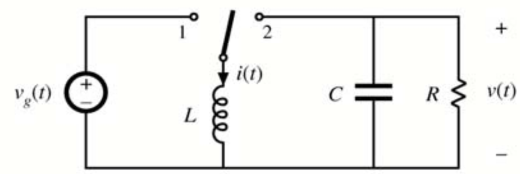
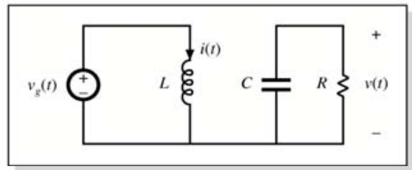
7.2. The basic AC modeling approach

Buck-boost converter example



Switch in position 1



Inductor voltage and capacitor current are:

$$v_L(t) = L\,\frac{di(t)}{dt} = v_g(t)$$

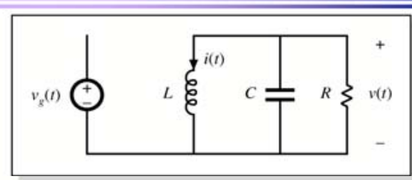
$$i_C(t) = C\,\frac{dv(t)}{dt} = -\,\frac{v(t)}{R}$$

Small ripple approximation: replace waveforms with their low-frequency averaged values:

$$v_L(t) = L\,\frac{di(t)}{dt} \approx \left\langle v_g(t) \right\rangle_{T_s}$$

$$i_C(t) = C\,\frac{dv(t)}{dt} \approx -\,\frac{\left\langle v(t) \right\rangle_{T_s}}{R}$$

Switch in position 2



Inductor voltage and capacitor current are:

$$v_L(t) = L\,\frac{di(t)}{dt} = v(t)$$

$$i_C(t) = C\,\frac{dv(t)}{dt} = -\,i(t) - \frac{v(t)}{R}$$

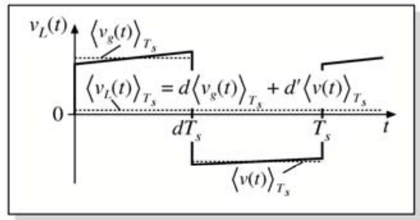
Small ripple approximation: replace waveforms with their low-frequency averaged values:

$$v_L(t) = L\,\frac{di(t)}{dt} \approx \left\langle v(t) \right\rangle_{T_s}$$

$$i_C(t) = C\,\frac{dv(t)}{dt} \approx -\left\langle i(t) \right\rangle_{T_s} - \frac{\left\langle v(t) \right\rangle_{T_s}}{R}$$

7.2.1 Averaging the inductor waveforms

Inductor voltage waveform



Low-frequency average is found by evaluation of

$$\langle x_L(t)\rangle_{T_s} = \frac{1}{T_s}\int_t^{t+T_s} x(\tau)\,d\tau$$

Average the inductor voltage in this manner:

$$\langle v_L(t)\rangle_{T_s} = \frac{1}{T_s}\int_t^{t+T_s} v_L(\tau)\,d\tau \approx d(t)\,\langle v_g(t)\rangle_{T_s} + d'(t)\,\langle v(t)\rangle_{T_s}$$

Insert into Eq. (7.2):

$$L\frac{d\langle i(t)\rangle_{T_s}}{dt} = d(t)\,\langle v_g(t)\rangle_{T_s} + d'(t)\,\langle v(t)\rangle_{T_s}$$

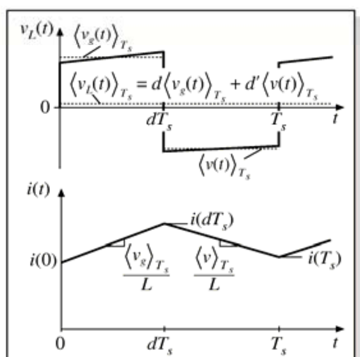
This equation describes how the low-frequency components of the inductor waveforms evolve in time.

7.2.2 Discussion of the averaging approximation

Use of the average inductor voltage allows us to determine the net change in inductor current over one switching period, while neglecting the switching ripple.

