Lecture 12: Dynamics and Control

ECE 481: Power Electronics

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Part II
Converter Dynamics and Control

7. AC equivalent circuit modeling
8. Converter transfer functions
9. Controller design
10. Input filter design
11. AC and DC equivalent circuit modeling of the discontinuous conduction mode
12. Current programmed control

Chapter 7. AC Equivalent Circuit Modeling

7.1 Introduction

7.2 The basic AC modeling approach

7.3 State-space averaging

7.4 Circuit averaging and averaged switch modeling

7.5 The canonical circuit model

7.6 Modeling the pulse-width modulator

7.7 Summary of key points

7.1. Introduction

Objective: maintain $v(t)$ equal to an accurate, constant value V.

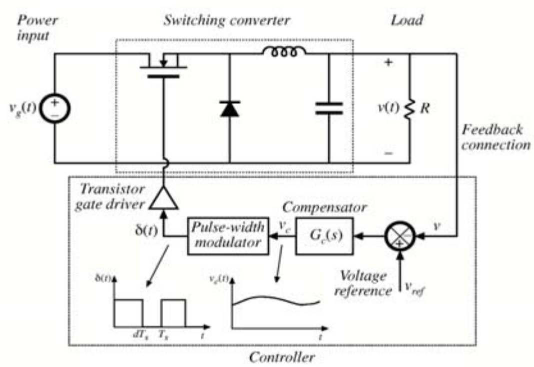
There are disturbances:

- in $v_g(t)$
- in R

There are uncertainties:

- in element values
- in V_g
- in R

A simple dc-dc regulator system, employing a buck converter



Applications of control in power electronics

DC-DC converters

Regulate dc output voltage.

Control the duty cycle $d(t)$ such that $v(t)$ accurately follows a reference signal v_{ref}.

DC-AC inverters

Regulate an ac output voltage.

Control the duty cycle $d(t)$ such that $v(t)$ accurately follows a reference signal $v_{ref}(t)$.

AC-DC rectifiers

Regulate the dc output voltage.

Regulate the ac input current waveform.

Control the duty cycle $d(t)$ such that $i_g(t)$ accurately follows a reference signal i (t), and $v(t)$ accurately follows a reference signal v

Converter Modeling

Applications

- Aerospace worst-case analysis
- Commercial high-volume production: design for reliability and yield

High quality design

- Ensure that the converter works well under worst-case conditions
  - Steady state (losses, efficiency, voltage regulation)
  - Small-signal ac (controller stability and transient response)

Engineering methodology

- Simulate model during preliminary design (design verification)
- Construct laboratory prototype converter system and make it work under nominal conditions
- Develop a converter model. Refine model until it predicts behavior of nominal laboratory prototype
- Use model to predict behavior under worst-case conditions
- Improve design until worst-case behavior meets specifications (or until reliability and production yield are acceptable)

Objective of Part II

Develop tools for modeling, analysis, and design of converter control systems

Need dynamic models of converters:

How do ac variations in $v_g(t)$, R, or $d(t)$ affect the output voltage $v(t)$?

What are the small-signal transfer functions of the converter?

- Extend the steady-state converter models of Chapters 2 and 3, to include CCM converter dynamics (Chapter 7)
- Construct converter small-signal transfer functions (Chapter 8)
- Design converter control systems (Chapter 9)
- Design input EMI filters that do not disrupt control system operation (Chapter 10)
- Model converters operating in DCM (Chapter 11)
- Current-programmed control of converters (Chapter 12)

Modeling

- Representation of physical behavior by mathematical means
- Model dominant behavior of system, ignore other insignificant phenomena
- Simplified model yields physical insight, allowing engineer to design system to operate in specified manner
- Approximations neglect small but complicating phenomena
- After basic insight has been gained, model can be refined (if it is judged worthwhile to expend the engineering effort to do so), to account for some of the previously neglected phenomena

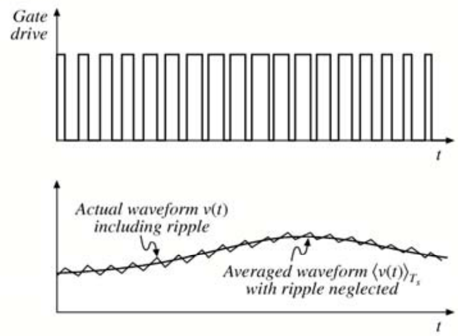
Neglecting the switching ripple

Suppose the duty cycle is modulated sinusoidally:

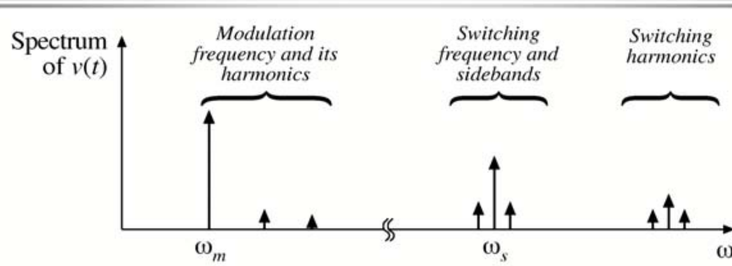
$$d(t) = D + D_m \cos \omega_m t$$

where D and D_m are constants, $|D_m| \ll D$, and the modulation frequency ω_m is much smaller than the converter switching frequency $\omega_s = 2\pi f_s$.

The resulting variations in transistor gate drive signal and converter output voltage:



Output voltage spectrum
with sinusoidal modulation of duty cycle



Contains frequency components at:

- Modulation frequency and its harmonics
- Switching frequency and its harmonics
- Sidebands of switching frequency

With small switching ripple, high-frequency components (switching harmonics and sidebands) are small.

If ripple is neglected, then only low-frequency components (modulation frequency and harmonics) remain.

Objective of ac converter modeling

- Predict how low-frequency variations in duty cycle induce low-frequency variations in the converter voltages and currents
- Ignore the switching ripple
- Ignore complicated switching harmonics and sidebands

Approach:

- Remove switching harmonics by averaging all waveforms over one switching period

Transient Volt-Second Balance

Averaging to remove switching ripple

Average over one switching period to remove switching ripple:

$$L\frac{d\left\langle i_L(t)\right\rangle_{T_s}}{dt} = \left\langle v_L(t)\right\rangle_{T_s}$$

$$C\frac{d\left\langle v_C(t)\right\rangle_{T_s}}{dt} = \left\langle i_C(t)\right\rangle_{T_s}$$

where

$$\left\langle x(t)\right\rangle_{T_s} = \frac{1}{T_s}\int_t^{t+T_s} x(\tau)\,d\tau$$

Note that, in steady-state,

$$\left\langle v_L(t)\right\rangle_{T_s} = 0$$

$$\left\langle i_C(t)\right\rangle_{T_s} = 0$$

by inductor volt-second balance and capacitor charge balance.

Nonlinear averaged equations

The averaged voltages and currents are, in general, nonlinear functions of the converter duty cycle, voltages, and currents. Hence, the averaged equations

$$L\frac{d\left\langle i_L(t)\right\rangle_{T_s}}{dt} = \left\langle v_L(t)\right\rangle_{T_s}$$

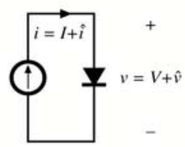
$$C\frac{d\left\langle v_C(t)\right\rangle_{T_s}}{dt} = \left\langle i_C(t)\right\rangle_{T_s}$$

constitute a system of nonlinear differential equations.

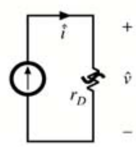
Hence, must linearize by constructing a small-signal converter model.

Small-signal modeling of the diode

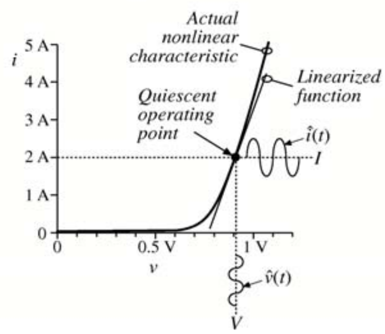
Nonlinear diode, driven by current source having a DC and small AC component



