steady-state

$$\sum_{i=0}^{N} b_i \frac{d^i}{dt^i} v_o(t) = \sum_{i=0}^{M} a_i \frac{d^i}{dt^i} v_i(t)$$

$$v_i(t) = \underline{V_I \cos(\omega t)} \rightarrow$$

if inputs are sinusoids @ ω
outputs are sinusoids @ ω

let's guess $v_o(t) = A\cos(\omega t + \varphi)$

$$\frac{d}{dt} v_o(t) = v_o'(t) = -A\omega \sin(\omega t + \varphi) \rightarrow v_o'(t) = A\omega \cos(\omega t + \varphi + 90^\circ)$$

$$v_o''(t) = -A\omega^2 \cos(\omega t + \varphi) \rightarrow v_o''(t) = A\omega^2 \cos(\omega t + \varphi + 180^\circ)$$

$$v_o'''(t) = A\omega^3 \sin(\omega t + \varphi) \rightarrow v_o'''(t) = A\omega^3 \cos(\omega t + \varphi + 270^\circ)$$

$$v_o''''(t) = A\omega^4 \cos(\omega t + \varphi) \rightarrow v_o''''(t) = A\omega^4 \cos(\omega t + \varphi + 360^\circ)$$

$\vdots$

every derivative
1) multiplies by ω
2) adds 90° of phase

Trig Identities (Review)

$$\sin(\theta) = \cos(\theta - 90°)$$
$$-\cos(\theta) = \cos(\theta \pm 180°)$$

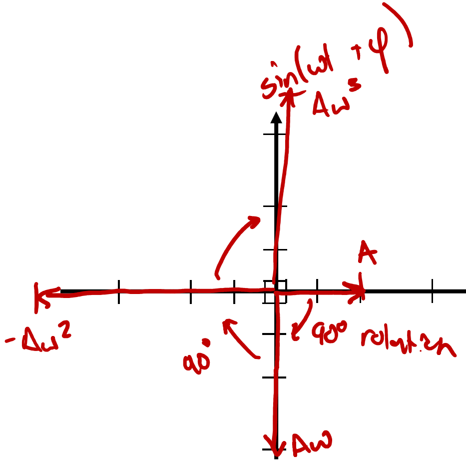
$$A\cos(\omega t) + B\sin(\omega t) = \sqrt{A^2 + B^2}\cos\left(\omega t - \tan^{-1}\left(\frac{B}{A}\right)\right)$$

$$Ae^{j\theta} = A\cos(\theta) + jA\sin(\theta)$$

Last page of book is a table of trig identities

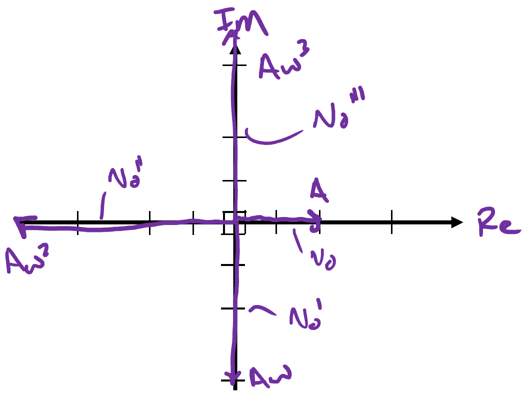
Sinusoidal Steady State

$$v(t) = A\sin(\omega t + \varphi) = A\cos(\omega t + \varphi - 90°)$$
$$v'(t) = A\omega\cos(\omega t + \varphi) = A\omega\cos(\omega t + \varphi)$$
$$v''(t) = -A\omega^2\sin(\omega t + \varphi) = A\omega^2\cos(\omega t + \varphi + 90°)$$
$$v'''(t) = -A\omega^3\cos(\omega t + \varphi) = A\omega^3\cos(\omega t + \varphi + 180°)$$
$$v'''(t) = A\omega^4\sin(\omega t + \varphi) = A\omega^4\cos(\omega t + \varphi + 270°)$$



