Lecture 14: Averaged SSM and Linearization

ECE 481: Power Electronics

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Department of Electrical Engineering and Computer Science

University of Tennessee Knoxville

Fall 2013

Dominium Power Job Opening

Seeking December graduates.

Engineer I (entry level engineer)

This role provides station and switchyard technical support to the generation plant for high voltage transmission and distribution equipment. Excellent entry level position for Electrical Engineers to gain Generation/Transmission knowledge.

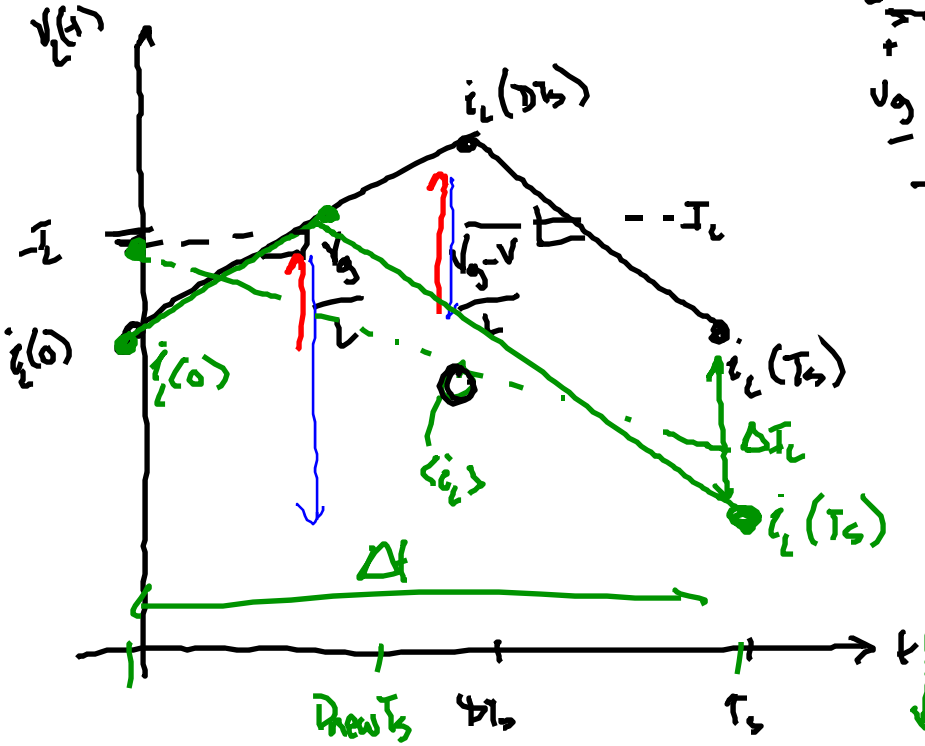
Major Duties:

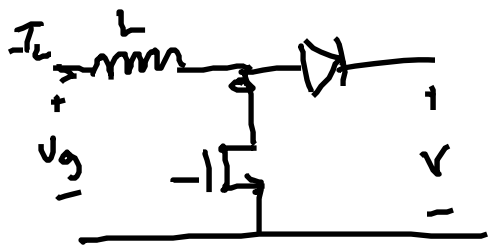
- -Responsible for substation/transmission procedure development to support plant human performance and work control requirements.
- -Insure work orders, work scheduling, Plant Issues, Root Cause / Apparent Cause Evaluations, and SOER reviews are completed to support station requirements.
- -Provide departmental representation at required station meetings as required by the Nuclear Generation scheduling processes.
- -Perform secondary engineering review of work prior to work evolutions.
- -Develop necessary training to support System Protection technicians during unit outages.
- -Support Electrical Equipment Specialist in substation inspections.
- -Work with transmission training and methods support personnel.
- -Attend applicable manufacturing training, review vendor manufacturing facilities and attend industry professional meetings as needed.
- -Perform other duties as assigned.

Is it full-time or part-time? Full time Where is it located? Mineral, Va. What are the required qualifications and the desired qualifications? B.S. EE required. Power systems experience/coursework highly desirable

If interested, send a resume to Prof. Liu, Liu@utk.edu ASAP.

Transient Volt-Second Balance





for an inductor

$$V_L = L \frac{di_L}{dt}$$

$$i_L(T_s) = \underbrace{i_L(0) + \frac{V_g}{L} D T_s}_{i_L(DT_s)} + \frac{V_g - V}{L} D' T_s$$

$$i_L(T_s) - i_L(0) = \Delta I_L = \frac{V_g}{L} D T_s + T_s \frac{V_g - V}{L} D'$$

If we are in steady-state

$$\Delta I_L = \emptyset = \frac{T_s}{L} \left(D V_g + D' V_g - D' V \right)$$

$$\emptyset = \left(V_g - D' V \right)$$

If we are not in steady-state

$$\Delta I_L = \frac{V_g}{L} D_{new} T_s + \frac{V_g - V}{L} D'_{new} T_s$$

$$L \frac{\Delta I_L}{\Delta t} = \left(V_g - D'_{new} V \right) = \langle v_L(t) \rangle |_{T_s}$$

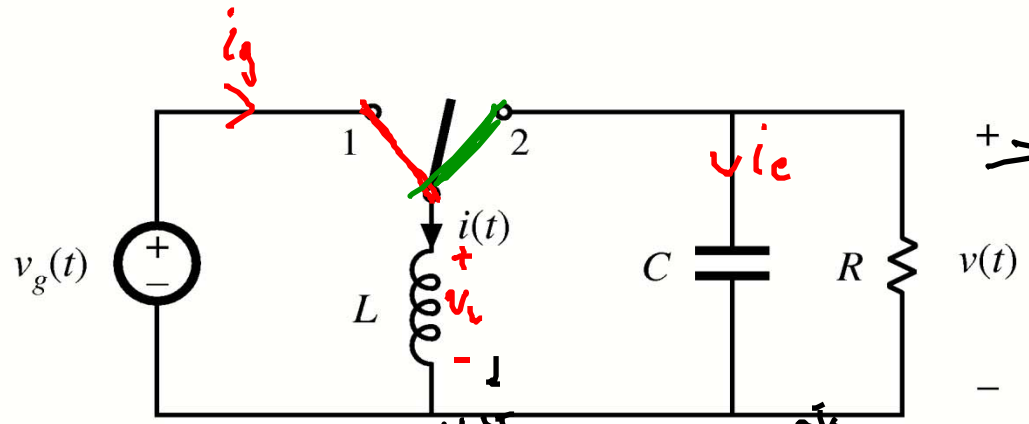
$\Delta t = T_s$

$$L \frac{d \langle i_L \rangle}{dt} \approx \langle v_L \rangle$$

9

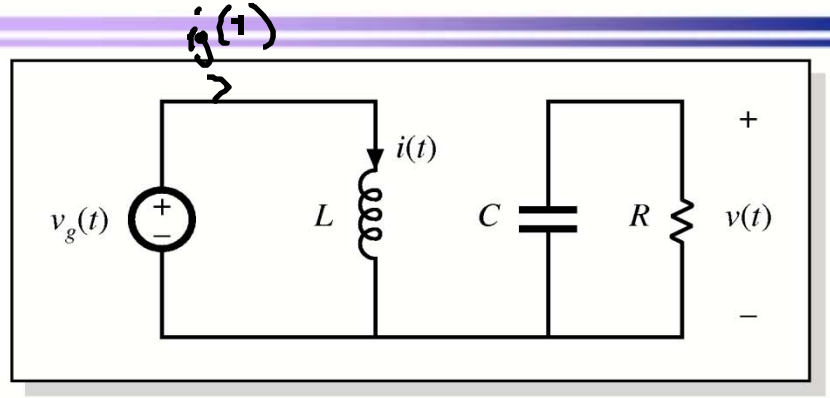
7.2. The basic AC modeling approach

Buck-boost converter example



$$\langle v_L(t) \rangle = L \frac{d\langle i_L(t) \rangle}{dt} = \frac{1}{T_s} \int_0^{d(t)T_s} v_g(t)\,dt + \frac{1}{T_s} \int_{d(t)T_s}^{T_s} v(t)\,dt$$

Switch in position 1



Inductor voltage and capacitor current are:

$$v_L(t) = L\frac{di(t)}{dt} = v_g(t)$$

$$i_C(t) = C\frac{dv(t)}{dt} = -\frac{v(t)}{R}$$

Small ripple approximation: replace waveforms with their low-frequency averaged values:

$$v_L(t) = L\frac{di(t)}{dt} \approx \left\langle v_g(t)\right\rangle_{T_s}$$

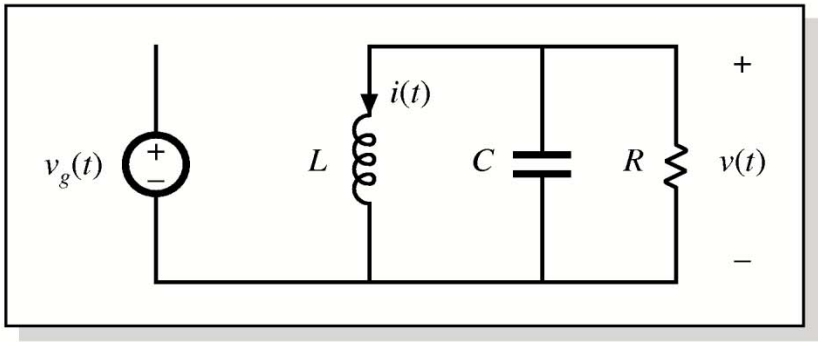
$$i_C(t) = C\frac{dv(t)}{dt} \approx -\frac{\left\langle v(t)\right\rangle_{T_s}}{R}$$

Switch in position 2

Inductor voltage and capacitor current are:

$$v_L(t) = L\frac{di(t)}{dt} = v(t)$$

$$i_C(t) = C\frac{dv(t)}{dt} = -i(t) - \frac{v(t)}{R}$$



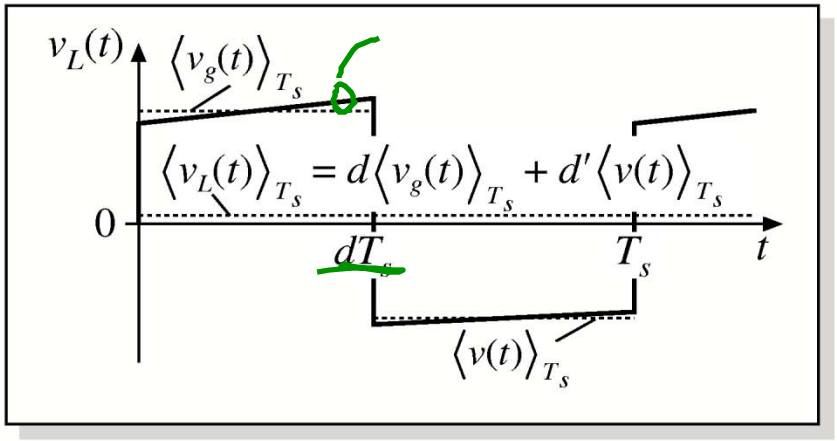
Small ripple approximation: replace waveforms with their low-frequency averaged values:

$$v_L(t) = L\frac{di(t)}{dt} \approx \langle v(t) \rangle_{T_s}$$

$$i_C(t) = C\frac{dv(t)}{dt} \approx -\langle i(t) \rangle_{T_s} - \frac{\langle v(t) \rangle_{T_s}}{R}$$

7.2.1 Averaging the inductor waveforms

Inductor voltage waveform



Low-frequency average is found by evaluation of

$$\left\langle x_L(t) \right\rangle_{T_s} = \frac{1}{T_s} \int_t^{t+T_s} x(\tau) d\tau$$

Average the inductor voltage in this manner:

$$\left\langle v_L(t) \right\rangle_{T_s} = \frac{1}{T_s} \int_t^{t+T_s} v_L(\tau) d\tau \approx d(t) \left\langle v_g(t) \right\rangle_{T_s} + d'(t) \left\langle v(t) \right\rangle_{T_s}$$

Insert into Eq. (7.2):

$$L \frac{d \left\langle i(t) \right\rangle_{T_s}}{dt} = d(t) \left\langle v_g(t) \right\rangle_{T_s} + d'(t) \left\langle v(t) \right\rangle_{T_s}$$

This equation describes how the low-frequency components of the inductor waveforms evolve in time.