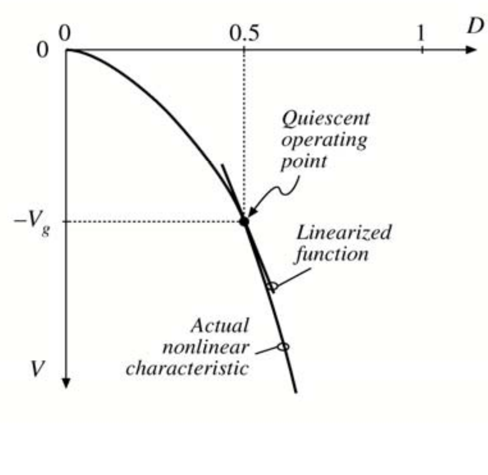
Small-signal AC model



Linearization of the diode *i-v* characteristic about a quiescent operating point



Buck-boost converter: nonlinear static control-to-output characteristic



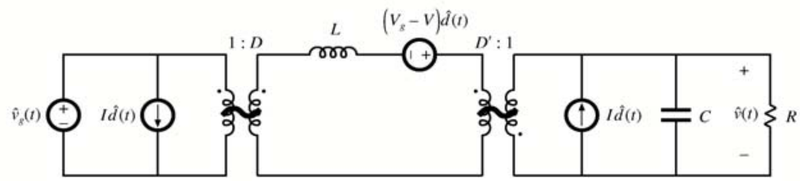
$V = V_g\, D/(1 - D)$

Example: linearization at the quiescent operating point

$D = 0.5$

Result of averaged small-signal ac modeling

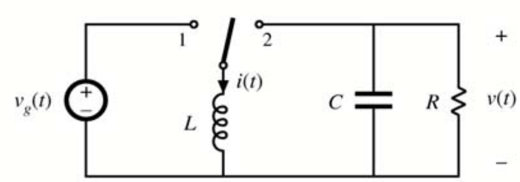
Small-signal ac equivalent circuit model



buck-boost example

7.2. The basic AC modeling approach

Buck-boost converter example

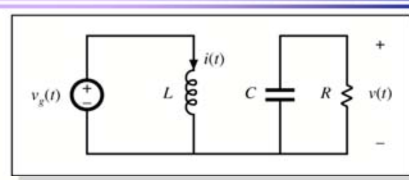


Switch in position 1

Inductor voltage and capacitor current are:

$$v_L(t) = L\,\frac{di(t)}{dt} = v_g(t)$$

$$i_C(t) = C\,\frac{dv(t)}{dt} = -\,\frac{v(t)}{R}$$

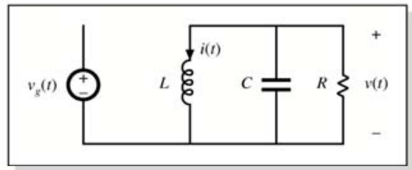


Small ripple approximation: replace waveforms with their low-frequency averaged values:

$$v_L(t) = L\,\frac{di(t)}{dt} \approx \left\langle v_g(t)\right\rangle_{T_s}$$

$$i_C(t) = C\,\frac{dv(t)}{dt} \approx -\,\frac{\left\langle v(t)\right\rangle_{T_s}}{R}$$

Switch in position 2



Inductor voltage and capacitor current are:

$$v_L(t) = L\frac{di(t)}{dt} = v(t)$$

$$i_C(t) = C\frac{dv(t)}{dt} = -\,i(t) - \frac{v(t)}{R}$$

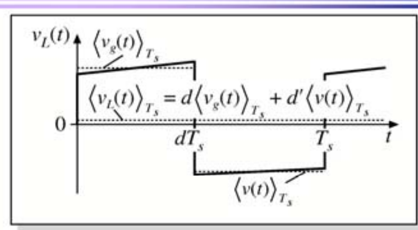
Small ripple approximation: replace waveforms with their low-frequency averaged values:

$$v_L(t) = L\frac{di(t)}{dt} \approx \left\langle v(t)\right\rangle_{T_s}$$

$$i_C(t) = C\frac{dv(t)}{dt} \approx -\left\langle i(t)\right\rangle_{T_s} - \frac{\left\langle v(t)\right\rangle_{T_s}}{R}$$

