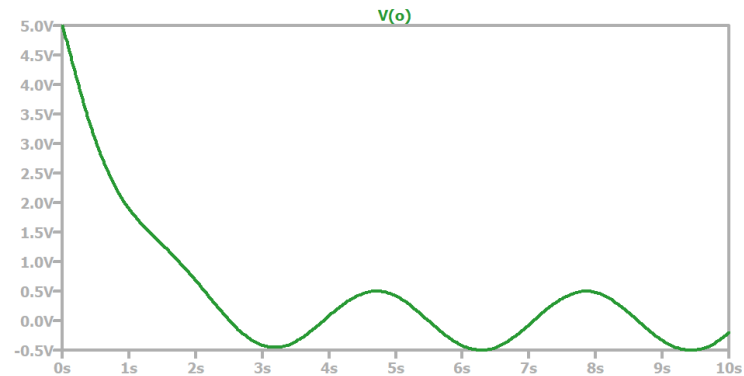
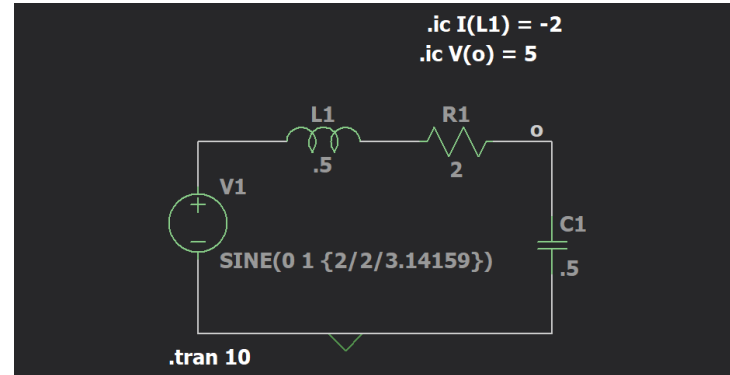
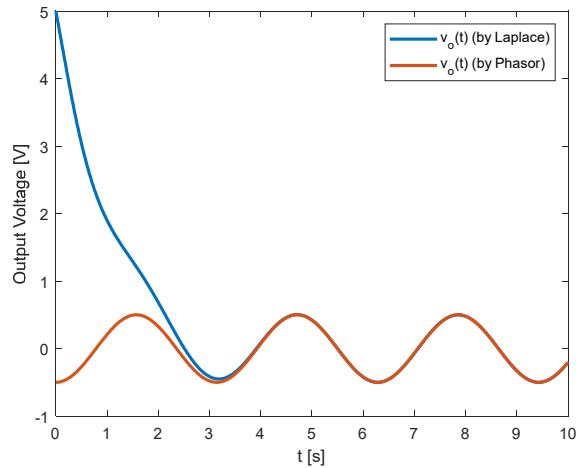
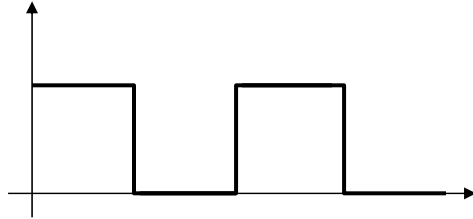


Comparison to Simulation



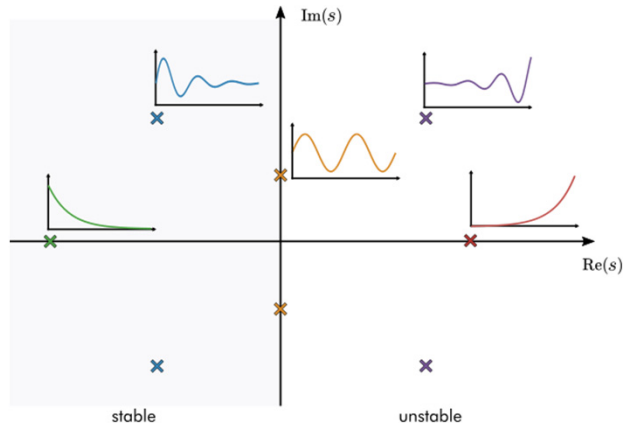
Laplace Transform of Periodic PWL Signals



