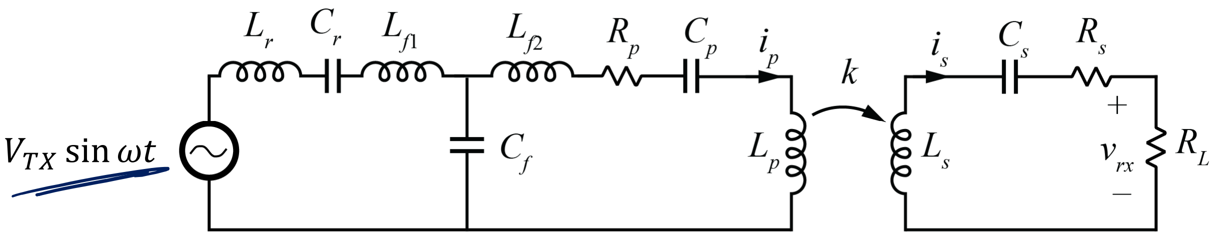
CHAPTER 10: SINUSOIDAL STEADY-STATE

Motivation



want to be able to analyze more complex circuits w/out complex & error-prone diff EQs analysis & algebra

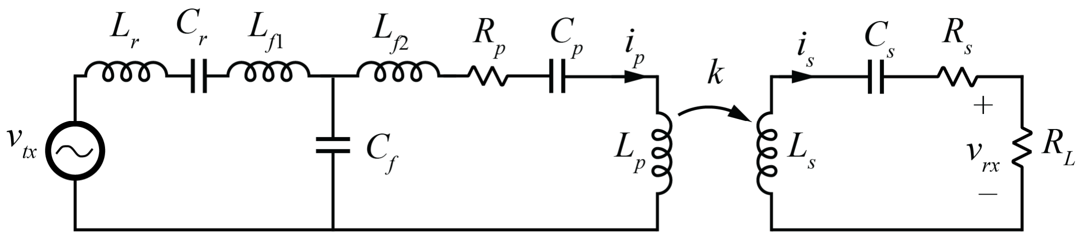
Sinusoidal → Good model for e.g. applications in RF, Power systems, oscillators, certain power conversion

later: extend to non-sinusoidal sources

steady-state → Ignore, initially any transient dynamics

later: extend to transient dynamics

Motivation



Numerical | **Analytical**

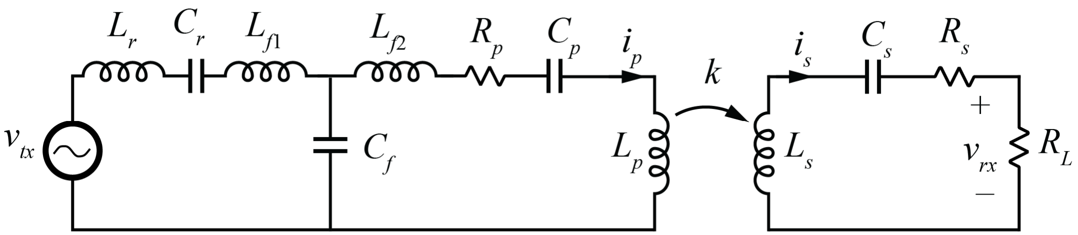
Time Domain | **Frequency Domain**

system of Diff EQs

$$
\begin{cases}
\dfrac{dA}{dt} = \beta_A \cdot \dfrac{D^n}{K_{DA}^n + D^n} \cdot \left(1 - \dfrac{C^n}{K_{CA}^n + C^n}\right) - \alpha_A \cdot A \\
\dfrac{dX}{dt} = \beta_X \cdot \dfrac{A^n}{K_{AX}^n + A^n} - \alpha_X \cdot X \\
\dfrac{dC}{dt} = \beta_C \cdot \dfrac{(A+X)^n}{K_{AXC}^n + (A+X)^n} - \alpha_C \cdot C \\
\dfrac{dB}{dt} = \beta_B \cdot \dfrac{C^n}{K_{CB}^n + C^n} \cdot \left(1 - \dfrac{D^n}{K_{DB}^n + D^n}\right) - \alpha_B \cdot B \\
\dfrac{dY}{dt} = \beta_Y \cdot \dfrac{B^n}{K_{BY}^n + B^n} - \alpha_Y \cdot Y \\
\dfrac{dD}{dt} = \beta_D \cdot \dfrac{(B+Y)^n}{K_{BYD}^n + (B+Y)^n} - \alpha_D \cdot D
\end{cases}
$$

$\downarrow$

Motivation



Numerical | **Analytical**

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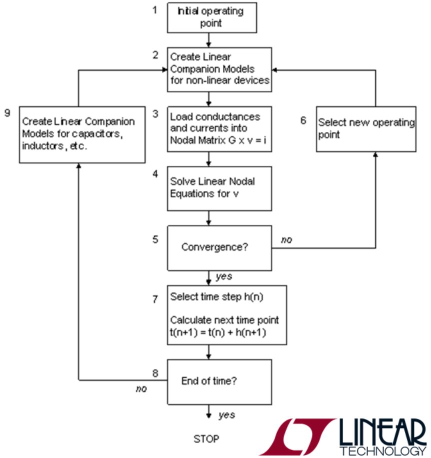