(1)

$v_L(t) = V_g(t)$

$i_C(t) = -\frac{v(t)}{R}$

$i_g(t) = i_1(t)$

(2)

$v_L(t) = v(t)$

$i_C(t) = -\frac{v(t)}{R} - i_1(t)$ ←

$i_g(t) = 0$

↓

$$\Rightarrow \quad \langle v_L(t) \rangle = L\frac{d\langle i_L(t)\rangle}{dt} = \frac{1}{T_s}\int_0^{d(t)T_s} v_g(t)\,dt + \frac{1}{T_s}\int_{d(t)T_s}^{T_s} v(t)\,dt$$

$$L\frac{d\langle i_L\rangle}{dt} \approx d(t)\langle v_g\rangle + d'(t)\langle v\rangle$$

$$C\frac{d\langle v\rangle}{dt} \approx -d'(t)\langle i_L\rangle - \frac{\langle v\rangle}{R}$$

$$\langle i_g\rangle \approx d(t)\langle i_L\rangle$$

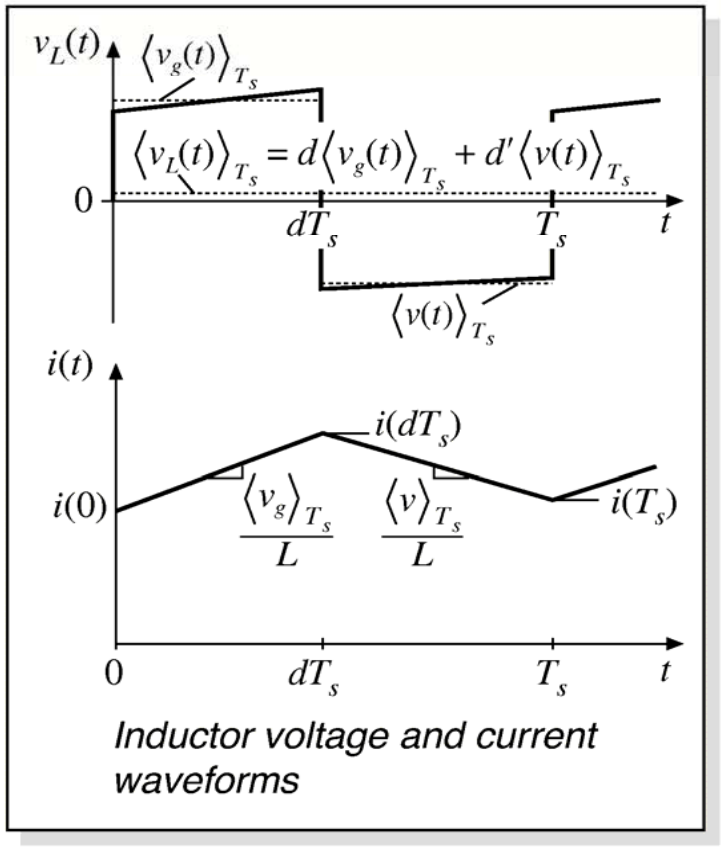
} Averaged, large-signal + small-signal nonlinear Equations

7.2.2 Discussion of the averaging approximation

Use of the average inductor voltage allows us to determine the net change in inductor current over one switching period, while neglecting the switching ripple.

In steady-state, the average inductor voltage is zero (volt-second balance), and hence the inductor current waveform is periodic: $i(t + T_s) = i(t)$. There is no net change in inductor current over one switching period.

During transients or ac variations, the average inductor voltage is not zero in general, and this leads to net variations in inductor current.



Inductor voltage and current waveforms

Net change in inductor current is correctly predicted by the average inductor voltage

Inductor equation:

$$L\frac{di(t)}{dt} = v_L(t)$$

Divide by L and integrate over one switching period:

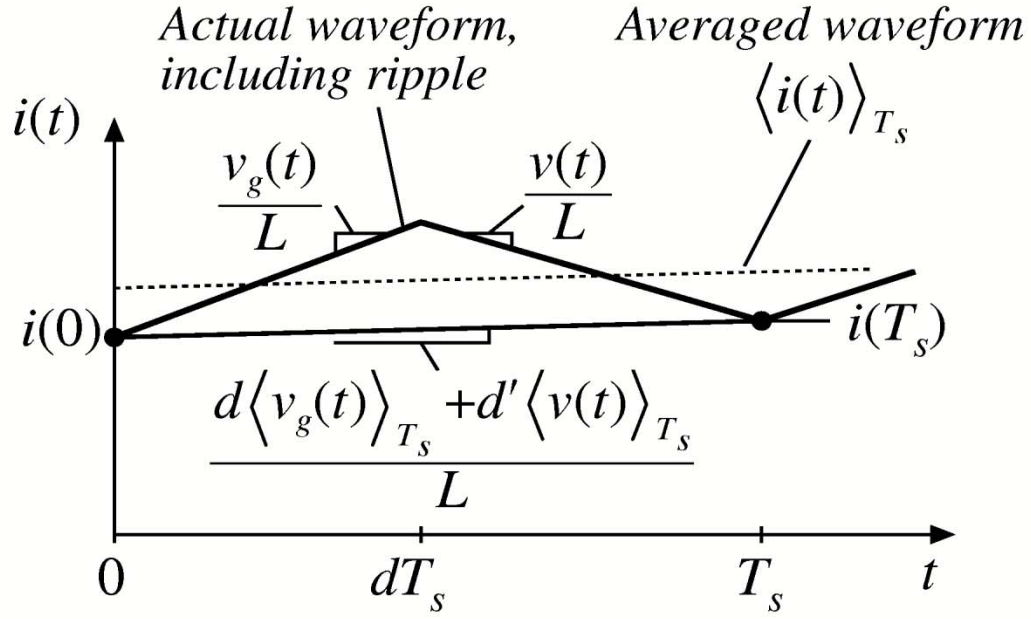
$$\int_t^{t+T_s} di = \frac{1}{L}\int_t^{t+T_s} v_L(\tau)d\tau$$

Left-hand side is the change in inductor current. Right-hand side can be related to average inductor voltage by multiplying and dividing by T_s as follows:

$$i(t+T_s) - i(t) = \frac{1}{L}\,T_s\left\langle v_L(t)\right\rangle_{T_s}$$

So the net change in inductor current over one switching period is exactly equal to the period T_s multiplied by the average slope $\langle v_L \rangle_{T_s}/L$.

Average inductor voltage correctly predicts average slope of $i_L(t)$



The net change in inductor current over one switching period is exactly equal to the period T_s multiplied by the average slope $\langle v_L \rangle_{T_s} / L$.

$$\frac{d\left\langle i(t)\right\rangle_{T_s}}{dt}$$

We have

$$i(t+T_s) - i(t) = \frac{1}{L}\, T_s \left\langle v_L(t)\right\rangle_{T_s}$$

Rearrange:

$$L\,\frac{i(t+T_s) - i(t)}{T_s} = \left\langle v_L(t)\right\rangle_{T_s}$$

Define the derivative of $\langle\, i \,\rangle_{T_s}$ as:

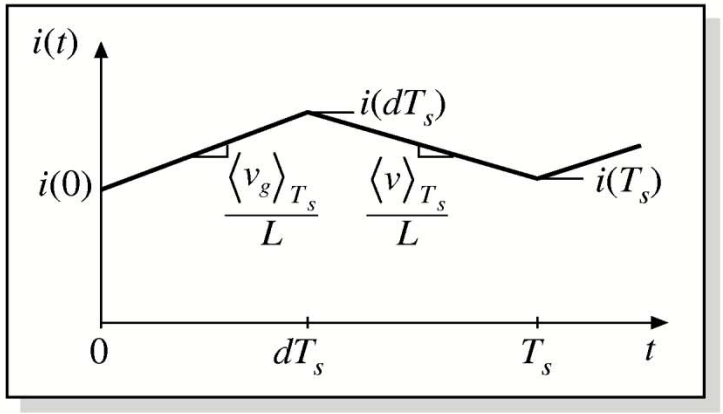
$$\frac{d\left\langle i(t)\right\rangle_{T_s}}{dt} = \frac{d}{dt}\left(\frac{1}{T_s}\int_t^{t+T_s} i(\tau)d\tau\right) = \frac{i(t+T_s) - i(t)}{T_s}$$

Hence,

$$L\,\frac{d\left\langle i(t)\right\rangle_{T_s}}{dt} = \left\langle v_L(t)\right\rangle_{T_s}$$

Computing how the inductor current changes over one switching period

Let's compute the actual inductor current waveform, using the linear ripple approximation.



With switch in position 1:

$$\underbrace{i(dT_s)}_{\text{(final value)}} = \underbrace{i(0)}_{\text{(initial value)}} + \underbrace{(dT_s)}_{\text{(length of interval)}} \underbrace{\left(\frac{\langle v_g(t)\rangle_{T_s}}{L}\right)}_{\text{(average slope)}}$$

With switch in position 2:

$$\underbrace{i(T_s)}_{\text{(final value)}} = \underbrace{i(dT_s)}_{\text{(initial value)}} + \underbrace{(d'T_s)}_{\text{(length of interval)}} \underbrace{\left(\frac{\langle v(t)\rangle_{T_s}}{L}\right)}_{\text{(average slope)}}$$

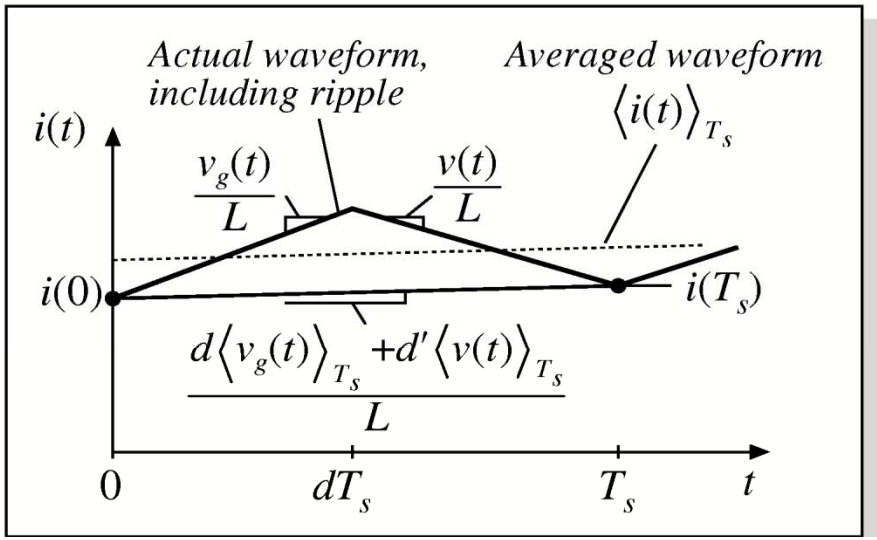
Net change in inductor current over one switching period

Eliminate $i(dT_s)$, to express $i(T_s)$ directly as a function of $i(0)$:

$$i(T_s) = i(0) + \frac{T_s}{L} \underbrace{\left(d(t) \left\langle v_g(t) \right\rangle_{T_s} + d'(t) \left\langle v(t) \right\rangle_{T_s} \right)}_{\left\langle v_L(t) \right\rangle_{T_s}}$$

The intermediate step of computing $i(dT_s)$ is eliminated.