7.2.1 Averaging the inductor waveforms

Inductor voltage waveform



Low-frequency average is found by evaluation of

$$\left\langle x_L(t)\right\rangle_{T_s} = \frac{1}{T_s}\int_t^{t+T_s} x(\tau)\,d\tau$$

Average the inductor voltage in this manner:

$$\left\langle v_L(t)\right\rangle_{T_s} = \frac{1}{T_s}\int_t^{t+T_s} v_L(\tau)\,d\tau \approx d(t)\left\langle v_g(t)\right\rangle_{T_s} + d'(t)\left\langle v(t)\right\rangle_{T_s}$$

Insert into Eq. (7.2):

$$L\frac{d\left\langle i(t)\right\rangle_{T_s}}{dt} = d(t)\left\langle v_g(t)\right\rangle_{T_s} + d'(t)\left\langle v(t)\right\rangle_{T_s}$$

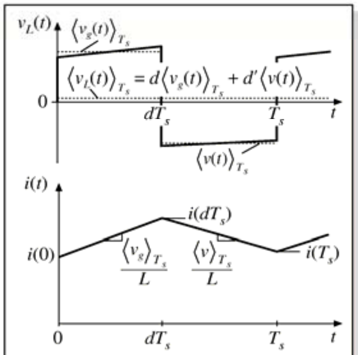
This equation describes how the low-frequency components of the inductor waveforms evolve in time.

7.2.2 Discussion of the averaging approximation

Use of the average inductor voltage allows us to determine the net change in inductor current over one switching period, while neglecting the switching ripple.

In steady-state, the average inductor voltage is zero (volt-second balance), and hence the inductor current waveform is periodic: $i(t + T_s) = i(t)$. There is no net change in inductor current over one switching period.

During transients or ac variations, the average inductor voltage is not zero in general, and this leads to net variations in inductor current.



Inductor voltage and current waveforms

Net change in inductor current is correctly predicted by the average inductor voltage

Inductor equation:

$$L \frac{di(t)}{dt} = v_L(t)$$

Divide by L and integrate over one switching period:

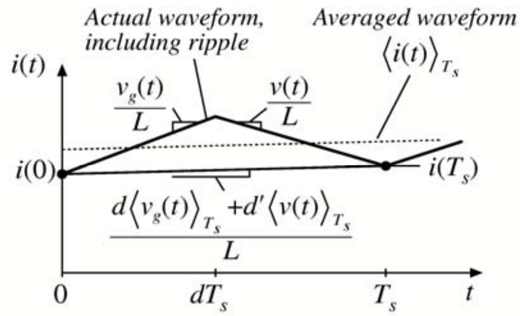
$$\int_t^{t+T_s} di = \frac{1}{L} \int_t^{t+T_s} v_L(\tau) d\tau$$

Left-hand side is the change in inductor current. Right-hand side can be related to average inductor voltage by multiplying and dividing by T_s as follows:

$$i(t + T_s) - i(t) = \frac{1}{L} T_s \langle v_L(t) \rangle_{T_s}$$

So the net change in inductor current over one switching period is exactly equal to the period T_s multiplied by the average slope $\langle v_L \rangle_{T_s} / L$.

Average inductor voltage correctly predicts average slope of $i_L(t)$



The net change in inductor current over one switching period is exactly equal to the period T_s multiplied by the average slope $\langle v_L \rangle_{T_s} / L$.

$\frac{d\langle i(t)\rangle_{T_s}}{dt}$

We have

$$i(t + T_s) - i(t) = \frac{1}{L} T_s \langle v_L(t) \rangle_{T_s}$$

Rearrange:

$$L \frac{i(t + T_s) - i(t)}{T_s} = \langle v_L(t) \rangle_{T_s}$$

Define the derivative of $\langle i \rangle_{T_s}$ as:

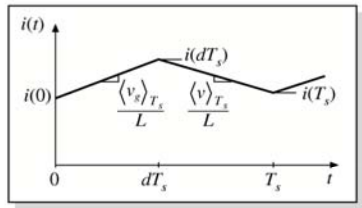
$$\frac{d\langle i(t)\rangle_{T_s}}{dt} = \frac{d}{dt}\left(\frac{1}{T_s}\int_t^{t+T_s} i(\tau)\,d\tau\right) = \frac{i(t + T_s) - i(t)}{T_s}$$

Hence,

$$L \frac{d\langle i(t)\rangle_{T_s}}{dt} = \langle v_L(t) \rangle_{T_s}$$

Computing how the inductor current changes over one switching period

Let's compute the actual inductor current waveform, using the linear ripple approximation.