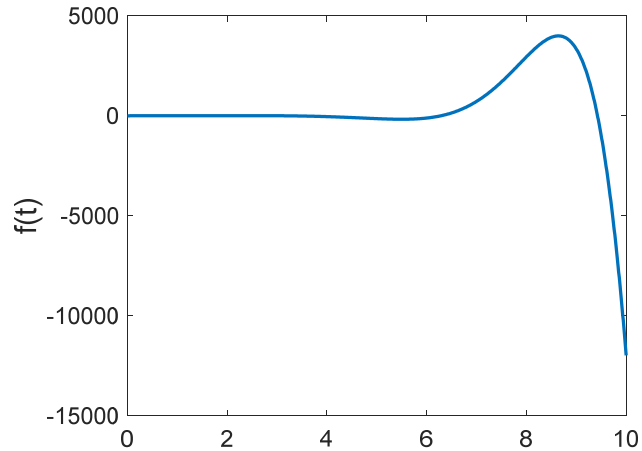
Pole Locations

$$x = N_H + N_I$$

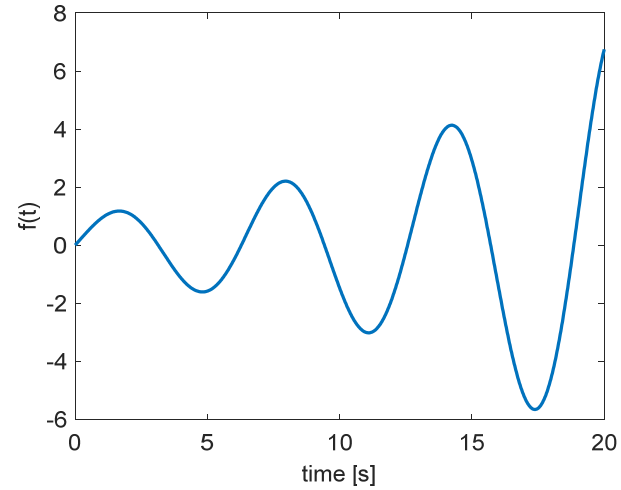
$$V_o(s) = V_I(s)H(s) = \left(\frac{\sum_{i=0}^{M_I} c_i s^i}{\sum_{i=0}^{N_I} d_i s^i} \right) \left(\frac{\sum_{i=0}^{M_H} a_i s^i}{\sum_{i=0}^{N_H} b_i s^i} \right) = \frac{(s - z_1)(s - z_2) \cdots (s - z_{M_H+M_I})}{(s - p_1)(s - p_2) \cdots (s - p_{N_H+N_I})}$$
$$= \frac{k_1}{(s - p_1)} + \frac{k_2}{(s - p_2)} + \cdots + \frac{k_x}{(s - p_x)}$$



Unbounded Signals & Unstable Systems



time [s]
 $f(t) = e^t \sin(t)$



time [s]
 $f(t) = e^{t/10} \sin(t)$

Laplace and Fourier Revisited

Fourier Transform:

$$F(j\omega) = \int_{-\infty}^{+\infty} e^{-j\omega t} f(t) dt$$

Laplace Transform (Bilateral):

$$F(s) = \int_{-\infty}^{+\infty} e^{-st} f(t) dt$$

Inverse Fourier Transform:

$$f(t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} F(j\omega) e^{j\omega t} d\omega$$

Inverse Laplace Transform (Bilateral):

$$f(t) = \frac{1}{2\pi j} \int_{\sigma_0 - j\infty}^{\sigma_0 + j\infty} F(s) e^{st} ds$$

Laplace Explanation

$$F(s) = \int_{0^-}^{+\infty} e^{-st} f(t) dt = \int_{0^-}^{+\infty} e^{-\sigma t} e^{-j\omega t} f(t) dt$$

$$f(t) = \frac{1}{2\pi j} \int_{\sigma_0 - j\infty}^{\sigma_0 + j\infty} e^{\sigma t} e^{j\omega t} F(s) ds$$

