Complex Roots: Complex Math

ex/ $F(s) = \dfrac{1}{s^2 - 2s + 2}$

$$P_{1,2} = \dfrac{2 \pm \sqrt{4 - 4(1)(2)}}{2(1)} = 1 \pm j$$

$$F(s) = \dfrac{1}{s^2 - 2s + 2} = \dfrac{1}{(s - (1+j))(s - (1-j))} = \dfrac{k_1}{(s - (1+j))} + \dfrac{k_2}{(s - (1-j))}$$

Complex roots will always occur in conjugate pairs $(s-p)(s-p^*)$ and their residues will always be complex conjugates $k_1 = k_2^*$, for any real signals & systems

$$k_1 = \dfrac{1}{s - (1-j)}\bigg|_{s = 1+j} = \dfrac{1}{2j}$$

$$k_2 = \dfrac{1}{s - (1+j)}\bigg|_{s = 1-j} = \dfrac{-1}{2j} = k_1^* \checkmark$$

$$F(s) = \frac{\frac{1}{2j}}{s - (1+j)} + \frac{-\frac{1}{2j}}{s - (1-j)}$$

$$f(t) = \mathcal{L}^{-1}\{F(s)\} = \frac{1}{2j} e^{(1+j)t} u(t) + \left(\frac{-1}{2j}\right) e^{(1-j)t} u(t)$$

$$f(t) = \frac{1}{2j} e^{t} u(t) \left[e^{jt} - e^{-jt} \right]$$

$$f(t) = e^{t} \sin(t) u(t)$$

Euler's Formula

$$\cos\theta = \frac{1}{2}\left(e^{j\theta} + e^{-j\theta}\right)$$

$$\sin\theta = \frac{1}{2j}\left(e^{j\theta} - e^{-j\theta}\right)$$

Complex Roots: General Case

$$\mathcal{L}^{-1}\left\{ \frac{k}{s-(\sigma+j\upsilon)} + \frac{k^*}{s-(\sigma-j\omega)} \right\}$$

$$= \left[k e^{(\sigma+j\upsilon)t} + k^* e^{(\sigma-j\upsilon)t} \right] u(t)$$

$$k = Re\{k\} + j\, Im\{k\}$$
$$k^* = Re\{k\} - j\, Im\{k\}$$

$$= e^{\sigma t}\left[2Re\{k\}\left(\frac{e^{j\omega t} + e^{-j\omega t}}{2} \right) + (2j)\, j\, Im\{k\}\left(\frac{e^{j\omega t} - e^{-j\omega t}}{2j} \right) \right] u(t)$$

$$= e^{\sigma t}\left[2Re\{k\}\cos(\omega t) - 2 Im\{k\}\sin(\omega t) \right] u(t)$$

$$= e^{\sigma t}\left[\sqrt{(2Re\{k\})^2 + (2Im\{k\})^2}\, \cos\left(\omega t + \tan^{-1}\left(\frac{-2 Im\{k\}}{2Re\{k\}} \right) \right) \right] u(t)$$

$$= e^{\sigma t}\, 2|k|\cos(\omega t - \angle k)\, u(t)$$

TABLE 14.2 Laplace Transform Operations

Operation	$f(t)$	$F(s)$
Addition	$f_1(t) \pm f_2(t)$	$\mathbf{F}_1(s) \pm \mathbf{F}_2(s)$
Scalar multiplication	$kf(t)$	$k\mathbf{F}(s)$
Time differentiation	$\dfrac{df}{dt}$	$s\mathbf{F}(s) - f(0^-)$
	$\dfrac{d^2 f}{dt^2}$	$s^2\mathbf{F}(s) - sf(0^-) - f'(0^-)$
	$\dfrac{d^3 f}{dt^3}$	$s^3\mathbf{F}(s) - s^2 f(0^-) - sf'(0^-) - f''(0^-)$
Time integration	$\displaystyle\int_{0^-}^{t} f(t)\,dt$	$\dfrac{1}{s}\mathbf{F}(s)$
	$\displaystyle\int_{-\infty}^{t} f(t)\,dt$	$\dfrac{1}{s}F(s) + \dfrac{1}{s}\displaystyle\int_{-\infty}^{0^-} f(t)\,dt$
Convolution	$f_1(t) * f_2(t)$	$\mathbf{F}_1(s)\mathbf{F}_2(s)$
Time shift	$f(t-a)u(t-a),\ a \ge 0$	$e^{-as}\mathbf{F}(s)$
Frequency shift	$f(t)e^{-at}$	$\mathbf{F}(s+a)$
Frequency differentiation	$t f(t)$	$-\dfrac{d\mathbf{F}(s)}{ds}$
Frequency integration	$\dfrac{f(t)}{t}$	$\displaystyle\int_{s}^{\infty} \mathbf{F}(s)\,ds$
Scaling	$f(at),\ a \ge 0$	$\dfrac{1}{a}\mathbf{F}\left(\dfrac{s}{a}\right)$
Initial value	$f(0^+)$	$\displaystyle\lim_{s \to \infty} s\mathbf{F}(s)$
Final value	$f(\infty)$	$\displaystyle\lim_{s \to 0} s\mathbf{F}(s)$, all poles of $s\mathbf{F}(s)$ in LHP
Time periodicity	$f(t) = f(t+nT),$ $n = 1, 2, \ldots$	$\dfrac{1}{1 - e^{-T_s}}\mathbf{F}_1(s),$ where $\mathbf{F}_1(s) = \displaystyle\int_{0^-}^{T} f(t) e^{-st}\,dt$