With switch in position 1:

$$\underbrace{i(dT_s)}_{\text{(final value)}} = \underbrace{i(0)}_{\text{(initial value)}} + \underbrace{(dT_s)}_{\text{(length of interval)}} \underbrace{\left(\frac{\langle v_g(t) \rangle_{T_s}}{L}\right)}_{\text{(average slope)}}$$

With switch in position 2:

$$\underbrace{i(T_s)}_{\text{(final value)}} = \underbrace{i(dT_s)}_{\text{(initial value)}} + \underbrace{(d'T_s)}_{\text{(length of interval)}} \underbrace{\left(\frac{\langle v(t) \rangle_{T_s}}{L}\right)}_{\text{(average slope)}}$$

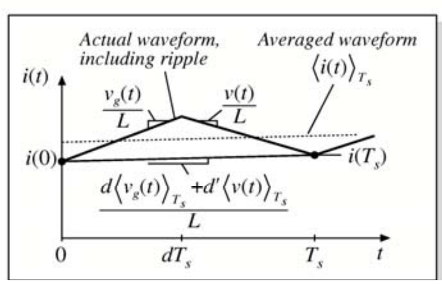
Net change in inductor current over one switching period

Eliminate $i(dT_s)$, to express $i(T_s)$ directly as a function of $i(0)$:

$$i(T_s) = i(0) + \frac{T_s}{L} \underbrace{\left(d(t)\,\langle v_g(t) \rangle_{T_s} + d'(t)\,\langle v(t) \rangle_{T_s}\right)}_{\langle v_L(t) \rangle_{T_s}}$$

The intermediate step of computing $i(dT_s)$ is eliminated.

The final value $i(T_s)$ is equal to the initial value $i(0)$, plus the switching period Ts multiplied by the average slope $\langle v_L \rangle_{T_s} / L$.



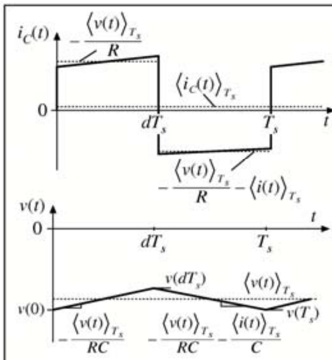
7.2.3 Averaging the capacitor waveforms

Average capacitor current:

$$\left\langle i_C(t)\right\rangle_{T_s} = d(t)\left(-\frac{\left\langle v(t)\right\rangle_{T_s}}{R}\right) + d'(t)\left(-\left\langle i(t)\right\rangle_{T_s} - \frac{\left\langle v(t)\right\rangle_{T_s}}{R}\right)$$

Collect terms, and equate to $C\, d\langle v \rangle_{T_s}/dt$:

$$C\,\frac{d\left\langle v(t)\right\rangle_{T_s}}{dt} = -\,d'(t)\left\langle i(t)\right\rangle_{T_s} - \frac{\left\langle v(t)\right\rangle_{T_s}}{R}$$



Capacitor voltage and current waveforms

7.2.4 The average input current

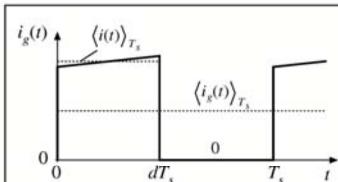
We found in Chapter 3 that it was sometimes necessary to write an equation for the average converter input current, to derive a complete dc equivalent circuit model. It is likewise necessary to do this for the ac model.