The final value $i(T_s)$ is equal to the initial value $i(0)$, plus the switching period Ts multiplied by the average slope $\langle v_L \rangle_{T_s} / L$.



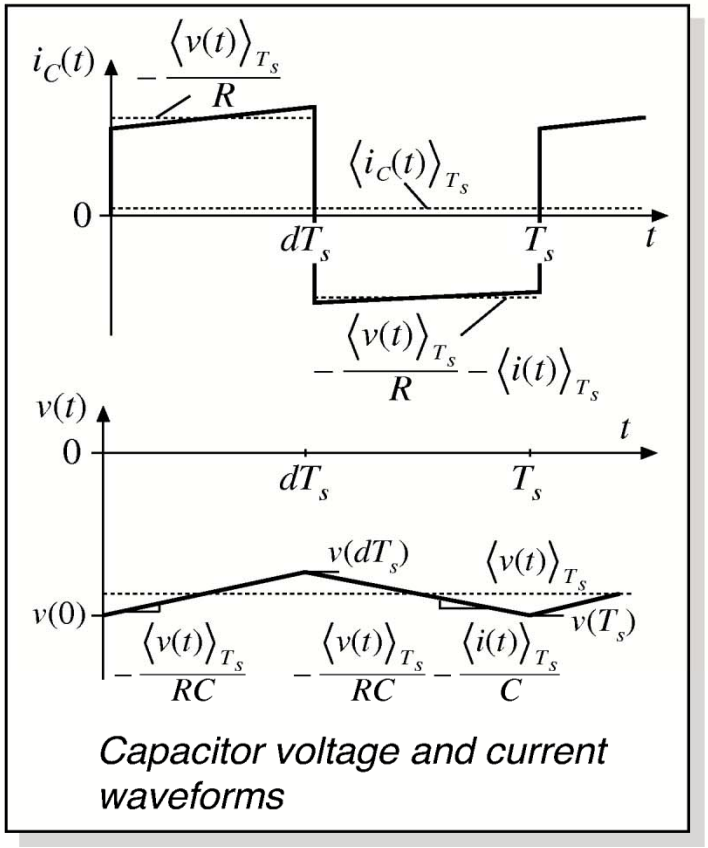
7.2.3 Averaging the capacitor waveforms

Average capacitor current:

$$\left\langle i_C(t)\right\rangle_{T_s} = d(t)\left(-\frac{\left\langle v(t)\right\rangle_{T_s}}{R}\right) + d'(t)\left(-\left\langle i(t)\right\rangle_{T_s} - \frac{\left\langle v(t)\right\rangle_{T_s}}{R}\right)$$

Collect terms, and equate to $C\, d\langle v \rangle_{T_s}/dt$:

$$C\frac{d\left\langle v(t)\right\rangle_{T_s}}{dt} = -d'(t)\left\langle i(t)\right\rangle_{T_s} - \frac{\left\langle v(t)\right\rangle_{T_s}}{R}$$



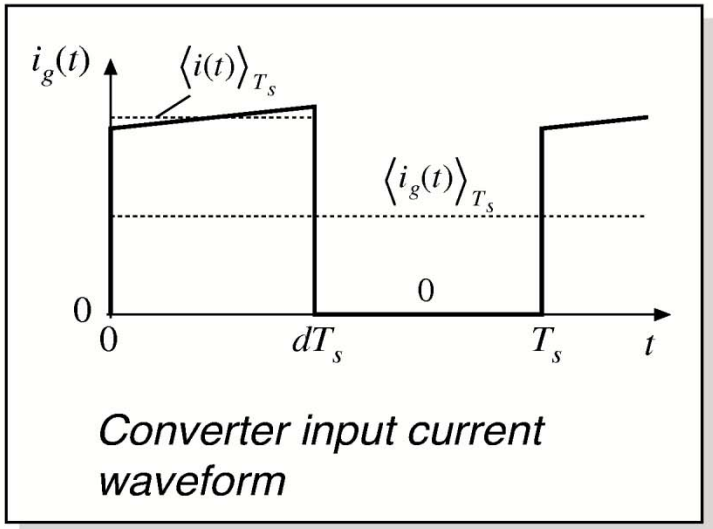
Capacitor voltage and current waveforms

7.2.4 The average input current

We found in Chapter 3 that it was sometimes necessary to write an equation for the average converter input current, to derive a complete dc equivalent circuit model. It is likewise necessary to do this for the ac model.

Buck-boost input current waveform is

$$i_g(t) = \begin{cases} \langle i(t) \rangle_{T_s} & \text{during subinterval 1} \\ 0 & \text{during subinterval 2} \end{cases}$$



Converter input current waveform

Average value:

$$\langle i_g(t) \rangle_{T_s} = d(t) \langle i(t) \rangle_{T_s}$$

7.2.5. Perturbation and linearization

Converter averaged equations:

$$\begin{cases} L\dfrac{d\langle i(t)\rangle_{T_s}}{dt} = d(t)\,\langle v_g(t)\rangle_{T_s} + d'(t)\,\langle v(t)\rangle_{T_s} \\ C\dfrac{d\langle v(t)\rangle_{T_s}}{dt} = -\,d'(t)\,\langle i(t)\rangle_{T_s} - \dfrac{\langle v(t)\rangle_{T_s}}{R} \end{cases}$$

$$\langle i_g(t)\rangle_{T_s} = d(t)\,\langle i(t)\rangle_{T_s}$$

—nonlinear because of multiplication of the time-varying quantity $d(t)$ with other time-varying quantities such as $i(t)$ and $v(t)$.

2 ways to linearize:

(1) perturb & linearize

(2) 1st order Taylor series

$$L\frac{d\langle i(t)\rangle_{T_s}}{dt} = d(t)\langle v_g(t)\rangle_{T_s} + d'(t)\langle v(t)\rangle_{T_s}$$

$d'(t) = 1 - d(t)$

$\langle v(t)\rangle_{T_s} = \hat{v}$

Perturb:

$\langle i(t)\rangle = I + \hat{i}$

$\langle v_g\rangle = V_g + \hat{v}_g$

$\langle v\rangle = V + \hat{v}$

$d(t) = D + \hat{d}$

$$L\frac{d}{dt}(I+\hat{i}) = (D+\hat{d})(V_g+\hat{v}_g) + (1-D-\hat{d})(V+\hat{v})$$

$$L\frac{d}{dt}I + L\frac{d}{dt}\hat{i} = DV_g + D\hat{v}_g + V_g\hat{d} + \hat{v}_g\hat{d} + V - DV - V\hat{d} + \hat{v} - D\hat{v} - \hat{d}\hat{v}$$

$L\frac{d}{dt}I = 0$

Neglect higher order terms (product of small signal terms)

→ Linearizing the system

→ Justification: $\hat{x}$ is small, $\hat{y}$ is small, $\hat{x}\hat{y}$ is very small

$(1-D)V$

DC:
$$0 = DV_g + V - DV = DV_g + D'V$$
→ steady-state
Averaged Large-signal (DC)
Linear

AC:
$$L\frac{d\hat{i}}{dt} = D\hat{v}_g + \hat{v} - D\hat{v} + V_g\hat{d} - V\hat{d}$$

$$L\frac{d\hat{i}}{dt} = D\hat{v}_g + D'\hat{v} + \hat{d}(V_g - V)$$
→ Averaged, small-signal linear equation

Construct small-signal model: Linearize about quiescent operating point

If the converter is driven with some steady-state, or quiescent, inputs

$$d(t) = D$$
$$\left\langle v_g(t) \right\rangle_{T_s} = V_g$$

then, from the analysis of Chapter 2, after transients have subsided the inductor current, capacitor voltage, and input current

$$\left\langle i(t) \right\rangle_{T_s}, \left\langle v(t) \right\rangle_{T_s}, \left\langle i_g(t) \right\rangle_{T_s}$$

reach the quiescent values I, V, and I_g, given by the steady-state analysis as

$$V = -\frac{D}{D'} V_g$$
$$I = -\frac{V}{D' R}$$
$$I_g = D I$$

Perturbation

So let us assume that the input voltage and duty cycle are equal to some given (dc) quiescent values, plus superimposed small ac variations:

$$\left\langle v_g(t) \right\rangle_{T_s} = V_g + \hat{v}_g(t)$$
$$d(t) = D + \hat{d}(t)$$

In response, and after any transients have subsided, the converter dependent voltages and currents will be equal to the corresponding quiescent values, plus small ac variations:

$$\left\langle i(t) \right\rangle_{T_s} = I + \hat{i}(t)$$
$$\left\langle v(t) \right\rangle_{T_s} = V + \hat{v}(t)$$
$$\left\langle i_g(t) \right\rangle_{T_s} = I_g + \hat{i}_g(t)$$

The small-signal assumption

If the ac variations are much smaller in magnitude than the respective quiescent values,

$$\left|\hat{v}_g(t)\right| << \left|V_g\right|$$
$$\left|\hat{d}(t)\right| << \left|D\right|$$
$$\left|\hat{i}(t)\right| << \left|I\right|$$
$$\left|\hat{v}(t)\right| << \left|V\right|$$
$$\left|\hat{i}_g(t)\right| << \left|I_g\right|$$

then the nonlinear converter equations can be linearized.

Perturbation of inductor equation

Insert the perturbed expressions into the inductor differential equation:

$$L\frac{d\left(I+\hat{i}(t)\right)}{dt}=\left(D+\hat{d}(t)\right)\left(V_g+\hat{v}_g(t)\right)+\left(D'-\hat{d}(t)\right)\left(V+\hat{v}(t)\right)$$

note that $d'(t)$ is given by

$$d'(t)=\left(1-d(t)\right)=1-\left(D+\hat{d}(t)\right)=D'-\hat{d}(t)$$

with $D' = 1 - D$

Multiply out and collect terms:

$$L\left(\frac{dI}{dt}^{0}+\frac{d\hat{i}(t)}{dt}\right)=\underbrace{\left(DV_g+D'V\right)}_{\text{Dc terms}}+\underbrace{\left(D\hat{v}_g(t)+D'\hat{v}(t)+\left(V_g-V\right)\hat{d}(t)\right)}_{1^{st}\text{ order ac terms (linear)}}+\underbrace{\hat{d}(t)\left(\hat{v}_g(t)-\hat{v}(t)\right)}_{2^{nd}\text{ order ac terms (nonlinear)}}$$

The perturbed inductor equation

$$L\left(\cancel{\frac{dI}{dt}}^{0} + \frac{d\hat{i}(t)}{dt}\right) = \underbrace{\left(DV_g + D'V\right)}_{\text{Dc terms}} + \underbrace{\left(D\hat{v}_g(t) + D'\hat{v}(t) + \left(V_g - V\right)\hat{d}(t)\right)}_{1^{st}\text{ order ac terms (linear)}} + \underbrace{\hat{d}(t)\left(\hat{v}_g(t) - \hat{v}(t)\right)}_{2^{nd}\text{ order ac terms (nonlinear)}}$$

Since I is a constant (dc) term, its derivative is zero

The right-hand side contains three types of terms:

- Dc terms, containing only dc quantities
- First-order ac terms, containing a single ac quantity, usually multiplied by a constant coefficient such as a dc term. These are linear functions of the ac variations
- Second-order ac terms, containing products of ac quantities. These are nonlinear, because they involve multiplication of ac quantities

Neglect of second-order terms

$$L\left(\frac{dI^{0}}{dt}+\frac{d\hat{i}(t)}{dt}\right)=\underbrace{\left(DV_g+D'V\right)}_{\text{Dc terms}}+\underbrace{\left(D\hat{v}_g(t)+D'\hat{v}(t)+\left(V_g-V\right)\hat{d}(t)\right)}_{1^{st}\text{ order ac terms (linear)}}+\underbrace{\hat{d}(t)\left(\hat{v}_g(t)-\hat{v}(t)\right)}_{2^{nd}\text{ order ac terms (nonlinear)}}$$

Provided $\left|\hat{v}_g(t)\right| << \left|V_g\right|$

$\left|\hat{d}(t)\right| << \left|D\right|$

$\left|\hat{i}(t)\right| << \left|I\right|$

$\left|\hat{v}(t)\right| << \left|V\right|$

$\left|\hat{i}_g(t)\right| << \left|I_g\right|$

then the second-order ac terms are much smaller than the first-order terms. For example,

$\left|\hat{d}(t)\,\hat{v}_g(t)\right| << \left|D\,\hat{v}_g(t)\right|$ when $\left|\hat{d}(t)\right| << D$

So neglect second-order terms.