In steady-state, the average inductor voltage is zero (volt-second balance), and hence the inductor current waveform is periodic: $i(t + T_s) = i(t)$. There is no net change in inductor current over one switching period.

During transients or ac variations, the average inductor voltage is not zero in general, and this leads to net variations in inductor current.



Inductor voltage and current waveforms

Net change in inductor current is correctly predicted by the average inductor voltage

Inductor equation:

$$L \frac{di(t)}{dt} = v_L(t)$$

Divide by L and integrate over one switching period:

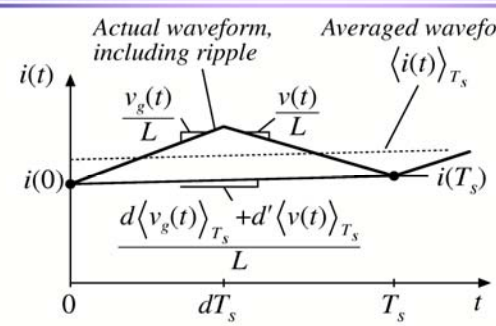
$$\int_t^{t+T_s} di = \frac{1}{L} \int_t^{t+T_s} v_L(\tau) d\tau$$

Left-hand side is the change in inductor current. Right-hand side can be related to average inductor voltage by multiplying and dividing by T_s as follows:

$$i(t + T_s) - i(t) = \frac{1}{L} T_s \left\langle v_L(t) \right\rangle_{T_s}$$

So the net change in inductor current over one switching period is exactly equal to the period T_s multiplied by the average slope $\langle v_L \rangle_{T_s} / L$.

Average inductor voltage correctly predicts average slope of $i_L(t)$



The net change in inductor current over one switching period is exactly equal to the period T_s multiplied by the average slope $\langle v_L \rangle_{T_s} / L$.

$$\frac{d\left\langle i(t)\right\rangle_{T_s}}{dt}$$

We have

$$i(t+T_s) - i(t) = \frac{1}{L}\, T_s \left\langle v_L(t)\right\rangle_{T_s}$$

Rearrange:

$$L\,\frac{i(t+T_s) - i(t)}{T_s} = \left\langle v_L(t)\right\rangle_{T_s}$$

Define the derivative of $\langle i \rangle_{T_s}$ as:

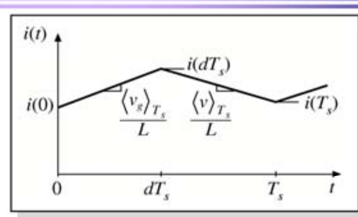
$$\frac{d\left\langle i(t)\right\rangle_{T_s}}{dt} = \frac{d}{dt}\left(\frac{1}{T_s}\int_t^{t+T_s} i(\tau)\,d\tau\right) = \frac{i(t+T_s) - i(t)}{T_s}$$

Hence,

$$L\,\frac{d\left\langle i(t)\right\rangle_{T_s}}{dt} = \left\langle v_L(t)\right\rangle_{T_s}$$

Computing how the inductor current changes over one switching period

Let's compute the actual inductor current waveform, using the linear ripple approximation.



With switch in position 1:

$$\underbrace{i(dT_s)}_{\text{(final value)}} = \underbrace{i(0)}_{\text{(initial value)}} + \underbrace{\left(dT_s\right)}_{\text{(length of interval)}}\ \underbrace{\left(\frac{\left\langle v_g(t)\right\rangle_{T_s}}{L}\right)}_{\text{(average slope)}}$$

With switch in position 2:

$$\underbrace{i(T_s)}_{\text{(final value)}} = \underbrace{i(dT_s)}_{\text{(initial value)}} + \underbrace{\left(d'T_s\right)}_{\text{(length of interval)}}\ \underbrace{\left(\frac{\left\langle v(t)\right\rangle_{T_s}}{L}\right)}_{\text{(average slope)}}$$

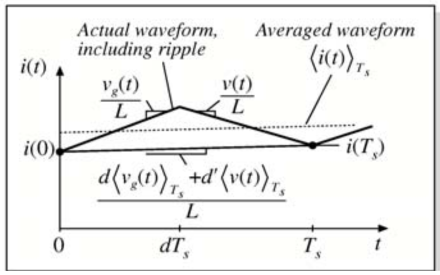
Net change in inductor current over one switching period

Eliminate $i(dT_s)$, to express $i(T_s)$ directly as a function of $i(0)$:

$$i(T_s) = i(0) + \frac{T_s}{L}\underbrace{\left(d(t)\left\langle v_g(t)\right\rangle_{T_s} + d'(t)\left\langle v(t)\right\rangle_{T_s}\right)}_{\left\langle v_L(t)\right\rangle_{T_s}}$$

The intermediate step of computing $i(dT_s)$ is eliminated.