Buck-boost input current waveform is

$$i_g(t) = \begin{cases} \left\langle i(t)\right\rangle_{T_s} & \text{during subinterval 1} \\ 0 & \text{during subinterval 2} \end{cases}$$

Average value:

$$\left\langle i_g(t)\right\rangle_{T_s} = d(t)\left\langle i(t)\right\rangle_{T_s}$$



Converter input current waveform

7.2.5. Perturbation and linearization

Converter averaged equations:

$$L\frac{d\left\langle i(t)\right\rangle_{T_s}}{dt} = d(t)\left\langle v_g(t)\right\rangle_{T_s} + d'(t)\left\langle v(t)\right\rangle_{T_s}$$
$$C\frac{d\left\langle v(t)\right\rangle_{T_s}}{dt} = -\,d'(t)\left\langle i(t)\right\rangle_{T_s} - \frac{\left\langle v(t)\right\rangle_{T_s}}{R}$$
$$\left\langle i_g(t)\right\rangle_{T_s} = d(t)\left\langle i(t)\right\rangle_{T_s}$$

—nonlinear because of multiplication of the time-varying quantity $d(t)$ with other time-varying quantities such as $i(t)$ and $v(t)$.

Construct small-signal model: Linearize about quiescent operating point

If the converter is driven with some steady-state, or quiescent, inputs

$$d(t) = D$$
$$\left\langle v_g(t)\right\rangle_{T_s} = V_g$$

then, from the analysis of Chapter 2, after transients have subsided the inductor current, capacitor voltage, and input current

$$\left\langle i(t)\right\rangle_{T_s},\ \left\langle v(t)\right\rangle_{T_s},\ \left\langle i_g(t)\right\rangle_{T_s}$$

reach the quiescent values I, V, and I_g, given by the steady-state analysis as

$$V = -\frac{D}{D'}V_g$$
$$I = -\frac{V}{D'\,R}$$
$$I_g = D\,I$$

Perturbation

So let us assume that the input voltage and duty cycle are equal to some given (dc) quiescent values, plus superimposed small ac variations:

$$\left\langle v_g(t) \right\rangle_{T_s} = V_g + \hat{v}_g(t)$$
$$d(t) = D + \hat{d}(t)$$

In response, and after any transients have subsided, the converter dependent voltages and currents will be equal to the corresponding quiescent values, plus small ac variations:

$$\left\langle i(t) \right\rangle_{T_s} = I + \hat{i}(t)$$
$$\left\langle v(t) \right\rangle_{T_s} = V + \hat{v}(t)$$
$$\left\langle i_g(t) \right\rangle_{T_s} = I_g + \hat{i}_g(t)$$

The small-signal assumption

If the ac variations are much smaller in magnitude than the respective quiescent values,

$$\left| \hat{v}_g(t) \right| << \left| V_g \right|$$
$$\left| \hat{d}(t) \right| << \left| D \right|$$
$$\left| \hat{i}(t) \right| << \left| I \right|$$
$$\left| \hat{v}(t) \right| << \left| V \right|$$
$$\left| \hat{i}_g(t) \right| << \left| I_g \right|$$

then the nonlinear converter equations can be linearized.

Perturbation of inductor equation

Insert the perturbed expressions into the inductor differential equation:

$$L\frac{d\left(I+\hat{i}(t)\right)}{dt}=\left(D+\hat{d}(t)\right)\left(V_g+\hat{v}_g(t)\right)+\left(D'-\hat{d}(t)\right)\left(V+\hat{v}(t)\right)$$

note that $d'(t)$ is given by

$$d'(t)=\left(1-d(t)\right)=1-\left(D+\hat{d}(t)\right)=D'-\hat{d}(t) \qquad \text{with } D'=1-D$$

Multiply out and collect terms:

$$L\left(\cancelto{0}{\frac{dI}{dt}}+\frac{d\hat{i}(t)}{dt}\right)=\underbrace{\left(DV_g+D'V\right)}_{\textit{Dc terms}}+\underbrace{\left(D\hat{v}_g(t)+D'\hat{v}(t)+\left(V_g-V\right)\hat{d}(t)\right)}_{\substack{1^{st}\ \textit{order ac terms}\\ (\textit{linear})}}+\underbrace{\hat{d}(t)\left(\hat{v}_g(t)-\hat{v}(t)\right)}_{\substack{2^{nd}\ \textit{order ac terms}\\ (\textit{nonlinear})}}$$

