Initial and Final Value Theorems

Initial Value Theorem

$$\mathcal{L}\left\{\frac{df}{dt}\right\} = \int_{0^-}^{\infty} e^{-st}\frac{df}{dt}\,dt = sF(s) - f(0^-)$$

$$\lim_{s\to\infty}\left[\int_{0^-}^{0^+} \underbrace{e^{-st}}_{=1 \text{ @ } t=0}\frac{df}{dt}\,dt + \cancel{\int_{0^+}^{\infty} e^{-st}\frac{df}{dt}\,dt}\right] = \lim_{s\to\infty}\left[sF(s) - f(0^-)\right]$$

$$= f(0^+) - \cancel{f(0^-)} = \lim_{s\to\infty}\left[sF(s)\right] - \cancel{f(0^-)}$$

$$\boxed{= \lim_{t\to 0^+} f(t) = \lim_{s\to\infty}\left[sF(s)\right]}$$

fastest part of response

highest frequencies

Final Value Theorem :

$$\lim_{s\to 0}\left[\int_{0^-}^{\infty} e^{-st}\frac{df}{dt}\,dt\right] = \lim_{s\to 0}\left[sF(s) - f(0^-)\right]$$

$$f(t\to\infty) - \cancel{f(0^-)} = \lim_{s\to 0}\left[sF(s)\right] - \cancel{f(0^-)}$$

$$\boxed{\lim_{t\to\infty} f(t) = \lim_{s\to 0}\left[sF(s)\right]}$$

All poles in LHP

(Final value is defined)

TABLE 14.2 Laplace Transform Operations

Operation	f(t)	F(s)
Addition	$f_1(t) \pm f_2(t)$	$\mathbf{F}_1(\mathbf{s}) \pm \mathbf{F}_2(\mathbf{s})$
Scalar multiplication	$kf(t)$	$k\mathbf{F}(\mathbf{s})$
Time differentiation	$\dfrac{df}{dt}$	$\mathbf{sF}(\mathbf{s}) - f(0^-)$
	$\dfrac{d^2f}{dt^2}$	$\mathbf{s}^2\mathbf{F}(\mathbf{s}) - \mathbf{s}f(0^-) - f'(0^-)$
	$\dfrac{d^3f}{dt^3}$	$\mathbf{s}^3\mathbf{F}(\mathbf{s}) - \mathbf{s}^2f(0^-) - \mathbf{s}f'(0^-) - f''(0^-)$
Time integration	$\int_{0^-}^{t} f(t)\,dt$	$\dfrac{1}{\mathbf{s}}\mathbf{F}(\mathbf{s})$
	$\int_{-\infty}^{t} f(t)\,dt$	$\dfrac{1}{\mathbf{s}}F(\mathbf{s}) + \dfrac{1}{\mathbf{s}}\int_{-\infty}^{0^-} f(t)\,dt$
Convolution	$f_1(t) * f_2(t)$	$\mathbf{F}_1(\mathbf{s})\mathbf{F}_2(\mathbf{s})$
Time shift	$f(t-a)u(t-a),\ a \geq 0$	$e^{-a\mathbf{s}}\mathbf{F}(\mathbf{s})$
Frequency shift	$f(t)e^{-at}$	$\mathbf{F}(\mathbf{s}+\mathbf{a})$
Frequency differentiation	$tf(t)$	$-\dfrac{d\mathbf{F}(\mathbf{s})}{d\mathbf{s}}$
Frequency integration	$\dfrac{f(t)}{t}$	$\int_{\mathbf{s}}^{\infty} \mathbf{F}(\mathbf{s})\,d\mathbf{s}$
Scaling	$f(at),\ a \geq 0$	$\dfrac{1}{a}\mathbf{F}\left(\dfrac{\mathbf{s}}{a}\right)$
Initial value	$f(0^+)$	$\lim_{\mathbf{s} \to \infty} \mathbf{sF}(\mathbf{s})$
Final value	$f(\infty)$	$\lim_{\mathbf{s} \to 0} \mathbf{sF}(\mathbf{s})$, all poles of $\mathbf{sF}(\mathbf{s})$ in LHP
Time periodicity	$f(t) = f(t+nT)$, $n = 1, 2, \ldots$	$\dfrac{1}{1-e^{-T_s}}\mathbf{F}_1(\mathbf{s})$, where $\mathbf{F}_1(\mathbf{s}) = \int_{0^-}^{T} f(t)e^{-\mathbf{s}t}\,dt$