The final value $i(T_s)$ is equal to the initial value $i(0)$, plus the switching period Ts multiplied by the average slope $\langle v_L \rangle_{T_s}/L$.



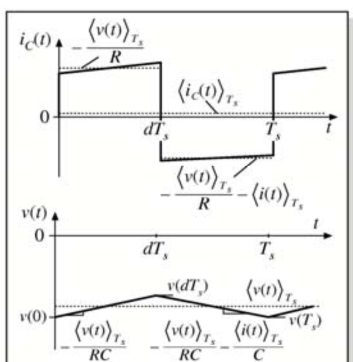
7.2.3 Averaging the capacitor waveforms

Average capacitor current:

$$\left\langle i_C(t)\right\rangle_{T_s} = d(t)\left(-\frac{\left\langle v(t)\right\rangle_{T_s}}{R}\right) + d'(t)\left(-\left\langle i(t)\right\rangle_{T_s} - \frac{\left\langle v(t)\right\rangle_{T_s}}{R}\right)$$

Collect terms, and equate to $C\, d\langle v \rangle_{T_s}/dt$:

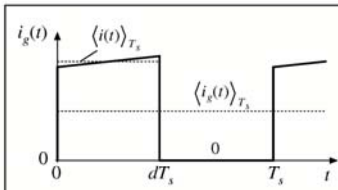
$$C\frac{d\left\langle v(t)\right\rangle_{T_s}}{dt} = -d'(t)\left\langle i(t)\right\rangle_{T_s} - \frac{\left\langle v(t)\right\rangle_{T_s}}{R}$$



Capacitor voltage and current waveforms

7.2.4 The average input current

We found in Chapter 3 that it was sometimes necessary to write an equation for the average converter input current, to derive a complete dc equivalent circuit model. It is likewise necessary to do this for the ac model.



Converter input current waveform

Buck-boost input current waveform is

$$i_g(t) = \begin{cases} \langle i(t) \rangle_{T_s} & \text{during subinterval 1} \\ 0 & \text{during subinterval 2} \end{cases}$$

Average value:

$$\langle i_g(t) \rangle_{T_s} = d(t) \langle i(t) \rangle_{T_s}$$

7.2.5. Perturbation and linearization

Converter averaged equations:

$$L \frac{d\langle i(t) \rangle_{T_s}}{dt} = d(t) \langle v_g(t) \rangle_{T_s} + d'(t) \langle v(t) \rangle_{T_s}$$

$$C \frac{d\langle v(t) \rangle_{T_s}}{dt} = - d'(t) \langle i(t) \rangle_{T_s} - \frac{\langle v(t) \rangle_{T_s}}{R}$$

$$\langle i_g(t) \rangle_{T_s} = d(t) \langle i(t) \rangle_{T_s}$$

—nonlinear because of multiplication of the time-varying quantity $d(t)$ with other time-varying quantities such as $i(t)$ and $v(t)$.

Construct small-signal model: Linearize about quiescent operating point

If the converter is driven with some steady-state, or quiescent, inputs

$$d(t) = D$$
$$\left\langle v_g(t) \right\rangle_{T_s} = V_g$$

then, from the analysis of Chapter 2, after transients have subsided the inductor current, capacitor voltage, and input current

$$\left\langle i(t) \right\rangle_{T_s}, \left\langle v(t) \right\rangle_{T_s}, \left\langle i_g(t) \right\rangle_{T_s}$$

reach the quiescent values I, V, and I_g, given by the steady-state analysis as

$$V = -\frac{D}{D'} V_g$$
$$I = -\frac{V}{D' R}$$
$$I_g = D\, I$$

Perturbation

So let us assume that the input voltage and duty cycle are equal to some given (dc) quiescent values, plus superimposed small ac variations:

$$\left\langle v_g(t) \right\rangle_{T_s} = V_g + \hat{v}_g(t)$$
$$d(t) = D + \hat{d}(t)$$