TABLE 14.1 Laplace Transform Pairs

$f(t) = \mathcal{L}^{-1}\{\mathbf{F}(s)\}$	$\mathbf{F}(s) = \mathcal{L}\{f(t)\}$
$\delta(t)$	1
$u(t)$	$\dfrac{1}{s}$
$tu(t)$	$\dfrac{1}{s^2}$
$\dfrac{t^{n-1}}{(n-1)!}u(t),\ n = 1, 2, \ldots$	$\dfrac{1}{s^n}$
$e^{-at}u(t)$	$\dfrac{1}{s+a}$
$te^{-at}u(t)$	$\dfrac{1}{(s+a)^2}$
$\dfrac{t^{n-1}}{(n-1)!}e^{-at}u(t),\ n = 1, 2, \ldots$	$\dfrac{1}{(s+a)^n}$
$\dfrac{1}{\beta - \alpha}(e^{-\alpha t} - e^{-\beta t})u(t)$	$\dfrac{1}{(s+\alpha)(s+\beta)}$
$\sin \omega t\, u(t)$	$\dfrac{\omega}{s^2 + \omega^2}$
$\cos \omega t\, u(t)$	$\dfrac{s}{s^2 + \omega^2}$
$\sin(\omega t + \theta)\, u(t)$	$\dfrac{s\sin\theta + \omega\cos\theta}{s^2 + \omega^2}$
$\cos(\omega t + \theta)\, u(t)$	$\dfrac{s\cos\theta - \omega\sin\theta}{s^2 + \omega^2}$
$e^{-at}\sin \omega t\, u(t)$	$\dfrac{\omega}{(s+\alpha)^2 + \omega^2}$
$e^{-at}\cos \omega t\, u(t)$	$\dfrac{s+\alpha}{(s+\alpha)^2 + \omega^2}$

$2|k|e^{\alpha t}\cos(\omega t - \measuredangle h)u(t)$

$\dfrac{k}{s - (\sigma + j\omega)} + \dfrac{t^{\measuredangle}}{s - (\sigma - j\omega)}$

Complex Roots: Table Lookup

when possible, manipulate into terms already in the table

ex/

complete
the
square

$$F(s) = \frac{s+2}{s^2-2s+2} = \frac{s+2}{(s-1)^2+1} = \frac{(s-1)+3}{(s-1)^2+1} = \frac{s-1}{(s-1)^2+1} + \frac{3(1)}{(s-1)^2+1}$$

$$\mathcal{L}^{-1}\{F(s)\} = e^t \cos(t) u(t) + 3 e^t \sin(t) u(t)$$

Example

$$v_i(t) = \sin(2t)\, u(t)$$
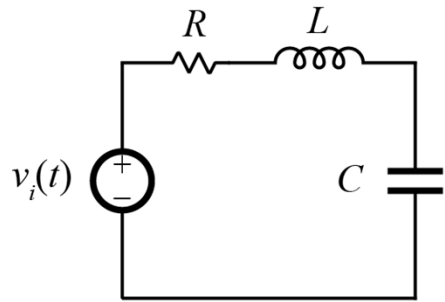
$$L = 500\text{mH}, \quad C = 500\text{mF}, \quad R = 2\Omega$$