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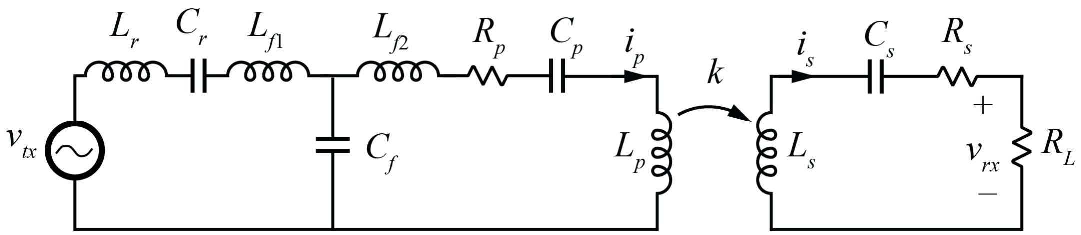

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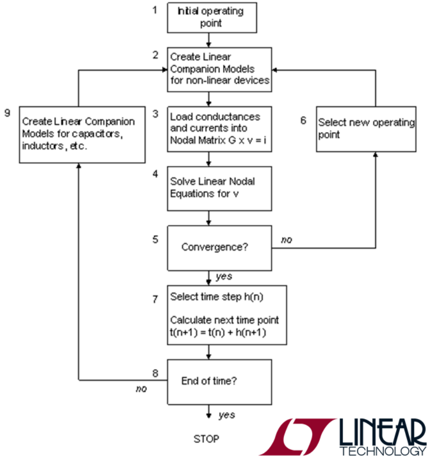

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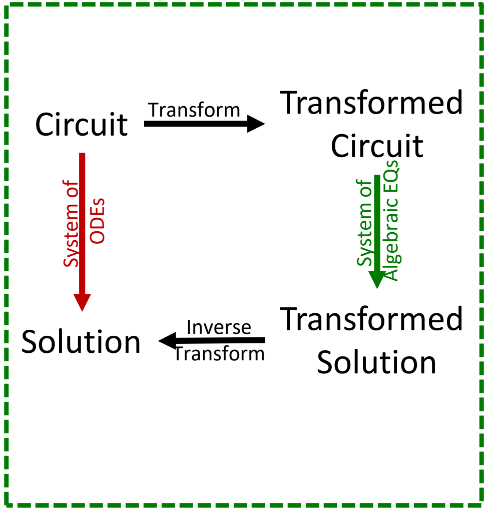


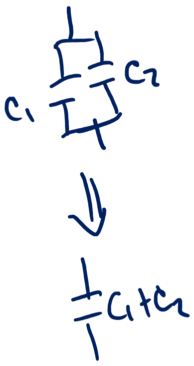
Form of the Solution

N derivatives of $v_o(t)$

M derivatives of $v_i(t)$

$M < N \approx$ # of capacitors & inductors

N^{th} order circuit with sinusoidal input described by



$$b_N \frac{d^N}{dt^N} v_o(t) + \cdots + b_1 \frac{d}{dt} v_o(t) + b_0 v_o(t) \quad = \quad a_M \frac{d^M}{dt^M} v_i(t) + \cdots + a_1 \frac{d}{dt} v_i(t) + a_0 v_i(t)$$

$$\sum_{i=0}^{N} b_i \frac{d^i}{dt^i} v_o(t) \quad = \quad \sum_{i=0}^{M} a_i \frac{d^i}{dt^i} v_i(t)$$

solution for $v_o(t)$ will be of the form

$$v_o(t) = v_{o,h}(t) + v_{o,p}(t)$$

Transient Response

$v_{o,h}(t)$ is the *homogeneous solution* to the differential equation, the *natural response* of the system, or the *transient response* of the system. For any non-ideal (damped) circuit, $v_{o,h}(t)$ will tend to zero over time

$v_{o,h}(t)$ is the solution to the equation

$$\sum_{i=0}^{N} b_i \frac{d^i}{dt^i} v_{o,h}(t) = 0$$

which will be of the form

$$v_{o,h}(t) = \sum_{i=0}^{N} A_i e^{s_i t}$$

$s_i \longrightarrow$ roots of characteristic polynomial

$A_i \longrightarrow$ determined by initial conditions

Note: some of the time constants (s_i) of a circuit are independent of the input

Steady-State Response

$v_{o,p}(t)$ is the *particular solution* to the differential equation, the *forced response* of the system, or the *steady-state response* of the system. In general, it does not tend to zero, if non-zero inputs are present.