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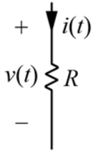
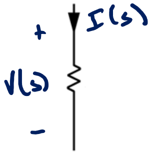
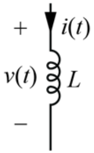
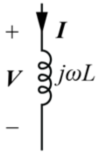
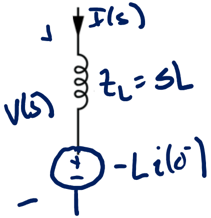
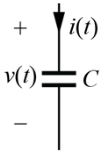
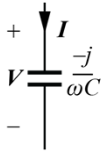
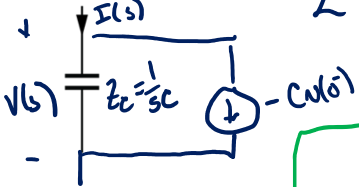
Circuit Laplace Transform

Time Domain	Phasor Domain	s-Domain
 $v(t) = i(t)R$	 $\boldsymbol{V} = \boldsymbol{I}R$	
 $v(t) = L\dfrac{di}{dt}$	 $\boldsymbol{V} = j\omega L\boldsymbol{I}$	
 $i(t) = C\dfrac{dv}{dt}$	 $\boldsymbol{V} = \dfrac{-j}{\omega C}\boldsymbol{I}$	

$\mathcal{L}\{v(t)\} = \mathcal{L}\{i(t)R\}$

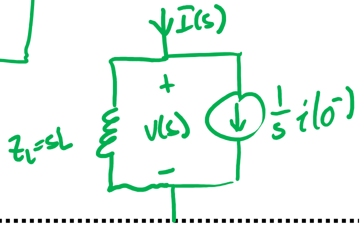
$V(s) = R\,I(s)$

$Z_R = R \rightarrow$ still called "Impedance"

$\mathcal{L}\{v(t)\} = \mathcal{L}\{L\frac{di(t)}{dt}\}$

$V(s) = sL\,I(s) - L\,i(0^-)$

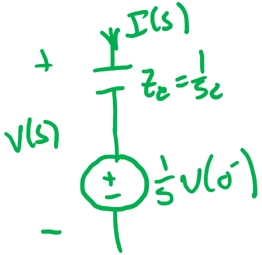
$I(s) = \dfrac{V(s)}{sL} + \dfrac{1}{s}i(0^-)$



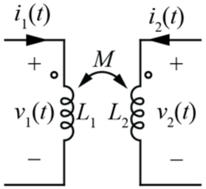
$\mathcal{L}\{i(t)\} = \mathcal{L}\{C\frac{dv(t)}{dt}\}$

$I(s) = sC\,V(s) - C\,v(0^-)$

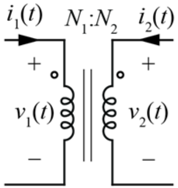
$V(s) = \dfrac{1}{sC}I(s) + \dfrac{1}{s}v(0^-)$



Time Domain

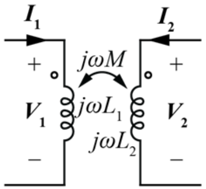


$$v_1(t) = L_1\frac{di_1}{dt} + M\frac{di_2}{dt}$$
$$v_2(t) = M\frac{di_1}{dt} + L_2\frac{di_2}{dt}$$

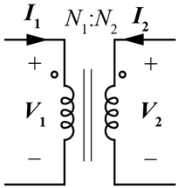


$$\frac{v_1(t)}{N_1} = \frac{v_2(t)}{N_2}$$
$$N_1 i_1(t) + N_2 i_2(t) = 0$$

Phasor Domain

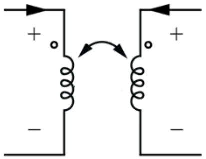


$$\boldsymbol{V}_1 = j\omega L_1 \boldsymbol{I}_1 + j\omega M \boldsymbol{I}_2$$
$$\boldsymbol{V}_2 = j\omega M \boldsymbol{I}_1 + j\omega L_2 \boldsymbol{I}_2$$

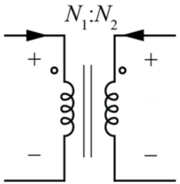


$$\frac{\boldsymbol{V}_1}{N_1} = \frac{\boldsymbol{V}_2}{N_2}$$
$$N_1 \boldsymbol{I}_1 + N_2 \boldsymbol{I}_2 = 0$$

s-Domain



$$V_1(s) = sL_1 I_1(s) - L_1 i_1(0^-) + sM I_2(s) - M i_2(0^-)$$
$$V_2(s) = sM I_1(s) - M i_1(0^-) + sL_2 I_2(s) - L_2 i_2(0^-)$$



$$\frac{V_1(s)}{N_1} = \frac{V_2(s)}{N_2}$$
$$N_1 I_1(s) + N_2 I_2(s) = 0$$

Laplace Transform of Diff EQs

N^{th} order circuit with sinusoidal input described by ($M \leq N$ for causality)

$$b_N \frac{d^N}{dt^N} v_o(t) + \cdots + b_1 \frac{d}{dt} v_o(t) + b_0 v_o(t) \quad = \quad a_M \frac{d^M}{dt^M} v_i(t) + \cdots + a_1 \frac{d}{dt} v_i(t) + a_0 v_i(t)$$

$$\sum_{i=0}^{N} b_i \frac{d^i}{dt^i} v_o(t) \quad = \quad \sum_{i=0}^{M} a_i \frac{d^i}{dt^i} v_i(t)$$