The perturbed inductor equation

$$L\left(\cancelto{0}{\frac{dI}{dt}}+\frac{d\hat{i}(t)}{dt}\right)=\underbrace{\left(DV_g+D'V\right)}_{\textit{Dc terms}}+\underbrace{\left(D\hat{v}_g(t)+D'\hat{v}(t)+\left(V_g-V\right)\hat{d}(t)\right)}_{\substack{1^{st}\ \textit{order ac terms}\\ (\textit{linear})}}+\underbrace{\hat{d}(t)\left(\hat{v}_g(t)-\hat{v}(t)\right)}_{\substack{2^{nd}\ \textit{order ac terms}\\ (\textit{nonlinear})}}$$

Since I is a constant (dc) term, its derivative is zero

The right-hand side contains three types of terms:

- Dc terms, containing only dc quantities
- First-order ac terms, containing a single ac quantity, usually multiplied by a constant coefficient such as a dc term. These are linear functions of the ac variations
- Second-order ac terms, containing products of ac quantities. These are nonlinear, because they involve multiplication of ac quantities

Neglect of second-order terms

$$L\left(\cancelto{0}{\frac{dI}{dt}} + \frac{d\hat{i}(t)}{dt}\right) = \underbrace{\left(DV_g + D'V\right)}_{\textit{Dc terms}} + \underbrace{\left(D\hat{v}_g(t) + D'\hat{v}(t) + \left(V_g - V\right)\hat{d}(t)\right)}_{\substack{1^{st}\ \textit{order ac terms} \\ (\textit{linear})}} + \underbrace{\hat{d}(t)\left(\hat{v}_g(t) - \hat{v}(t)\right)}_{\substack{2^{nd}\ \textit{order ac terms} \\ (\textit{nonlinear})}}$$

Provided

$$\left|\hat{v}_g(t)\right| << \left|V_g\right|$$
$$\left|\hat{d}(t)\right| << \left|D\right|$$
$$\left|\hat{i}(t)\right| << \left|I\right|$$
$$\left|\hat{v}(t)\right| << \left|V\right|$$
$$\left|\hat{i}_g(t)\right| << \left|I_g\right|$$

then the second-order ac terms are much smaller than the first-order terms. For example,

$\left|\hat{d}(t)\ \hat{v}_g(t)\right| << \left|D\ \hat{v}_g(t)\right|$ when $\left|\hat{d}(t)\right| << D$

So neglect second-order terms.
Also, dc terms on each side of equation are equal.

Linearized inductor equation

Upon discarding second-order terms, and removing dc terms (which add to zero), we are left with

$$L\frac{d\hat{i}(t)}{dt} = D\hat{v}_g(t) + D'\hat{v}(t) + \left(V_g - V\right)\hat{d}(t)$$

This is the desired result: a linearized equation which describes small-signal ac variations.

Note that the quiescent values D, D', V, V_g, are treated as given constants in the equation.

Capacitor equation

Perturbation leads to

$$C\frac{d\left(V+\hat{v}(t)\right)}{dt} = -\left(D' - \hat{d}(t)\right)\left(I+\hat{i}(t)\right) - \frac{\left(V+\hat{v}(t)\right)}{R}$$

Collect terms:

$$C\left(\frac{dV^{\,0}}{dt} + \frac{d\hat{v}(t)}{dt}\right) = \underbrace{\left(-D'I - \frac{V}{R}\right)}_{\textit{Dc terms}} + \underbrace{\left(-D'\hat{i}(t) - \frac{\hat{v}(t)}{R} + I\hat{d}(t)\right)}_{\substack{1^{st}\ \textit{order ac terms} \\ (\textit{linear})}} + \underbrace{\hat{d}(t)\hat{i}(t)}_{\substack{2^{nd}\ \textit{order ac term} \\ (\textit{nonlinear})}}$$

Neglect second-order terms. Dc terms on both sides of equation are equal. The following terms remain:

$$C\frac{d\hat{v}(t)}{dt} = -D'\hat{i}(t) - \frac{\hat{v}(t)}{R} + I\hat{d}(t)$$

This is the desired small-signal linearized capacitor equation.