Also, dc terms on each side of equation are equal.

Linearized inductor equation

Upon discarding second-order terms, and removing dc terms (which add to zero), we are left with

$$L \frac{d\hat{i}(t)}{dt} = D\hat{v}_g(t) + D'\hat{v}(t) + \left(V_g - V\right)\hat{d}(t)$$

This is the desired result: a linearized equation which describes small-signal ac variations.

Note that the quiescent values D, D', V, V_g, are treated as given constants in the equation.

$$C\frac{d\langle v(t)\rangle_{T_s}}{dt} = -d'(t)\langle i(t)\rangle_{T_s} - \frac{\langle v(t)\rangle_{T_s}}{R}$$

f_1

$1-d(t)$

1st order Taylor Series

$f_1(x,y)$ Expansion around (x_0, Y_0)

$$f_1(x_0+\hat{x}, Y_0+\hat{y}) = f_1(x_0, Y_0) + \left.\frac{\partial f_1}{\partial x}\right|_{(x_0,Y_0)}\hat{x} + \left.\frac{\partial f_1}{\partial y}\right|_{(x_0,Y_0)}\hat{y} + \left.\frac{1}{2!}\frac{\partial f_1^2}{\partial x^2}\right|_{(x_0,Y_0)}\hat{x}^2 + \cdots$$

Linear terms

Higher order terms

$$C\frac{dV}{dt} + C\frac{d\hat{v}}{dt} \simeq -D'I - \frac{V}{R} + -D'\hat{i} - \frac{1}{R}\hat{v} + I\hat{d}$$

$\left.\frac{\partial f_1}{\partial i}\right|_{DC}$ $\left.\frac{\partial f_1}{\partial v}\right|_{DC}$ $\left.\frac{\partial f_1}{\partial d}\right|_{DC}$

DC: $0 = -D'I - \frac{V}{R}$

AC: $C\frac{d\hat{v}}{dt} = -D'\hat{i} - \hat{v}\frac{1}{R} + I\hat{d}$

Capacitor equation

Perturbation leads to

$$C\frac{d\left(V+\hat{v}(t)\right)}{dt}=-\left(D'-\hat{d}(t)\right)\left(I+\hat{i}(t)\right)-\frac{\left(V+\hat{v}(t)\right)}{R}$$

Collect terms:

$$C\left(\cancel{\frac{dV}{dt}}^{0}+\frac{d\hat{v}(t)}{dt}\right)=\underbrace{\left(-D'I-\frac{V}{R}\right)}_{Dc\ terms}+\underbrace{\left(-D'\hat{i}(t)-\frac{\hat{v}(t)}{R}+I\hat{d}(t)\right)}_{1^{st}\ order\ ac\ terms\ (linear)}+\underbrace{\hat{d}(t)\hat{i}(t)}_{2^{nd}\ order\ ac\ term\ (nonlinear)}$$

Neglect second-order terms. Dc terms on both sides of equation are equal. The following terms remain:

$$C\frac{d\hat{v}(t)}{dt}=-D'\hat{i}(t)-\frac{\hat{v}(t)}{R}+I\hat{d}(t)$$

This is the desired small-signal linearized capacitor equation.

f_2

$$\left\langle i_g(t)\right\rangle_{T_s} = d(t)\left\langle i(t)\right\rangle_{T_s}$$

$$\hat{i}_g = \left.\frac{\partial f_2}{\partial d}\right|_{DC}\hat{d} + \left.\frac{\partial f_2}{\partial i}\right|_{DC}\hat{i}$$

Taylor Series (neglect DC)

$$\hat{i}_g = I\hat{d} + D\hat{i}$$

$$I_g + \hat{i}_g = (D + \hat{d})(I + \hat{i})$$

$$I_g + \hat{i}_g = DI + D\hat{i} + I\hat{d} + \cancel{\hat{d}\hat{i}}$$

Perturb & Linearize

$$\begin{cases} I_g = DI \\ \hat{i}_g = D\hat{i} + I\hat{d} \end{cases}$$

Average input current

Perturbation leads to

$$I_g + \hat{i}_g(t) = \left(D + \hat{d}(t)\right)\left(I + \hat{i}(t)\right)$$

Collect terms:

$$\underbrace{I_g}_{\text{Dc term}} + \underbrace{\hat{i}_g(t)}_{1^{st}\text{ order ac term}} = \underbrace{(DI)}_{\text{Dc term}} + \underbrace{\left(D\hat{i}(t) + I\hat{d}(t)\right)}_{1^{st}\text{ order ac terms (linear)}} + \underbrace{\hat{d}(t)\hat{i}(t)}_{2^{nd}\text{ order ac term (nonlinear)}}$$

Neglect second-order terms. Dc terms on both sides of equation are equal. The following first-order terms remain:

$$\hat{i}_g(t) = D\hat{i}(t) + I\hat{d}(t)$$

This is the linearized small-signal equation which described the converter input port.

7.2.6. Construction of small-signal equivalent circuit model

The linearized small-signal converter equations:

$$L \frac{d\hat{i}(t)}{dt} = D\hat{v}_g(t) + D'\hat{v}(t) + \left(V_g - V\right)\hat{d}(t)$$

$$C \frac{d\hat{v}(t)}{dt} = -D'\hat{i}(t) - \frac{\hat{v}(t)}{R} + I\hat{d}(t)$$

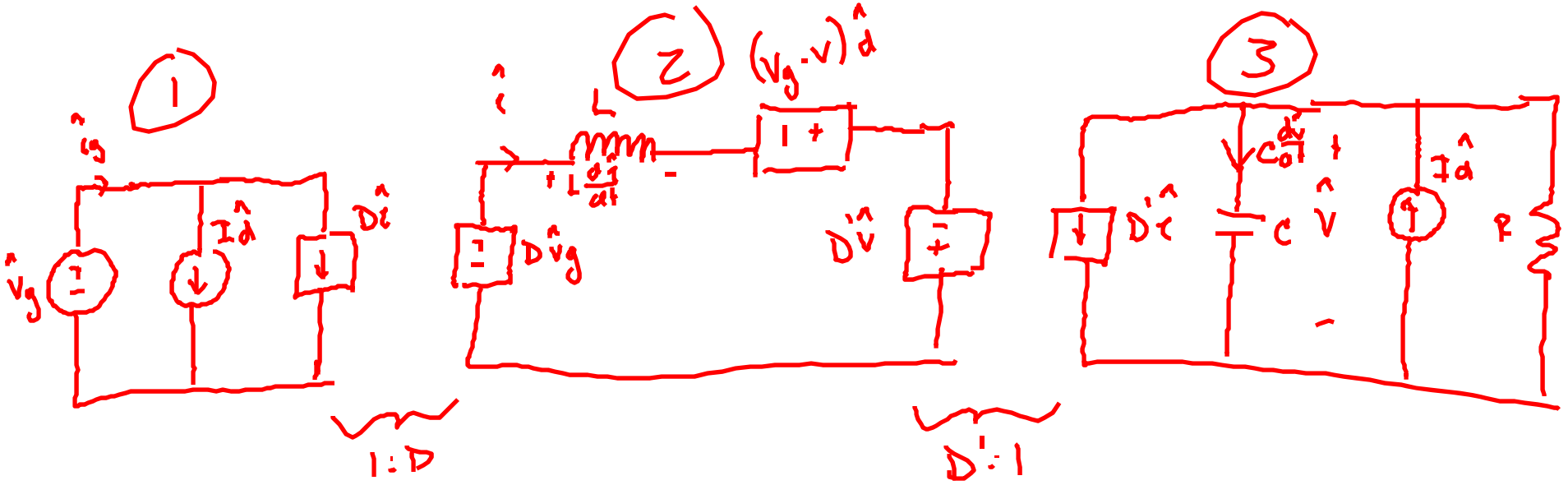
$$\hat{i}_g(t) = D\hat{i}(t) + I\hat{d}(t)$$

Reconstruct equivalent circuit corresponding to these equations, in manner similar to the process used in Chapter 3.

$$L\frac{d\hat{i}(t)}{dt} = D\hat{v}_g(t) + D'\hat{v}(t) + \left(V_g - V\right)\hat{d}(t)$$

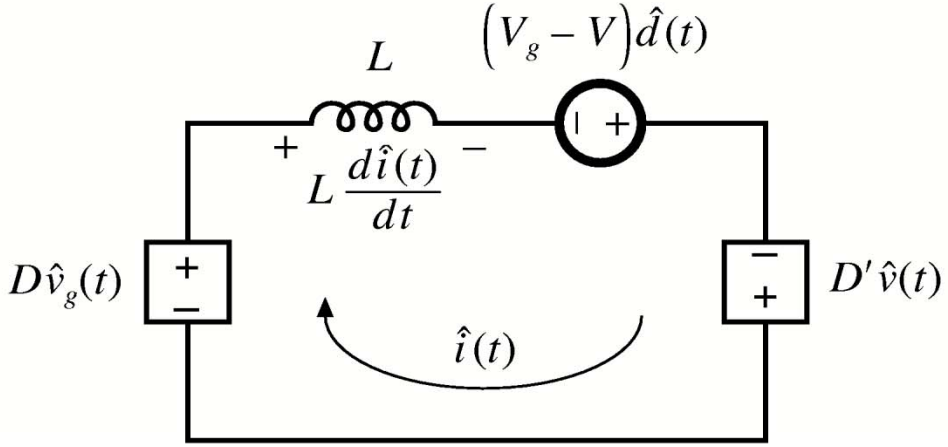
$$C\frac{d\hat{v}(t)}{dt} = -D'\hat{i}(t) - \frac{\hat{v}(t)}{R} + I\hat{d}(t)$$

$$\hat{i}_g(t) = D\hat{i}(t) + I\hat{d}(t)$$



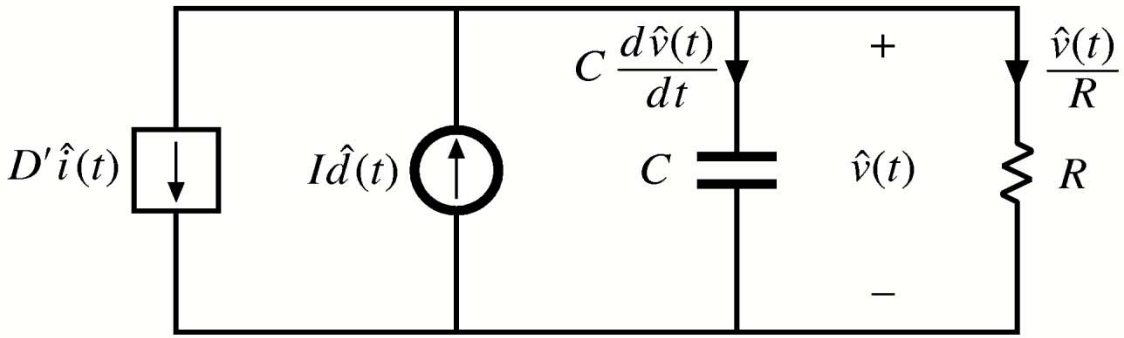
Inductor loop equation

$$L \frac{d\hat{i}(t)}{dt} = D\hat{v}_g(t) + D'\hat{v}(t) + \left(V_g - V\right)\hat{d}(t)$$