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Fourier Series

Assume we have some function $f(t)$ which is periodic with period $T_0 = \frac{2\pi}{\omega_0}$

$$f(t) = a_0 + \sum_{n=1}^{\infty} a_n \cos(k\omega_0 t) + b_n \sin(k\omega_0 t)$$

- $f(t)$ can be expressed this way if
- $f(t)$ is single-valued
- $\int_{-\infty}^{+\infty} |f(t)| dt < \infty$
- $f(t)$ had finite discontinuities and maxima/min per period

Need to find a_0, a_n, b_n for some function $f(t)$
for a_0 : $a_0 = \frac{1}{T} \int_0^T f(t) dt$ a_0 is average / DC value of $f(t)$

For a_n : $a_n \rightarrow$ but not $\frac{1}{T} \int_0^T f(t) \cos(n\omega_0 t) dt$

plugging in Fourier Series for $f(t)$:

$$\frac{1}{T} \int_0^T f(t) \cos(n\omega_0 t) dt = \frac{1}{T} \int_0^T \left[a_0 + \sum_{k=1}^{\infty} a_k \cos(k\omega_0 t) + b_k \sin(k\omega_0 t) \right] \cos(n\omega_0 t) dt$$

$$= \frac{1}{T} \int_0^T a_0 \cos(n\omega_0 t) dt + \frac{1}{T} \int_0^T \left[\sum_{k=1}^{\infty} a_k \cos(k\omega_0 t) \cos(n\omega_0 t) + b_k \sin(k\omega_0 t) \cos(n\omega_0 t) \right] dt$$

<https://www.khanacad.com/a/fourier-series.html>

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Non-periodic Waveforms: Fourier Transform

Fourier Series \rightarrow work only for periodic waveforms

Fourier Transform \rightarrow for non-periodic signals

Idea: treat any non-periodic signal as if it was periodic with $T \rightarrow \infty$

Fourier Series: $C_n = \frac{1}{T} \int_0^T f(t) e^{-jn\omega_0 t} dt$
Fourier Transform: $T C_n = F(\omega) = \int_{-\infty}^{+\infty} f(t) e^{-j\omega t} dt$

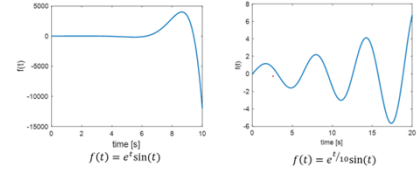
Fourier Series: Summation $f(t) = \sum_{n=-\infty}^{+\infty} C_n e^{jn\omega_0 t}$
Smart Fourier Transform: $f(t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} F(\omega) e^{j\omega t} d\omega$

- $f(t)$ can be expressed this way if
- $f(t)$ is single-valued
- $\int_{-\infty}^{+\infty} |f(t)| dt < \infty$
- $f(t)$ had finite discontinuities and maxima/min in any closed interval

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Unbounded Signals & Unstable Systems



TENNESSEE T

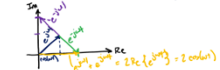
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Complex Form of Fourier Series

Euler: $e^{j\omega t} = \cos(\omega t) + j \sin(\omega t)$

$$\cos(\omega t) = \frac{1}{2}(e^{j\omega t} + e^{-j\omega t})$$

$$\sin(\omega t) = \frac{1}{2j}(e^{j\omega t} - e^{-j\omega t})$$



Plug into Fourier Series:

$$f(t) = a_0 + \sum_{n=1}^{\infty} a_n \cos(n\omega_0 t) + b_n \sin(n\omega_0 t)$$

$$= a_0 + \sum_{n=1}^{\infty} \frac{a_n}{2} (e^{jn\omega_0 t} + e^{-jn\omega_0 t}) + \frac{b_n}{2j} (e^{jn\omega_0 t} - e^{-jn\omega_0 t})$$