$Ae^{j\theta} = A\cos\theta + Aj\sin\theta$

$A\,\mathrm{cis}\,\theta$



Complex Numbers (Review)

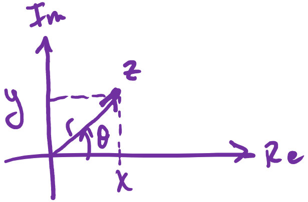
$j = \sqrt{-1}$, $j^2 = -1$

Rectangular — Polar

$z = x + jy = re^{j\theta}$

complex # → z; Real → x; imaginary → y; amplitude/magnitude → r; phase → θ

$$\begin{cases} |z| = r = \sqrt{x^2 + y^2} \\ \measuredangle z = \theta = \tan^{-1}\left(\frac{y}{x}\right) \end{cases}$$



$$\begin{cases} Re\{z\} = x = r\cos(\theta) \\ Im\{z\} = y = r\sin(\theta) \end{cases}$$

Euler's Formula

$re^{j\theta} = r\cos\theta + rj\sin\theta$

Complex Conjugate

for $z = x + jy$,

$z^* = x - jy$

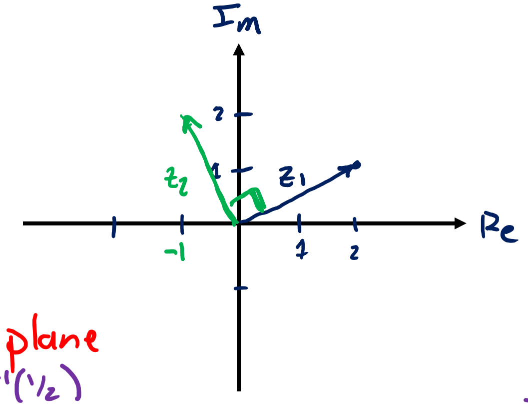
e.g. $zz^* = x^2 + y^2 + \cancel{jxy} - \cancel{jxy}$

$= r^2 = |z|^2$

Complex Number Arithmetic

ex/ $z_1 = 2 + j1 = \underline{\sqrt{5}\, e^{j\tan^{-1}(\frac{1}{2})}}$

recall: be careful with sign on $\tan^{-1}(\ )$



ex/ $z_2 = j z_1$

$z_2 = 2j + 1(-1) = -1 + j2$

Multiplying by j is a $90°$ rotation in the complex plane

$j = \emptyset + j1 = 1 e^{j\frac{\pi}{2}} \longrightarrow$

$jz_1 = e^{j\frac{\pi}{2}} \sqrt{5}\, e^{j\tan^{-1}(1/2)}$

$= \sqrt{5}\, e^{j\tan^{-1}(1/2) + j\frac{\pi}{2}} = \sqrt{5}\, e^{j\underbrace{\left[\tan^{-1}(\frac{1}{2}) + \frac{\pi}{2}\right]}_{\theta}}$

Usually easiest to

Addition

$z_1 + z_2 = (x_1 + x_2) + j(y_1 + y_2)$

Multiplication

$z_1 \cdot z_2 = r_1 r_2\, e^{j(\theta_1 + \theta_2)}$

Sinusoids as Complex Numbers

$$v(t) = A\cos(\omega t + \varphi)$$

$$= \mathrm{Re}\left\{ A e^{j(\omega t + \varphi)} \right\}$$

$$= \underbrace{\mathrm{Re}}_{\text{cosine}}\left\{ \underbrace{A}_{\text{magnitude}} \; \underbrace{e^{j\omega t}}_{\text{sinusoid}} \; \underbrace{e^{j\varphi}}_{\text{phase}} \right\}$$

$$\frac{d}{dt} v(t) = -A\omega \sin(\omega t + \varphi) = A\omega \cos(\omega t + \varphi + 90^\circ)$$

$$= \mathrm{Re}\left\{ A\omega \, e^{j\omega t} e^{j\varphi} e^{j\frac{\pi}{2}} \right\}$$

$$= \mathrm{Re}\left\{ A\underline{(j\omega)}\, e^{j\omega t} e^{j\varphi} \right\}$$