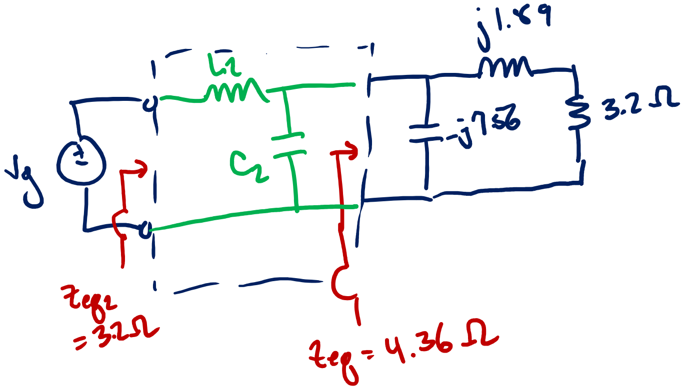
$$Z_c = \frac{V_g}{I_c} = \frac{170\angle 0^\circ}{22.5\angle +90^\circ} = 7.56\angle -90^\circ = -j7.56\,\Omega = \frac{-j}{\omega C} \rightarrow C = 350\,\mu F$$

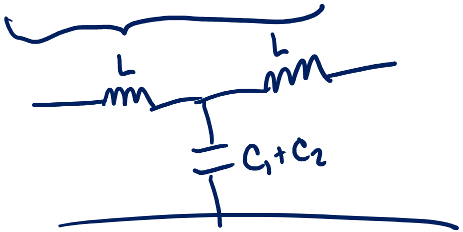


$$\rightarrow \quad S_g = \tfrac{1}{2}(V_g)(I_L + I_c)^* = 3.3kW + j0\,VAR$$

$$I_g = 45\cos(30^\circ) = 39A$$

$$Z_{L,eq} = \frac{V_g}{I_g} = 4.36\,\Omega \neq 3.2\,\Omega$$





from L-Network Analysis:

$$3.2\Omega = \frac{X_{c2}^2(4.3)}{(4.3)^2 + (X_{c2})^2}$$

$$\phi = X_{l2} - \frac{X_{c2}(4.3)}{(4.3)^2 + (X_{c2})^2}$$

Cap: $X_{c2} = 7.4\Omega \rightarrow C_2 = 360\mu F$

Ind: $X_{l2} = 1.58\Omega \rightarrow L_2 = 5mH$

Simulation Example