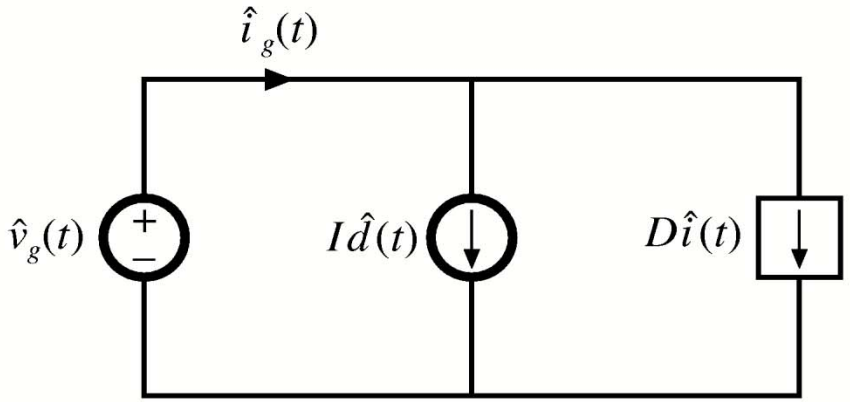
Capacitor node equation

$$C\frac{d\hat{v}(t)}{dt} = -D'\hat{i}(t) - \frac{\hat{v}(t)}{R} + I\hat{d}(t)$$



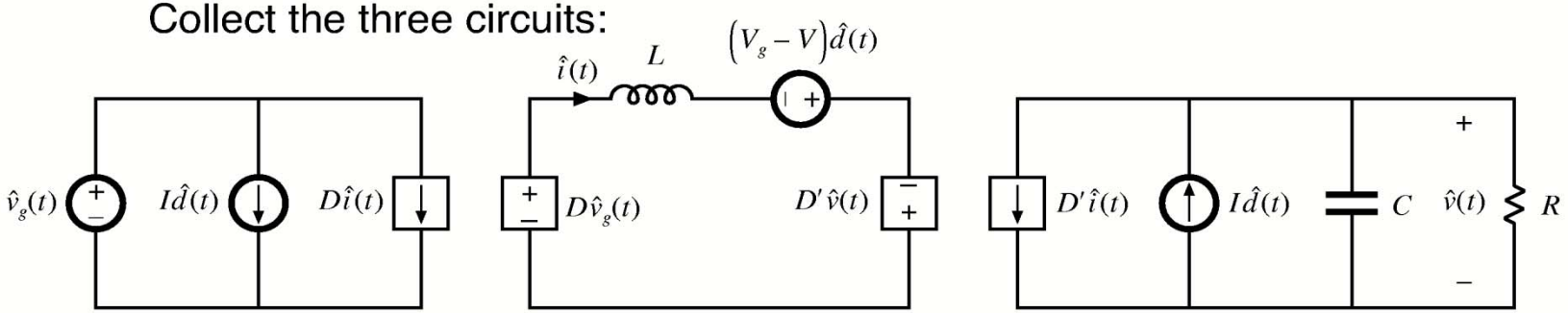
Input port node equation

$$\hat{i}_g(t) = D\hat{i}(t) + I\hat{d}(t)$$

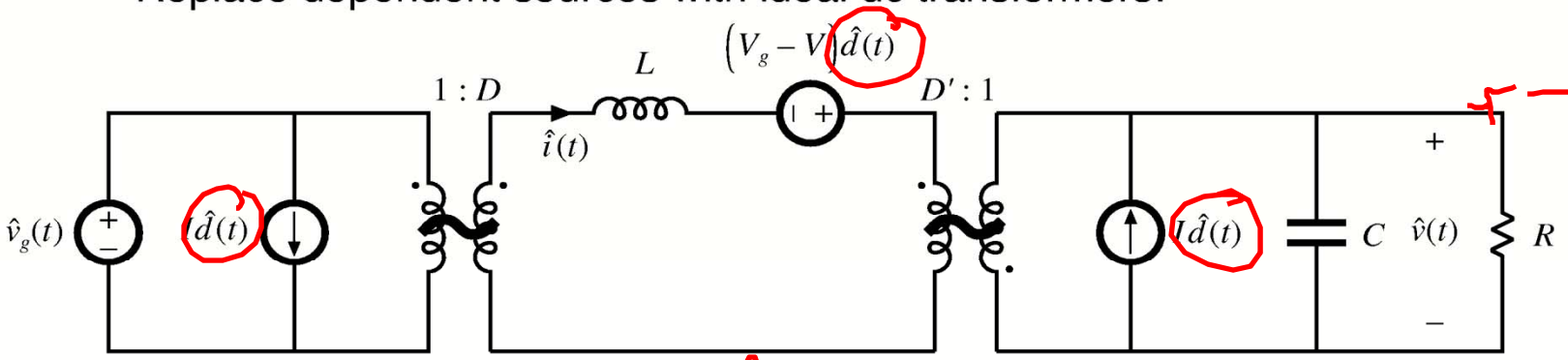


Complete equivalent circuit

Collect the three circuits:



Replace dependent sources with ideal dc transformers:



Small-signal ac equivalent circuit model of the buck-boost converter

7.2.7 Discussion of the perturbation and linearization step

The linearization step amounts to taking the Taylor expansion of the original nonlinear equation, about a quiescent operating point, and retaining only the constant and linear terms.

Inductor equation, buck-boost example:

$$L \frac{d\left\langle i(t) \right\rangle_{T_s}}{dt} = d(t) \left\langle v_g(t) \right\rangle_{T_s} + d'(t) \left\langle v(t) \right\rangle_{T_s} = f_1\left(\left\langle v_g(t) \right\rangle_{T_s}, \left\langle v(t) \right\rangle_{T_s}, d(t) \right)$$

Three-dimensional Taylor series expansion:

$$L\left(\frac{dI}{dt} + \frac{d\hat{i}(t)}{dt} \right) = f_1\left(V_g, V, D \right) + \hat{v}_g(t) \left. \frac{\partial f_1\left(v_g, V, D \right)}{\partial v_g} \right|_{v_g = V_g}$$

$$+ \hat{v}(t) \left. \frac{\partial f_1\left(V_g, v, D \right)}{\partial v} \right|_{v = V} + \hat{d}(t) \left. \frac{\partial f_1\left(V_g, V, d \right)}{\partial d} \right|_{d = D}$$

+ higher-order nonlinear terms

Linearization via Taylor series

Equate DC terms:

$$0 = f_1\left(V_g, V, D\right)$$

Coefficients of linear terms are:

$$\left.\frac{\partial f_1\left(v_g, V, D\right)}{\partial v_g}\right|_{v_g = V_g} = D$$

$$\left.\frac{\partial f_1\left(V_g, v, D\right)}{\partial v}\right|_{v = V} = D'$$

$$\left.\frac{\partial f_1\left(V_g, V, d\right)}{\partial d}\right|_{d = D} = V_g - V$$

$$L\left(\frac{dI}{dt} + \frac{d\hat{i}(t)}{dt}\right) = f_1\left(V_g, V, D\right) + \hat{v}_g(t) \left.\frac{\partial f_1\left(v_g, V, D\right)}{\partial v_g}\right|_{v_g = V_g}$$

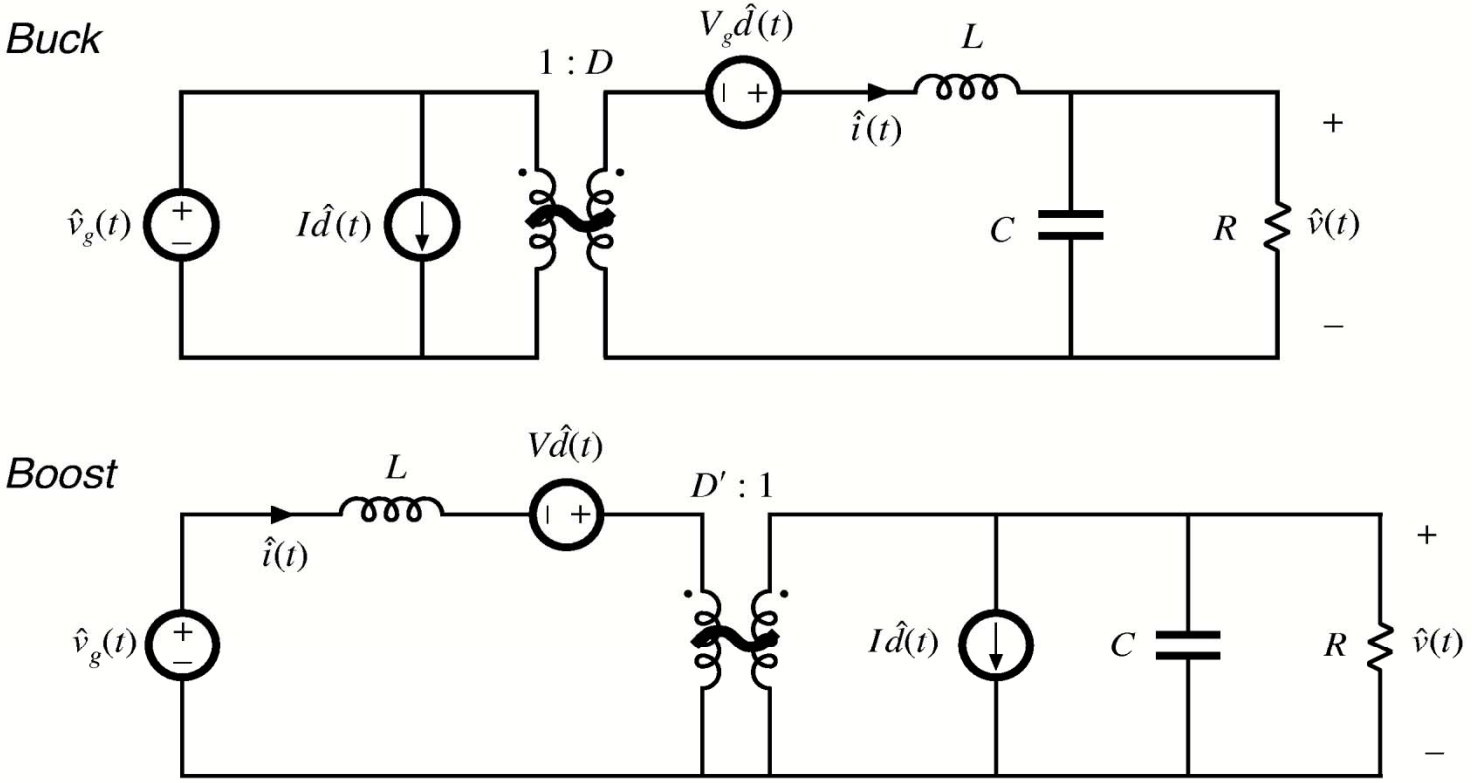
$$+ \hat{v}(t) \left.\frac{\partial f_1\left(V_g, v, D\right)}{\partial v}\right|_{v = V} + \hat{d}(t) \left.\frac{\partial f_1\left(V_g, V, d\right)}{\partial d}\right|_{d = D}$$