$$= a_0 + \sum_{n=1}^{\infty} \left(\frac{a_n}{2} - \frac{b_n}{2j} \right) e^{jn\omega_0 t} + \left(\frac{a_n}{2} + \frac{b_n}{2j} \right) e^{-jn\omega_0 t}$$

$$f(t) = \sum_{n=-\infty}^{+\infty} C_n e^{jn\omega_0 t} \rightarrow C_n = \frac{1}{T} \int_0^T f(t) e^{-jn\omega_0 t} dt \quad C_n^* = C_{-n}$$

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Example Signal Laplace Transforms

$$f(t) = u(t) \quad \mathcal{L}\{f(t)\} = F(s) = \int_0^{\infty} e^{-st} u(t) dt = \int_0^{\infty} e^{-st} dt$$

$$= \left[-\frac{1}{s} e^{-st} \right]_0^{\infty} = \left[0 - \left(-\frac{1}{s}\right) \right]$$

$$F(s) = \mathcal{L}\{u(t)\} = \frac{1}{s} \quad \text{if } \operatorname{Re}\{s\} > 0$$

Region of convergence for $\mathcal{L}\{u(t)\} \rightarrow \operatorname{Re}\{s\} > 0$
 $s = \sigma + j\omega \rightarrow \sigma > 0$

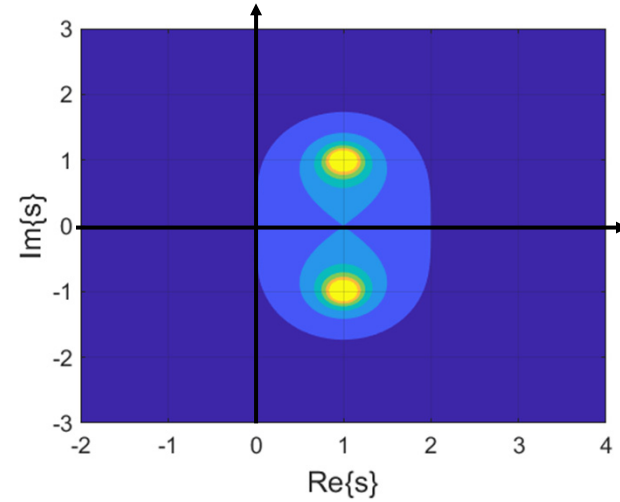
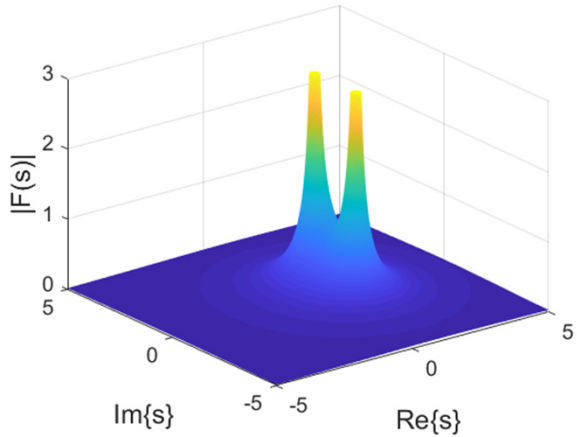
$$f(t) = e^{-at} u(t) \quad \mathcal{L}\{f(t)\} = \int_0^{\infty} e^{-st} e^{-at} u(t) dt = \int_0^{\infty} e^{-(s+a)t} dt$$

$$\mathcal{L}\{e^{-at} u(t)\} = \frac{1}{s+a} \quad \text{if } \operatorname{Re}\{s+a\} > 0$$

Generalize: $\mathcal{L}\{e^{-at} f(t)\} = F(s+a)$
(where $F(s) = \mathcal{L}\{f(t)\}$)

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The s-plane



$$F(s) = \frac{1}{s^2 - 2s + 2} = \frac{1}{(s - (1 + j))(s - (1 - j))}$$