In response, and after any transients have subsided, the converter dependent voltages and currents will be equal to the corresponding quiescent values, plus small ac variations:

$$\left\langle i(t) \right\rangle_{T_s} = I + \hat{i}(t)$$
$$\left\langle v(t) \right\rangle_{T_s} = V + \hat{v}(t)$$
$$\left\langle i_g(t) \right\rangle_{T_s} = I_g + \hat{i}_g(t)$$

The small-signal assumption

If the ac variations are much smaller in magnitude than the respective quiescent values,

$$\left|\hat{v}_g(t)\right| << \left|V_g\right|$$
$$\left|\hat{d}(t)\right| << \left|D\right|$$
$$\left|\hat{i}(t)\right| << \left|I\right|$$
$$\left|\hat{v}(t)\right| << \left|V\right|$$
$$\left|\hat{i}_g(t)\right| << \left|I_g\right|$$

then the nonlinear converter equations can be linearized.

Perturbation of inductor equation

Insert the perturbed expressions into the inductor differential equation:

$$L\frac{d\left(I+\hat{i}(t)\right)}{dt} = \left(D+\hat{d}(t)\right)\left(V_g+\hat{v}_g(t)\right) + \left(D'-\hat{d}(t)\right)\left(V+\hat{v}(t)\right)$$

note that $d'(t)$ is given by

$$d'(t) = \left(1-d(t)\right) = 1-\left(D+\hat{d}(t)\right) = D'-\hat{d}(t) \qquad \text{with } D' = 1-D$$

Multiply out and collect terms:

$$L\left(\frac{dI}{dt}^{\,0} + \frac{d\hat{i}(t)}{dt}\right) = \underbrace{\left(DV_g + D'V\right)}_{Dc\ terms} + \underbrace{\left(D\hat{v}_g(t) + D'\hat{v}(t) + \left(V_g - V\right)\hat{d}(t)\right)}_{\substack{1^{st}\ order\ ac\ terms \\ (linear)}} + \underbrace{\hat{d}(t)\left(\hat{v}_g(t)-\hat{v}(t)\right)}_{\substack{2^{nd}\ order\ ac\ terms \\ (nonlinear)}}$$

The perturbed inductor equation

$$L\left(\frac{dI}{dt}^{0}+\frac{d\hat{i}(t)}{dt}\right)=\underbrace{\left(DV_g+D'V\right)}_{Dc\ terms}+\underbrace{\left(D\hat{v}_g(t)+D'\hat{v}(t)+\left(V_g-V\right)\hat{d}(t)\right)}_{1^{st}\ order\ ac\ terms\ (linear)}+\underbrace{\hat{d}(t)\left(\hat{v}_g(t)-\hat{v}(t)\right)}_{2^{nd}\ order\ ac\ terms\ (nonlinear)}$$

Since I is a constant (dc) term, its derivative is zero

The right-hand side contains three types of terms:

- Dc terms, containing only dc quantities
- First-order ac terms, containing a single ac quantity, usually multiplied by a constant coefficient such as a dc term. These are linear functions of the ac variations
- Second-order ac terms, containing products of ac quantities. These are nonlinear, because they involve multiplication of ac quantities

Neglect of second-order terms

$$L\left(\frac{dI}{dt}^{0}+\frac{d\hat{i}(t)}{dt}\right)=\underbrace{\left(DV_g+D'V\right)}_{Dc\ terms}+\underbrace{\left(D\hat{v}_g(t)+D'\hat{v}(t)+\left(V_g-V\right)\hat{d}(t)\right)}_{1^{st}\ order\ ac\ terms\ (linear)}+\underbrace{\hat{d}(t)\left(\hat{v}_g(t)-\hat{v}(t)\right)}_{2^{nd}\ order\ ac\ terms\ (nonlinear)}$$