+ higher-order nonlinear terms

Hence the small-signal ac linearized equation is:

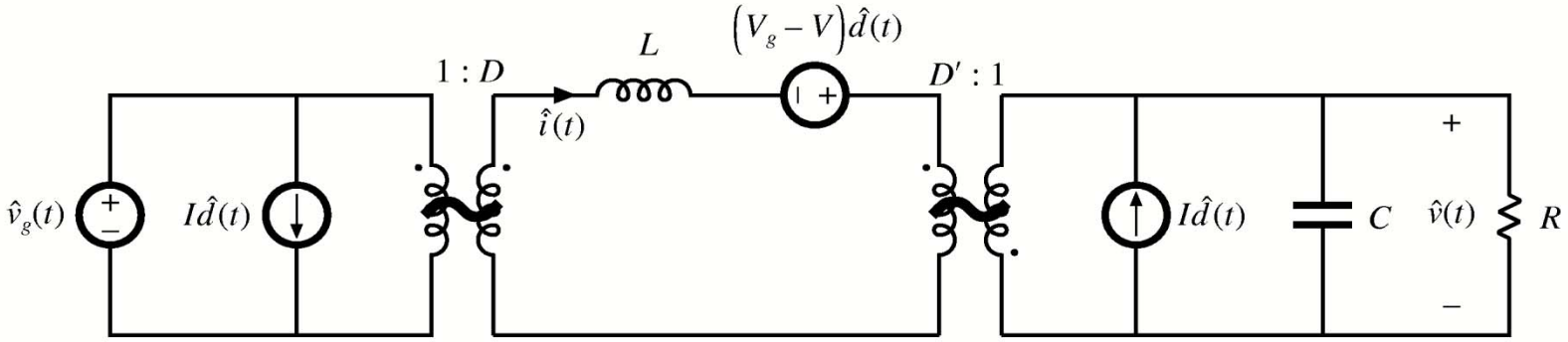
$$L\frac{d\hat{i}(t)}{dt} = D\hat{v}_g(t) + D'\hat{v}(t) + \left(V_g - V\right)\hat{d}(t)$$

7.2.8. Results for several basic converters



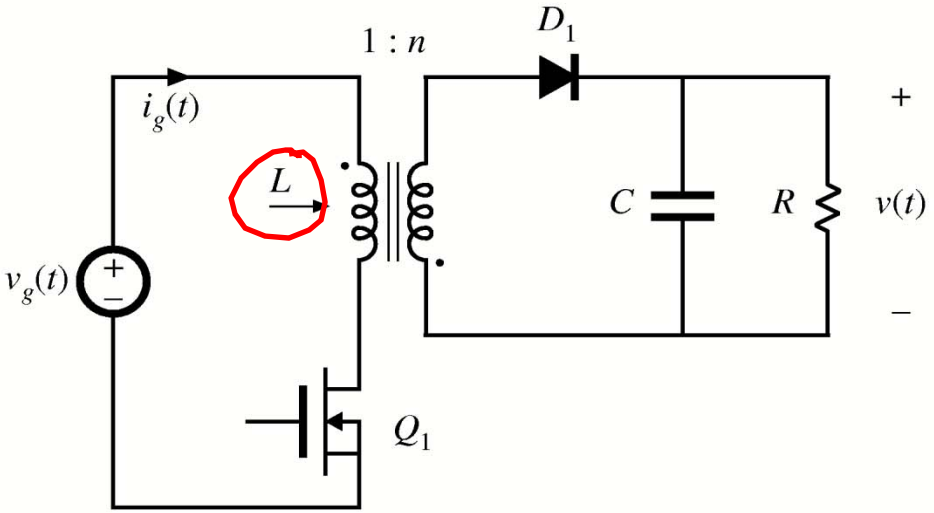
Results for several basic converters

Buck-boost



7.2.9 Example: a nonideal flyback converter

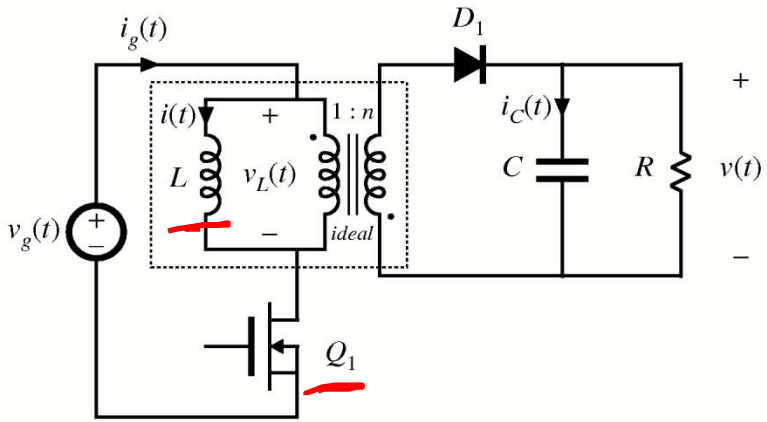
Flyback converter example



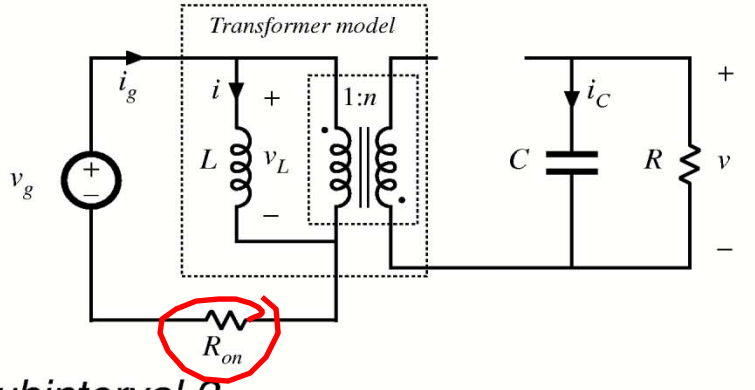
- MOSFET has on-resistance R_{on}
- Flyback transformer has magnetizing inductance L, referred to primary

Circuits during subintervals 1 and 2

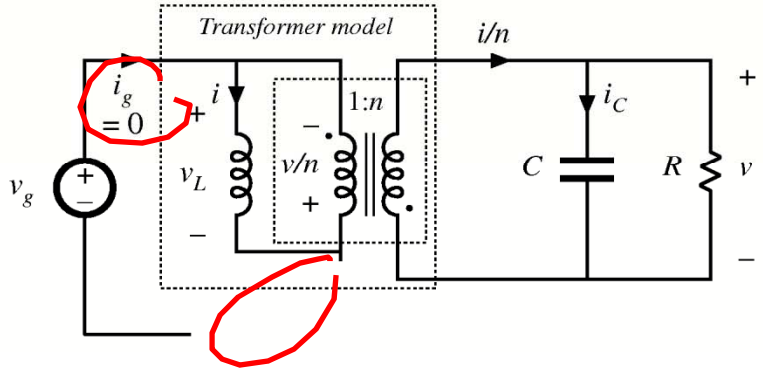
Flyback converter, with transformer equivalent circuit



Subinterval 1



Subinterval 2



Subinterval 1

Circuit equations:

$$v_L(t) = v_g(t) - i(t)\, R_{on}$$

$$i_C(t) = -\frac{v(t)}{R}$$

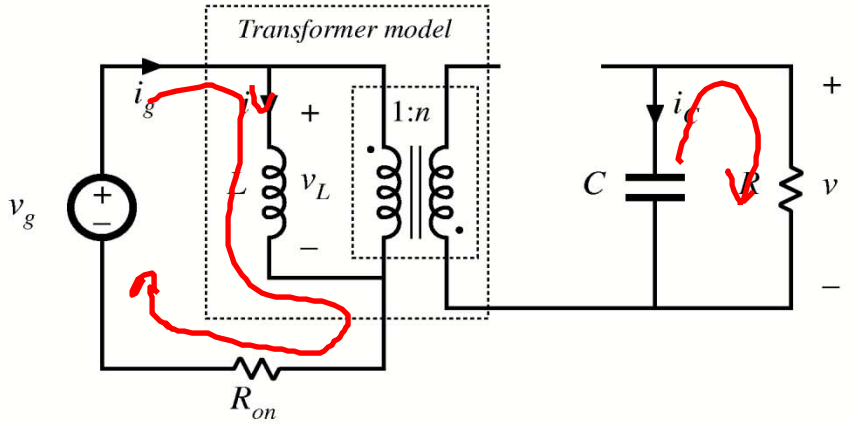
$$i_g(t) = i(t)$$

Small ripple approximation:

$$v_L(t) = \left\langle v_g(t) \right\rangle_{T_s} - \left\langle i(t) \right\rangle_{T_s} R_{on}$$

$$i_C(t) = -\frac{\left\langle v(t) \right\rangle_{T_s}}{R}$$

$$i_g(t) = \left\langle i(t) \right\rangle_{T_s}$$



MOSFET conducts, diode is reverse-biased

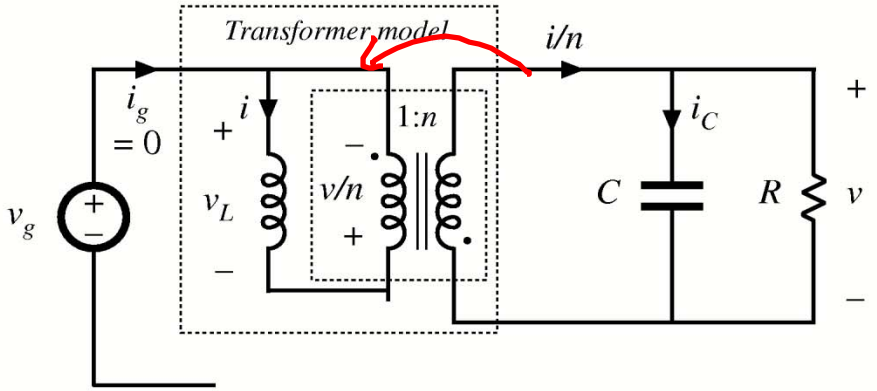
Subinterval 2

Circuit equations:

$$v_L(t) = -\frac{v(t)}{n}$$

$$i_C(t) = -\frac{i(t)}{n} - \frac{v(t)}{R}$$

$$i_g(t) = 0$$



Small ripple approximation:

$$v_L(t) = -\frac{\langle v(t) \rangle_{T_s}}{n}$$

$$i_C(t) = -\frac{\langle i(t) \rangle_{T_s}}{n} - \frac{\langle v(t) \rangle_{T_s}}{R}$$

$$i_g(t) = 0$$

MOSFET is off, diode conducts

$$\langle v_L \rangle = L\frac{d\langle i \rangle}{dt} = d(t)\left(\langle v_g \rangle - \langle i \rangle R_{on}\right) + d'(t)\left(-\frac{\langle v \rangle}{n}\right)$$

$$\langle i_C \rangle = C\frac{d\langle v \rangle}{dt} = -\frac{\langle v \rangle}{R} + d'(t)\frac{\langle i \rangle}{n}$$

$$\langle i_g \rangle = d(t)\langle i \rangle$$

Linearize:

$$L\frac{d\hat{i}}{dt} = \hat{d}\left(V_g - IR_{on} + \frac{V}{n}\right) + \hat{v}_g D + \hat{i}\left(-DR_{on}\right) + \hat{v}\left(\frac{-D'}{n}\right)$$