$$v_c(0) = 5\text{V}, \quad i_L(0) = -2\text{A}$$
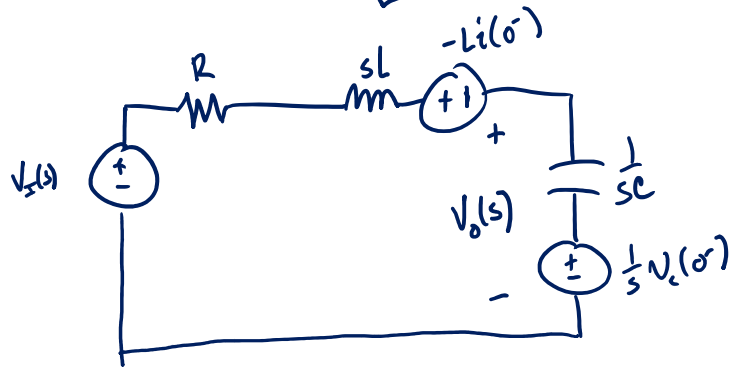
$$V_I(s) = \mathcal{L}\{v_i(t)\} = \frac{2}{s^2+4}$$

Solve circuit in s-domain:

$$V_o(s) = H_I(s)\, V_I(s) + H_L(s)\left(-Li(0^-)\right) + H_C(s)\left(\tfrac{1}{s}V_c(0^-)\right)$$

$$= H_I(s)\, V_I(s) + \left(-Li(0^-)\right)\left(-H_I(s)\right) + \left(\tfrac{1}{s}V_c(0^-)\right)\left(1 - H_I(s)\right)$$

$$H_I(s) = \frac{\frac{1}{sc}}{\frac{1}{sc} + sL + R} = \frac{1}{s^2 LC + sCR + 1}$$

$$H_I(s) = \frac{1}{s^2 LC + sCR + 1} = \frac{1}{\frac{s^2}{4} + s + 1} = \frac{4}{s^2 + 4s + 4} = \frac{4}{(s+2)^2}$$

$$V_I(s) = \frac{2}{s^2 + 4} = \frac{2}{(s+2j)(s-2j)}$$

$$V_o(s) = \underbrace{\frac{4}{(s+2)^2} \frac{2}{s^2+4}}_{①} + \underbrace{(-1)\frac{4}{(s+2)^2} + \frac{5}{5} - \frac{5}{5}\frac{4}{(s+2)^2}}_{\text{Look up in table}} \quad ②$$

① ②

Ⓟ PFE: $\dfrac{4}{(s+2)^2} \dfrac{2}{\underset{(s^2+4)}{(s+2j)(s-2j)}} = \dfrac{k_1}{s+2} + \dfrac{k_2}{(s+2)^2} + \dfrac{k_3}{s+2j} + \dfrac{k_3^*}{s-2j}$

$$k_2 = \frac{8}{s^2+4}\Big|_{s=-2} = 1$$

$$k_3 = \frac{8}{(s+2)^2(s-2j)}\Big|_{s=-2j} = \frac{8}{(-4-8j+4)(-4j)} = \frac{-1}{4} = k_3^*$$

$$k_1 = \frac{d}{ds}\left[\frac{8}{s^2+4}\right]\Big|_{s=-2} = \left[8(-1)(s^2+4)^{-2}(2s)\right]\Big|_{s=-2} = \frac{-8}{8^2}(-4) = \frac{1}{2}$$

② $\quad \dfrac{5}{5} \dfrac{4}{(s+2)^2} = \dfrac{k_1}{s} , \dfrac{k_2}{s+2} + \dfrac{k_3}{(s+2)^2}$

$k_1 = \dfrac{20}{(s+2)^2}\Big|_{s=0} = 5 \qquad\qquad k_2 = \dfrac{d}{ds}\left[\dfrac{20}{s}\right]\Big|_{s=-2} = \dfrac{-20}{s^2}\Big|_{s=-2} = -5$

$k_3 = \dfrac{20}{s}\Big|_{s=-2} = -10$

$V_0(s) = \dfrac{1/2}{s+2} + \dfrac{1}{(s+2)^2} + \dfrac{-1/4}{s+2j} + \dfrac{-1/4}{s-2j} + \dfrac{-4}{(s+2)^2} + \dfrac{5}{5} \cancel{-\dfrac{5}{s}} - \dfrac{-5}{s+2} - \dfrac{-10}{(s+2)^2}$

$V_0(s) = \dfrac{5+1/2}{s+2} + \dfrac{7}{(s+2)^2} + \dfrac{-1/4}{s+2j} + \dfrac{-1/4}{s-2j}$

$v_0(t) = \mathcal{L}^{-1}\{V_0(s)\} = \left[5.5e^{-2t} + 7te^{-2t} + 2\tfrac{1}{4}\cos(2t - 0^\circ) \right] u(t)$