Provided

$$\left|\hat{v}_g(t)\right| << \left|V_g\right|$$
$$\left|\hat{d}(t)\right| << \left|D\right|$$
$$\left|\hat{i}(t)\right| << \left|I\right|$$
$$\left|\hat{v}(t)\right| << \left|V\right|$$
$$\left|\hat{i}_g(t)\right| << \left|I_g\right|$$

then the second-order ac terms are much smaller than the first-order terms. For example,

$\left|\hat{d}(t)\ \hat{v}_g(t)\right| << \left|D\ \hat{v}_g(t)\right|$ when $\left|\hat{d}(t)\right| << D$

So neglect second-order terms.
Also, dc terms on each side of equation are equal.

Linearized inductor equation

Upon discarding second-order terms, and removing dc terms (which add to zero), we are left with

$$L\frac{d\hat{i}(t)}{dt} = D\hat{v}_g(t) + D'\hat{v}(t) + \left(V_g - V\right)\hat{d}(t)$$

This is the desired result: a linearized equation which describes small-signal ac variations.

Note that the quiescent values D, D', V, V_g, are treated as given constants in the equation.

Capacitor equation

Perturbation leads to

$$C\frac{d\left(V + \hat{v}(t)\right)}{dt} = -\left(D' - \hat{d}(t)\right)\left(I + \hat{i}(t)\right) - \frac{\left(V + \hat{v}(t)\right)}{R}$$

Collect terms:

$$C\left(\frac{dV}{dt}^{0} + \frac{d\hat{v}(t)}{dt}\right) = \underbrace{\left(-D'I - \frac{V}{R}\right)}_{\text{Dc terms}} + \underbrace{\left(-D'\hat{i}(t) - \frac{\hat{v}(t)}{R} + I\hat{d}(t)\right)}_{\substack{1^{st} \text{ order ac terms} \\ \text{(linear)}}} + \underbrace{\hat{d}(t)\hat{i}(t)}_{\substack{2^{nd} \text{ order ac term} \\ \text{(nonlinear)}}}$$

Neglect second-order terms. Dc terms on both sides of equation are equal. The following terms remain:

$$C\frac{d\hat{v}(t)}{dt} = -D'\hat{i}(t) - \frac{\hat{v}(t)}{R} + I\hat{d}(t)$$

This is the desired small-signal linearized capacitor equation.

Average input current

Perturbation leads to

$$I_g + \hat{i}_g(t) = \left(D + \hat{d}(t)\right)\left(I + \hat{i}(t)\right)$$

Collect terms:

$$\underbrace{I_g}_{Dc\ term} + \underbrace{\hat{i}_g(t)}_{1^{st}\ order\ ac\ term} = \underbrace{(DI)}_{Dc\ term} + \underbrace{\left(D\hat{i}(t) + I\hat{d}(t)\right)}_{\substack{1^{st}\ order\ ac\ terms \\ (linear)}} + \underbrace{\hat{d}(t)\hat{i}(t)}_{\substack{2^{nd}\ order\ ac\ term \\ (nonlinear)}}$$

Neglect second-order terms. Dc terms on both sides of equation are equal. The following first-order terms remain:

$$\hat{i}_g(t) = D\hat{i}(t) + I\hat{d}(t)$$

This is the linearized small-signal equation which described the converter input port.

7.2.6. Construction of small-signal equivalent circuit model

The linearized small-signal converter equations:

$$L\frac{d\hat{i}(t)}{dt} = D\hat{v}_g(t) + D'\hat{v}(t) + \left(V_g - V\right)\hat{d}(t)$$

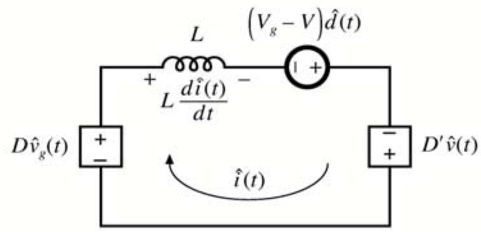
$$C\frac{d\hat{v}(t)}{dt} = -D'\hat{i}(t) - \frac{\hat{v}(t)}{R} + I\hat{d}(t)$$

$$\hat{i}_g(t) = D\hat{i}(t) + I\hat{d}(t)$$

Reconstruct equivalent circuit corresponding to these equations, in manner similar to the process used in Chapter 3.