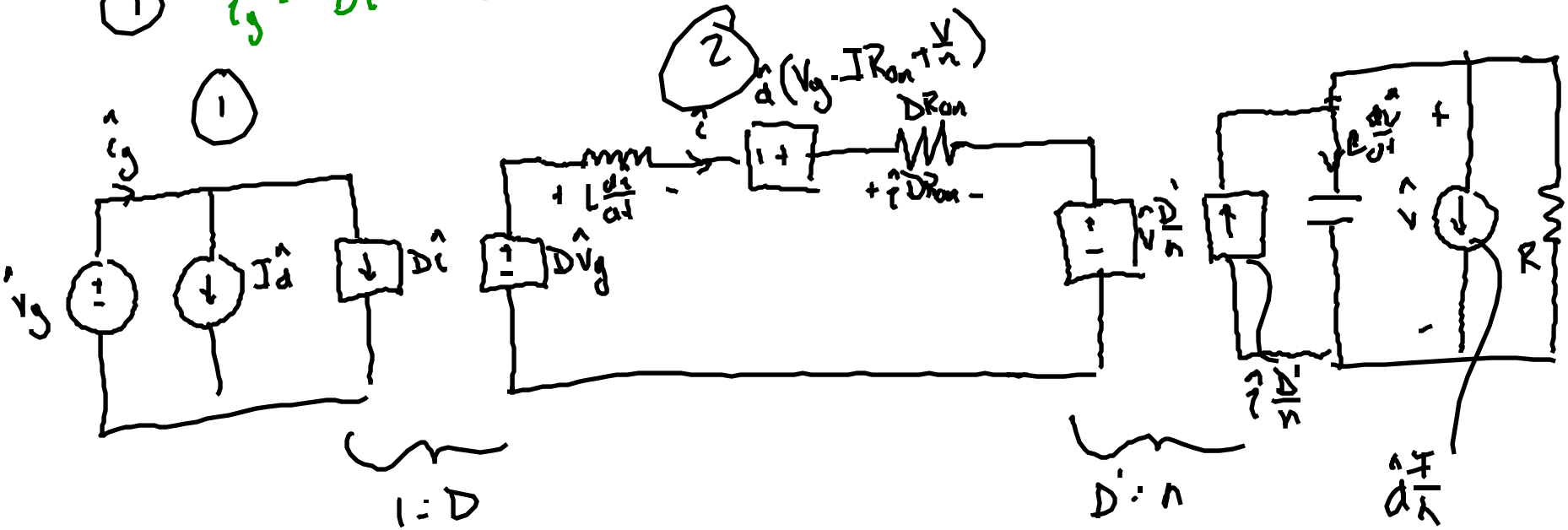
$$C\frac{d\hat{v}}{dt} = \hat{v}\left(-\frac{1}{R}\right) + \hat{i}\left(\frac{+D'}{n}\right) - \hat{d}\left(\frac{I}{n}\right)$$

$$\hat{i}_g = D\hat{i} + I\hat{d}$$

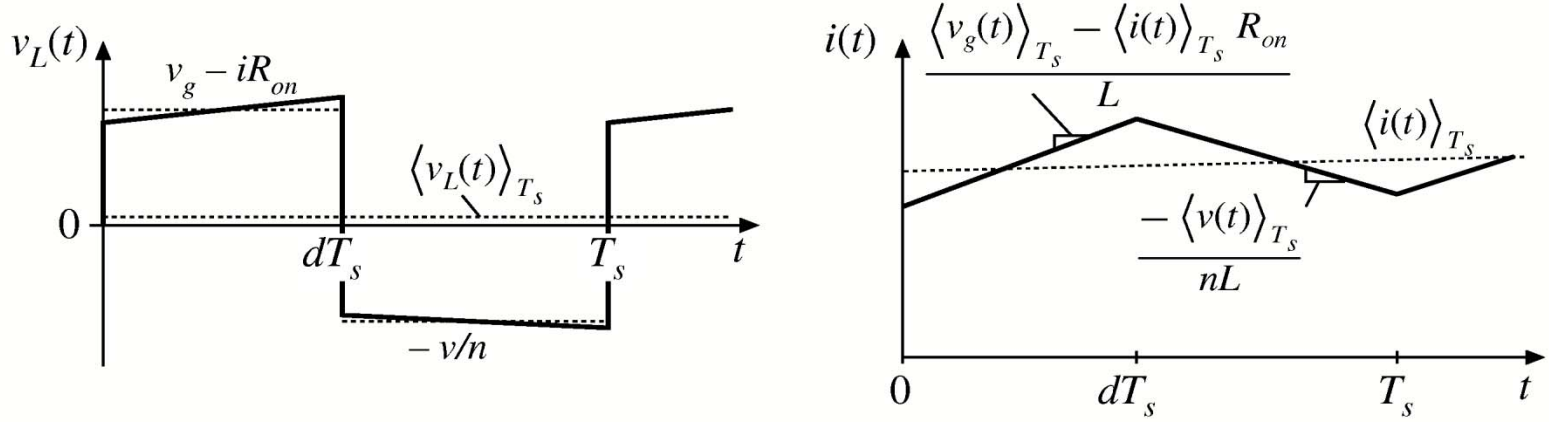
$$L\frac{d\hat{i}}{dt} = \hat{d}\left(V_g - IR_{on} + \frac{V}{n}\right) + \hat{v}_g D + \hat{i}\left(-DR_{on}\right) + \hat{v}\left(\frac{-D'}{n}\right)$$

$$C\frac{d\hat{v}}{dt} = \hat{v}\left(-\frac{1}{R}\right) + \hat{i}\left(\frac{+D'}{n}\right) - \hat{d}\left(\frac{I}{n}\right)$$

$$\hat{i}_g = D\hat{i} + I\hat{d}$$



Inductor waveforms



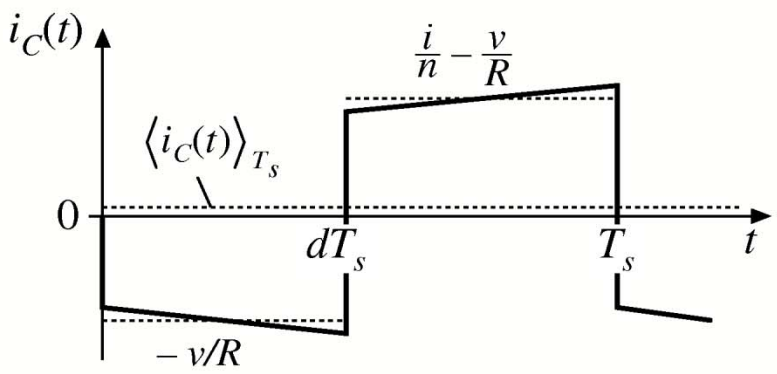
Average inductor voltage:

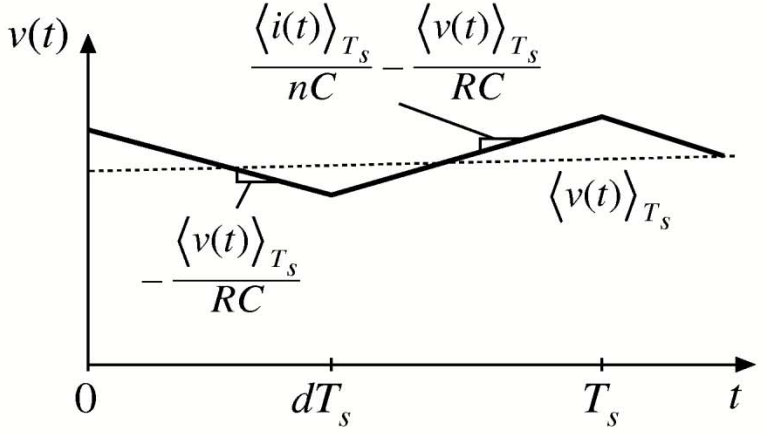
$$\left\langle v_L(t)\right\rangle_{T_s} = d(t)\left(\left\langle v_g(t)\right\rangle_{T_s} - \left\langle i(t)\right\rangle_{T_s} R_{on}\right) + d'(t)\left(\frac{-\left\langle v(t)\right\rangle_{T_s}}{n}\right)$$

Hence, we can write:

$$L\frac{d\left\langle i(t)\right\rangle_{T_s}}{dt} = d(t)\left\langle v_g(t)\right\rangle_{T_s} - d(t)\left\langle i(t)\right\rangle_{T_s} R_{on} - d'(t)\frac{\left\langle v(t)\right\rangle_{T_s}}{n}$$

Capacitor waveforms





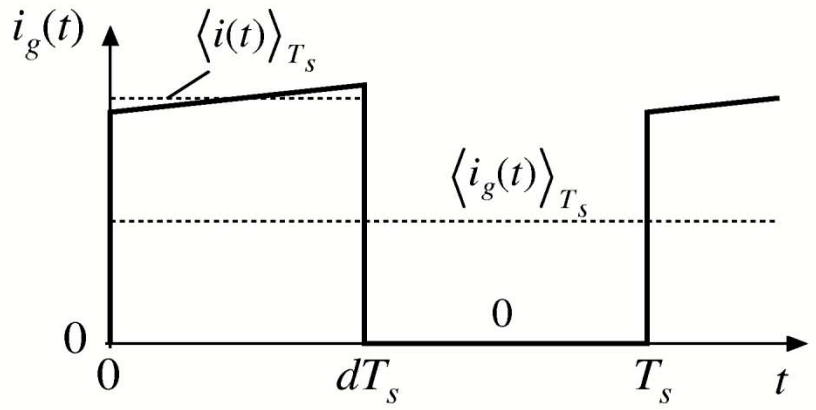
Average capacitor current:

$$\langle i_C(t) \rangle_{T_s} = d(t) \left(\frac{-\langle v(t) \rangle_{T_s}}{R} \right) + d'(t) \left(\frac{\langle i(t) \rangle_{T_s}}{n} - \frac{\langle v(t) \rangle_{T_s}}{R} \right)$$

Hence, we can write:

$$C \frac{d\langle v(t) \rangle_{T_s}}{dt} = d'(t) \frac{\langle i(t) \rangle_{T_s}}{n} - \frac{\langle v(t) \rangle_{T_s}}{R}$$

Input current waveform



Average input current:

$$\langle i_g(t) \rangle_{T_s} = d(t) \langle i(t) \rangle_{T_s}$$

The averaged converter equations

$$L \frac{d\left\langle i(t) \right\rangle_{T_s}}{dt} = d(t) \left\langle v_g(t) \right\rangle_{T_s} - d(t) \left\langle i(t) \right\rangle_{T_s} R_{on} - d'(t) \frac{\left\langle v(t) \right\rangle_{T_s}}{n}$$

$$C \frac{d\left\langle v(t) \right\rangle_{T_s}}{dt} = d'(t) \frac{\left\langle i(t) \right\rangle_{T_s}}{n} - \frac{\left\langle v(t) \right\rangle_{T_s}}{R}$$

$$\left\langle i_g(t) \right\rangle_{T_s} = d(t) \left\langle i(t) \right\rangle_{T_s}$$