Average input current

Perturbation leads to

$$I_g + \hat{i}_g(t) = \left(D + \hat{d}(t)\right)\left(I + \hat{i}(t)\right)$$

Collect terms:

$$\underbrace{I_g}_{\textit{Dc term}} + \underbrace{\hat{i}_g(t)}_{1^{st}\ \textit{order ac term}} = \underbrace{\left(DI\right)}_{\textit{Dc term}} + \underbrace{\left(D\hat{i}(t) + I\hat{d}(t)\right)}_{\substack{1^{st}\ \textit{order ac terms} \\ (\textit{linear})}} + \underbrace{\hat{d}(t)\hat{i}(t)}_{\substack{2^{nd}\ \textit{order ac term} \\ (\textit{nonlinear})}}$$

Neglect second-order terms. Dc terms on both sides of equation are equal. The following first-order terms remain:

$$\hat{i}_g(t) = D\hat{i}(t) + I\hat{d}(t)$$

This is the linearized small-signal equation which described the converter input port.

7.2.6. Construction of small-signal equivalent circuit model

The linearized small-signal converter equations:

$$L\frac{d\hat{i}(t)}{dt} = D\hat{v}_g(t) + D'\hat{v}(t) + \left(V_g - V\right)\hat{d}(t)$$

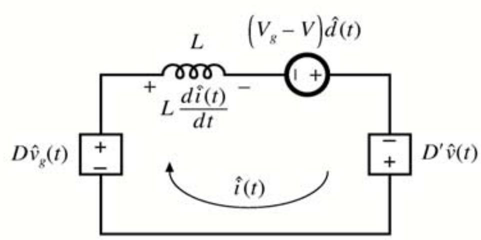
$$C\frac{d\hat{v}(t)}{dt} = -D'\hat{i}(t) - \frac{\hat{v}(t)}{R} + I\hat{d}(t)$$

$$\hat{i}_g(t) = D\hat{i}(t) + I\hat{d}(t)$$

Reconstruct equivalent circuit corresponding to these equations, in manner similar to the process used in Chapter 3.

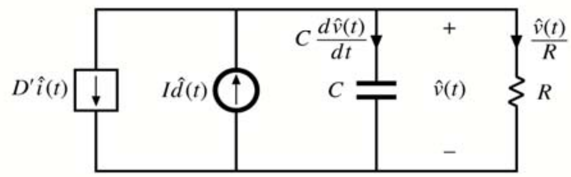
Inductor loop equation

$$L\frac{d\hat{i}(t)}{dt} = D\hat{v}_g(t) + D'\hat{v}(t) + \left(V_g - V\right)\hat{d}(t)$$



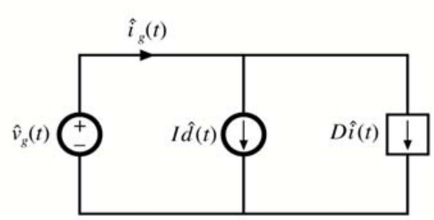
Capacitor node equation

$$C\frac{d\hat{v}(t)}{dt} = -D'\hat{i}(t) - \frac{\hat{v}(t)}{R} + I\hat{d}(t)$$



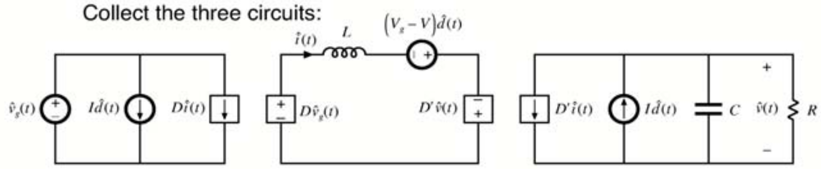
Input port node equation

$$\hat{i}_g(t) = D\hat{i}(t) + I\hat{d}(t)$$

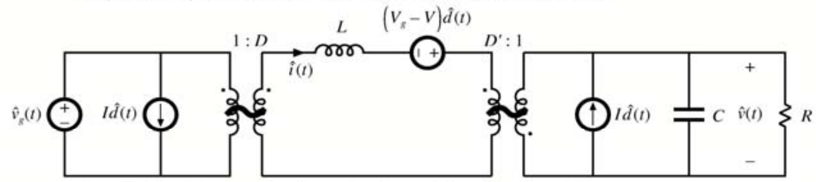


Complete equivalent circuit

Collect the three circuits:



Replace dependent sources with ideal dc transformers:



Small-signal ac equivalent circuit model of the buck-boost converter