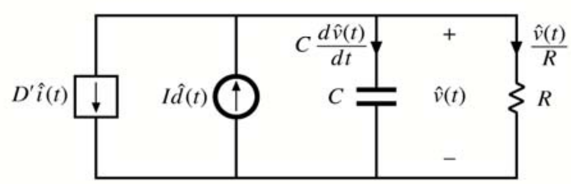
Inductor loop equation

$$L\frac{d\hat{i}(t)}{dt} = D\hat{v}_g(t) + D'\hat{v}(t) + \left(V_g - V\right)\hat{d}(t)$$



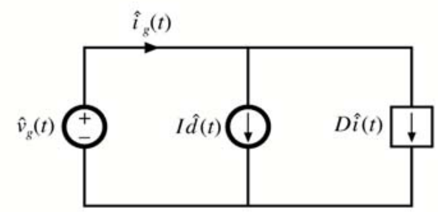
Capacitor node equation

$$C\frac{d\hat{v}(t)}{dt} = -D'\hat{i}(t) - \frac{\hat{v}(t)}{R} + I\hat{d}(t)$$



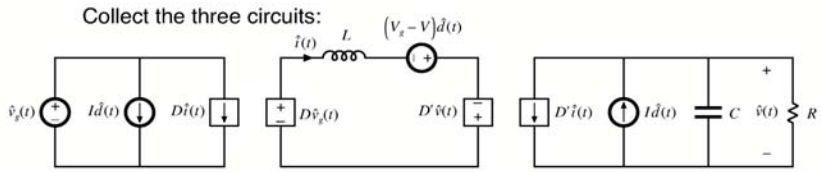
Input port node equation

$$\hat{i}_g(t) = D\hat{i}(t) + I\hat{d}(t)$$

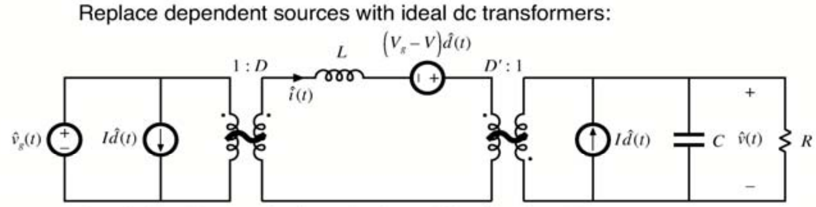


Complete equivalent circuit

Collect the three circuits:



Replace dependent sources with ideal dc transformers:



Small-signal ac equivalent circuit model of the buck-boost converter

7.2.7 Discussion of the perturbation and linearization step

The linearization step amounts to taking the Taylor expansion of the original nonlinear equation, about a quiescent operating point, and retaining only the constant and linear terms.

Inductor equation, buck-boost example:

$$L\frac{d\langle i(t)\rangle_{T_s}}{dt} = d(t)\,\langle v_g(t)\rangle_{T_s} + d'(t)\,\langle v(t)\rangle_{T_s} = f_1\Big(\langle v_g(t)\rangle_{T_s}, \langle v(t)\rangle_{T_s}, d(t)\Big)$$

Three-dimensional Taylor series expansion:

$$L\left(\frac{dI}{dt} + \frac{d\hat{i}(t)}{dt}\right) = f_1\big(V_g, V, D\big) + \hat{v}_g(t)\left.\frac{\partial f_1\big(v_g, V, D\big)}{\partial v_g}\right|_{v_g = V_g}$$

$$+ \hat{v}(t)\left.\frac{\partial f_1\big(V_g, v, D\big)}{\partial v}\right|_{v = V} + \hat{d}(t)\left.\frac{\partial f_1\big(V_g, V, d\big)}{\partial d}\right|_{d = D}$$