Then the Laplace transform of the circuit, neglecting initial conditions, is

$$\mathcal{L}\left\{\sum_{i=0}^{N} b_i \frac{d^i}{dt^i} v_o(t)\right\} \quad = \quad \mathcal{L}\left\{\sum_{i=0}^{M} a_i \frac{d^i}{dt^i} v_i(t)\right\}$$

$$\sum_{i=0}^{N} b_i s^i V_o(s) \quad = \quad \sum_{i=0}^{M} a_i s^i V_i(s)$$

if we replace $s \to j\omega$ in $H(s)$ we get $H(j\omega)$, the frequency response

Rearranging:

Transfer function

$$\boxed{\frac{V_o(s)}{V_i(s)} \quad = \quad H(s) \quad = \quad \frac{\sum_{i=0}^{M} a_i s^i}{\sum_{i=0}^{N} b_i s^i}}$$

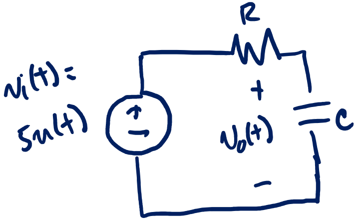
Laplace Circuit Solution Algorithm

1. Transform all sources, signals into Laplace Domain
2. Transform circuit components (including initial conditions) into Laplace Domain
3. Solve the circuit using 201 techniques

$$\boxed{V_o(s) = H(s)V_i(s) = \frac{\sum_{i=0}^{M} a_i s^i}{\sum_{i=0}^{N} b_i s^i}}$$

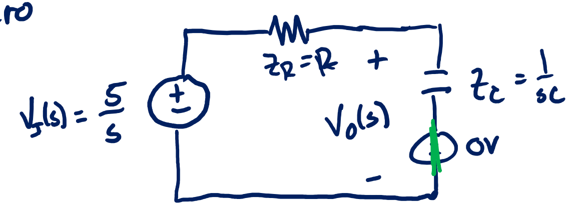
4. Inverse Laplace Transform to get back to time domain

Example Laplace Circuit Analysis



$v_o(t)$ @ $t \le 0$ is zero

$v_o(t) = (5 - 5e^{-\frac{1}{RC}t})u(t)$

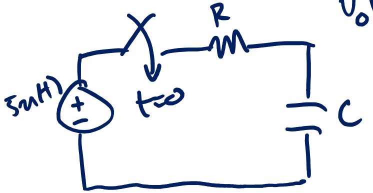


$$V_o(s) = V_I(s)\frac{Z_C}{Z_C + Z_R} = \frac{5}{s}\frac{\frac{1}{sC}}{\frac{1}{sC} + R} = \frac{5}{s}\frac{\frac{1}{RC}}{\frac{1}{RC} + s}$$

$$v_o(t) = \mathcal{L}^{-1}\{V_o(s)\} = \mathcal{L}^{-1}\left\{\frac{A}{s} + \frac{B}{\frac{1}{RC} + s}\right\} = \mathcal{L}^{-1}\left\{\frac{5}{s} + \frac{-5}{\frac{1}{RC} + s}\right\}$$

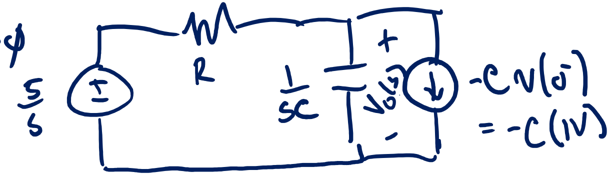
PFE (more review coming)

$$v(t) = \mathcal{L}^{-1}\left\{\frac{5}{s}\right\} + \mathcal{L}^{-1}\left\{\frac{-5}{\frac{1}{RC} + s}\right\} = \boxed{5u(t) - 5e^{-\frac{1}{RC}t}u(t)}$$



$V_o(t) = 1V$ for $t \le 0$

for $t > 0$



$$V_o(s) = V_I(s)\frac{Z_C}{Z_C + Z_R} + C(1V)\left[R \| \frac{1}{sC}\right]$$