$v_{o,p}(t)$ is the non-zeroing solution to the equation

$$\sum_{i=0}^{N} b_i \frac{d^i}{dt^i} v_o(t) \quad = \quad \sum_{i=0}^{M} a_i \frac{d^i}{dt^i} v_i(t)$$

LTI Systems

LTI

For a function $f(\cdot)$

Linearity

if: $v_{o,1}(t) = f\left(v_{i,1}(t)\right)$ $\quad$ $v_{o,2}(t) = f\left(v_{i,2}(t)\right)$

superposition applies

then: $av_{o,1}(t) + bv_{o,2}(t) = f\left(av_{i,1}(t) + bv_{i,2}(t)\right)$

Time Invariance

if: $v_o(t) = f\left(v_i(t)\right)$

then: $v_o(t-T) = f\left(v_i(t-T)\right)$

LTI: $v_x(t) + v_y(t)$, $\alpha v_x(t)$, $\frac{dv_x(t)}{dt}$, $\int_0^t v_x(t)dt$ (neglecting ICs)

Not LTI: $v_x(t) \cdot v_y(t)$, $v_x(t)^2$, $|v_x(t)|$ $\rightarrow$ power isn't a linear calculation

Causality

- Causal systems cannot predict the future
- $v_o(t_0)$ does not depend on values of $v_i(t)$, t $> t_0$

$$v_{o,1}(t) = f\left(v_{i,1}(t)\right) \qquad v_{o,2}(t) = f\left(v_{i,2}(t)\right)$$

if: $\quad v_{i,1}(t) = v_{i,2}(t) \quad \forall \quad t < t_0$

then: $\quad v_{o,1}(t) = v_{o,2}(t) \quad \forall \quad t < t_0$

Preview of Frequency Domain

for LTI sytems

- Develop "phasor analysis" → for sinusoidal inputs @ some frequency ω & steady state only
- we know superposition applies

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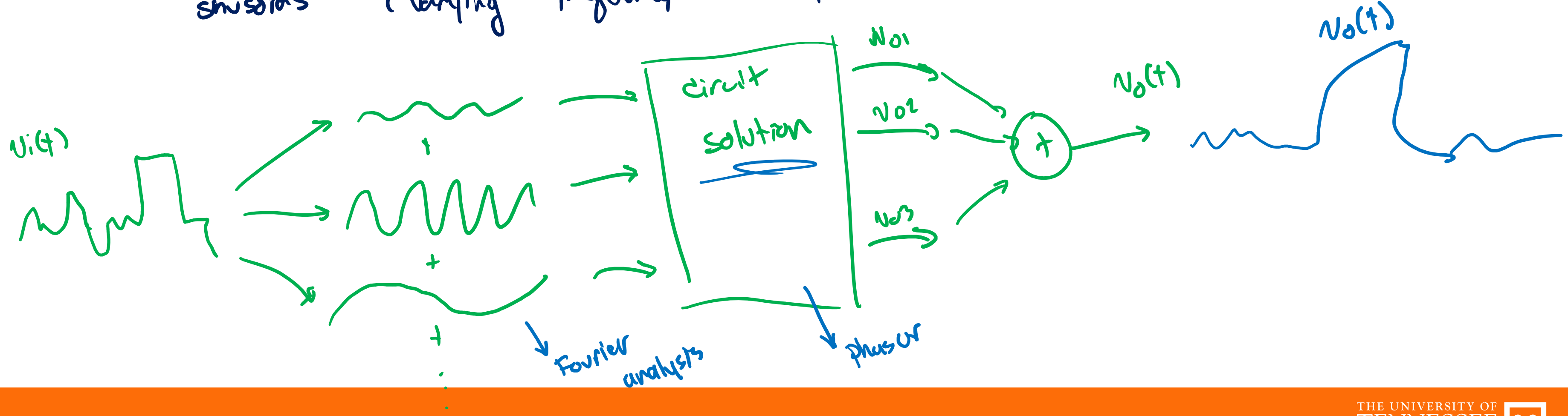
Remainder of course

(1) Make it very easy to solve circuits with sinusoidal inputs

(2) Find a way to express any arbitrary input as a sum of sinusoids (varying frequency & amplitude)



steady-state
$$\sum_{i=0}^{N} b_i \frac{d^i}{dt^i} v_o(t) = \sum_{i=0}^{M} a_i \frac{d^i}{dt^i} v_i(t)$$

$$v_i(t) = \underline{V_I \cos(\omega t)}$$ $\rightarrow$ if inputs are sinusoids @ ω
outputs are sinusoids @ ω

let's guess $v_o(t) = A\cos(\omega t + \varphi)$

$$\frac{d}{dt} v_o(t) = v_o'(t) = -A\omega \sin(\omega t + \varphi)$$

$$v_o''(t) = -A\omega^2 \cos(\omega t + \varphi)$$

$$v_o'''(t) = A\omega^3 \sin(\omega t + \varphi)$$

$$v_o''''(t) = A\omega^4 \cos(\omega t + \varphi)$$

$\vdots$