+ higher-order nonlinear terms

Linearization via Taylor series

Equate DC terms:

$$0 = f_1\big(V_g, V, D\big)$$

Coefficients of linear terms are:

$$\left.\frac{\partial f_1\big(v_g, V, D\big)}{\partial v_g}\right|_{v_g = V_g} = D$$

$$\left.\frac{\partial f_1\big(V_g, v, D\big)}{\partial v}\right|_{v = V} = D'$$

$$\left.\frac{\partial f_1\big(V_g, V, d\big)}{\partial d}\right|_{d = D} = V_g - V$$

$$L\left(\frac{dI}{dt} + \frac{d\hat{i}(t)}{dt}\right) = f_1\big(V_g, V, D\big) + \hat{v}_g(t)\left.\frac{\partial f_1\big(v_g, V, D\big)}{\partial v_g}\right|_{v_g = V_g}$$

$$+ \hat{v}(t)\left.\frac{\partial f_1\big(V_g, v, D\big)}{\partial v}\right|_{v = V} + \hat{d}(t)\left.\frac{\partial f_1\big(V_g, V, d\big)}{\partial d}\right|_{d = D}$$

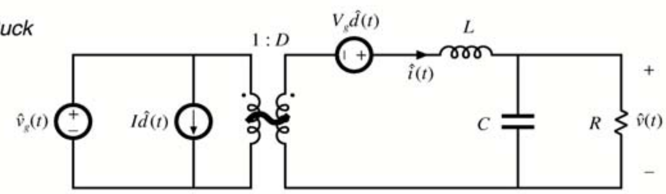
+ higher-order nonlinear terms

Hence the small-signal ac linearized equation is:

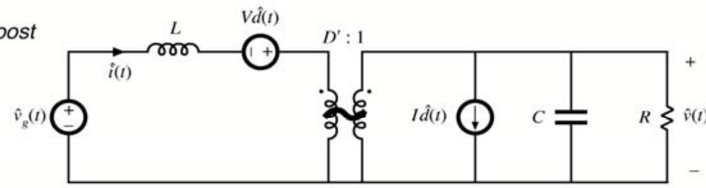
$$L\frac{d\hat{i}(t)}{dt} = D\hat{v}_g(t) + D'\hat{v}(t) + \big(V_g - V\big)\,\hat{d}(t)$$

7.2.8. Results for several basic converters

Buck

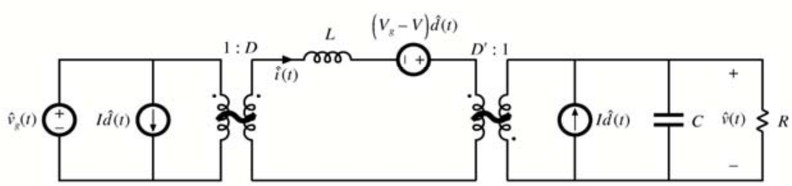


Boost



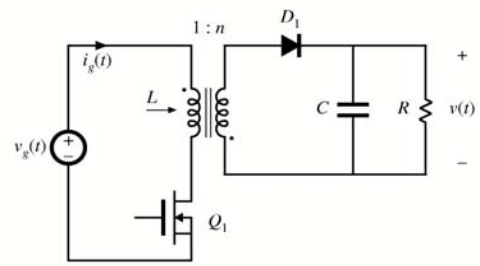
Results for several basic converters

Buck-boost



7.2.9 Example: a nonideal flyback converter

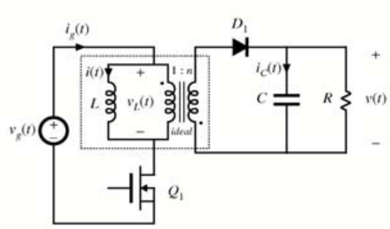
Flyback converter example



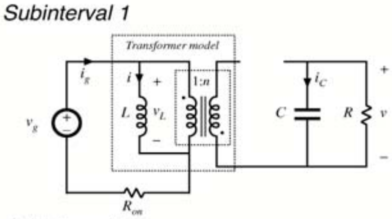
- MOSFET has on-resistance R_{on}
- Flyback transformer has magnetizing inductance L, referred to primary

Circuits during subintervals 1 and 2

Flyback converter, with transformer equivalent circuit



Subinterval 1



Subinterval 2