— a system of nonlinear differential equations

Next step: perturbation and linearization. Let

$$\left\langle v_g(t) \right\rangle_{T_s} = V_g + \hat{v}_g(t)$$

$$d(t) = D + \hat{d}(t)$$

$$\left\langle i(t) \right\rangle_{T_s} = I + \hat{i}(t)$$

$$\left\langle v(t) \right\rangle_{T_s} = V + \hat{v}(t)$$

$$\left\langle i_g(t) \right\rangle_{T_s} = I_g + \hat{i}_g(t)$$

Perturbation of the averaged inductor equation

$$L\frac{d\left\langle i(t)\right\rangle_{T_s}}{dt} = d(t)\left\langle v_g(t)\right\rangle_{T_s} - d(t)\left\langle i(t)\right\rangle_{T_s} R_{on} - d'(t)\frac{\left\langle v(t)\right\rangle_{T_s}}{n}$$

$$L\frac{d\left(I + \hat{i}(t)\right)}{dt} = \left(D + \hat{d}(t)\right)\left(V_g + \hat{v}_g(t)\right) - \left(D' - \hat{d}(t)\right)\frac{\left(V + \hat{v}(t)\right)}{n} - \left(D + \hat{d}(t)\right)\left(I + \hat{i}(t)\right)R_{on}$$

$$L\left(\cancel{\frac{dI}{dt}}^{0} + \frac{d\hat{i}(t)}{dt}\right) = \underbrace{\left(DV_g - D'\frac{V}{n} - DR_{on}I\right)}_{Dc\ terms} + \underbrace{\left(D\hat{v}_g(t) - D'\frac{\hat{v}(t)}{n} + \left(V_g + \frac{V}{n} - IR_{on}\right)\hat{d}(t) - DR_{on}\hat{i}(t)\right)}_{1^{st}\ order\ ac\ terms\ (linear)}$$

$$+ \underbrace{\left(\hat{d}(t)\hat{v}_g(t) + \hat{d}(t)\frac{\hat{v}(t)}{n} - \hat{d}(t)\hat{i}(t)R_{on}\right)}_{2^{nd}\ order\ ac\ terms\ (nonlinear)}$$

Linearization of averaged inductor equation

Dc terms:

$$0 = DV_g - D'\frac{V}{n} - DR_{on}I$$

Second-order terms are small when the small-signal assumption is satisfied. The remaining first-order terms are:

$$L\frac{d\hat{i}(t)}{dt} = D\hat{v}_g(t) - D'\frac{\hat{v}(t)}{n} + \left(V_g + \frac{V}{n} - IR_{on}\right)\hat{d}(t) - DR_{on}\hat{i}(t)$$

This is the desired linearized inductor equation.

Perturbation of averaged capacitor equation

Original averaged equation:

$$C\frac{d\left\langle v(t)\right\rangle_{T_s}}{dt} = d'(t)\frac{\left\langle i(t)\right\rangle_{T_s}}{n} - \frac{\left\langle v(t)\right\rangle_{T_s}}{R}$$

Perturb about quiescent operating point:

$$C\frac{d\left(V + \hat{v}(t)\right)}{dt} = \left(D' - \hat{d}(t)\right)\frac{\left(I + \hat{i}(t)\right)}{n} - \frac{\left(V + \hat{v}(t)\right)}{R}$$

Collect terms:

$$C\left(\frac{dV}{dt}^{0} + \frac{d\hat{v}(t)}{dt}\right) = \underbrace{\left(\frac{D'I}{n} - \frac{V}{R}\right)}_{\text{Dc terms}} + \underbrace{\left(\frac{D'\hat{i}(t)}{n} - \frac{\hat{v}(t)}{R} - \frac{I\hat{d}(t)}{n}\right)}_{1^{st}\text{ order ac terms (linear)}} - \underbrace{\frac{\hat{d}(t)\hat{i}(t)}{n}}_{2^{nd}\text{ order ac term (nonlinear)}}$$

Linearization of averaged capacitor equation

Dc terms:

$$0 = \left(\frac{D'I}{n} - \frac{V}{R} \right)$$

Second-order terms are small when the small-signal assumption is satisfied. The remaining first-order terms are:

$$C \frac{d\hat{v}(t)}{dt} = \frac{D'\hat{i}(t)}{n} - \frac{\hat{v}(t)}{R} - \frac{I\hat{d}(t)}{n}$$

This is the desired linearized capacitor equation.

Perturbation of averaged input current equation

Original averaged equation:

$$\left\langle i_g(t) \right\rangle_{T_s} = d(t) \left\langle i(t) \right\rangle_{T_s}$$

Perturb about quiescent operating point:

$$I_g + \hat{i}_g(t) = \left(D + \hat{d}(t)\right)\left(I + \hat{i}(t)\right)$$

Collect terms:

$$\underbrace{I_g}_{\text{Dc term}} + \underbrace{\hat{i}_g(t)}_{1^{st}\text{ order ac term}} = \underbrace{(DI)}_{\text{Dc term}} + \underbrace{\left(D\hat{i}(t) + I\hat{d}(t)\right)}_{1^{st}\text{ order ac terms (linear)}} + \underbrace{\hat{d}(t)\hat{i}(t)}_{2^{nd}\text{ order ac term (nonlinear)}}$$

Linearization of averaged input current equation

Dc terms:

$$I_g = DI$$

Second-order terms are small when the small-signal assumption is satisfied. The remaining first-order terms are:

$$\hat{i}_g(t) = D\hat{i}(t) + I\hat{d}(t)$$

This is the desired linearized input current equation.

Summary: dc and small-signal ac converter equations

Dc equations:

$$0 = DV_g - D'\frac{V}{n} - DR_{on}I$$

$$0 = \left(\frac{D'I}{n} - \frac{V}{R}\right)$$

$$I_g = DI$$

Small-signal ac equations:

$$L\frac{d\hat{i}(t)}{dt} = D\hat{v}_g(t) - D'\frac{\hat{v}(t)}{n} + \left(V_g + \frac{V}{n} - IR_{on}\right)\hat{d}(t) - DR_{on}\hat{i}(t)$$

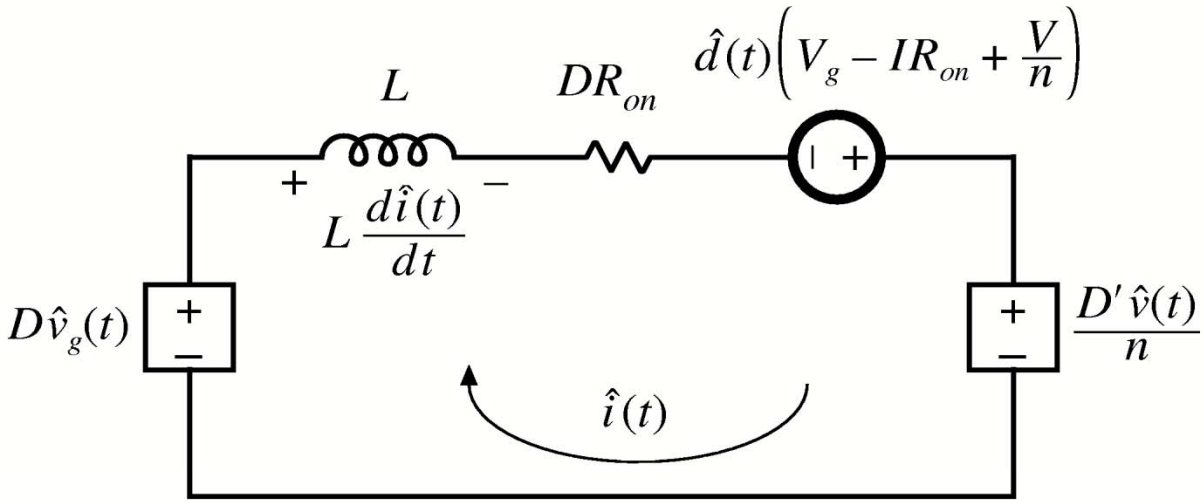
$$C\frac{d\hat{v}(t)}{dt} = \frac{D'\hat{i}(t)}{n} - \frac{\hat{v}(t)}{R} - \frac{I\hat{d}(t)}{n}$$

$$\hat{i}_g(t) = D\hat{i}(t) + I\hat{d}(t)$$

Next step: construct equivalent circuit models.

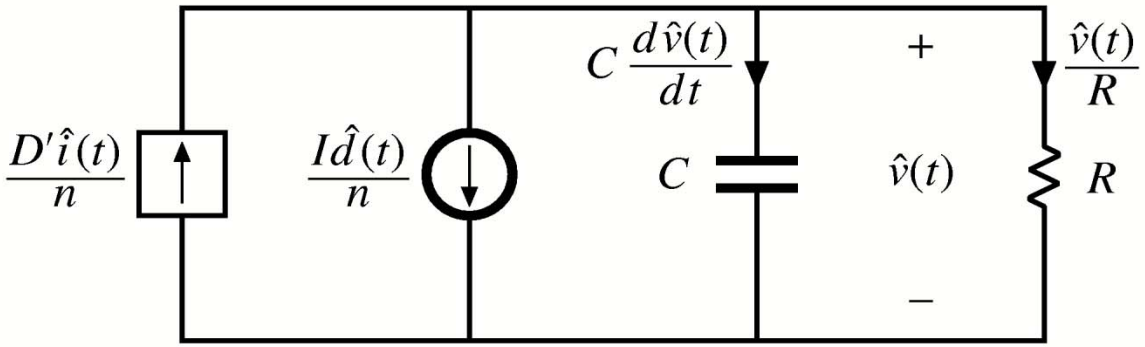
Small-signal ac equivalent circuit: inductor loop

$$L\frac{d\hat{i}(t)}{dt} = D\hat{v}_g(t) - D'\frac{\hat{v}(t)}{n} + \left(V_g + \frac{V}{n} - IR_{on}\right)\hat{d}(t) - DR_{on}\hat{i}(t)$$



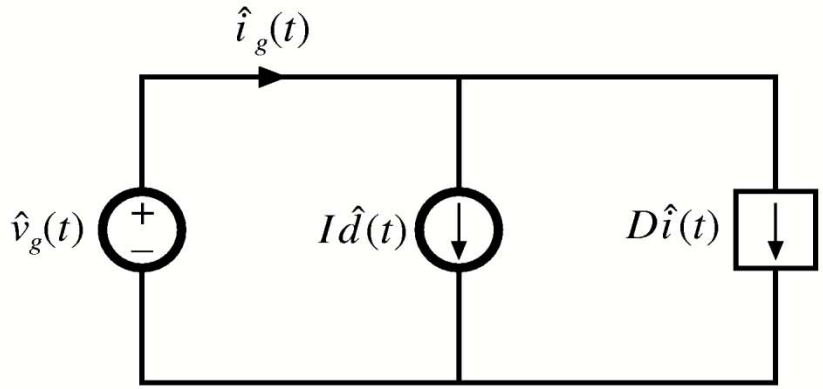
Small-signal ac equivalent circuit: capacitor node

$$C\frac{d\hat{v}(t)}{dt} = \frac{D'\hat{i}(t)}{n} - \frac{\hat{v}(t)}{R} - \frac{I\hat{d}(t)}{n}$$



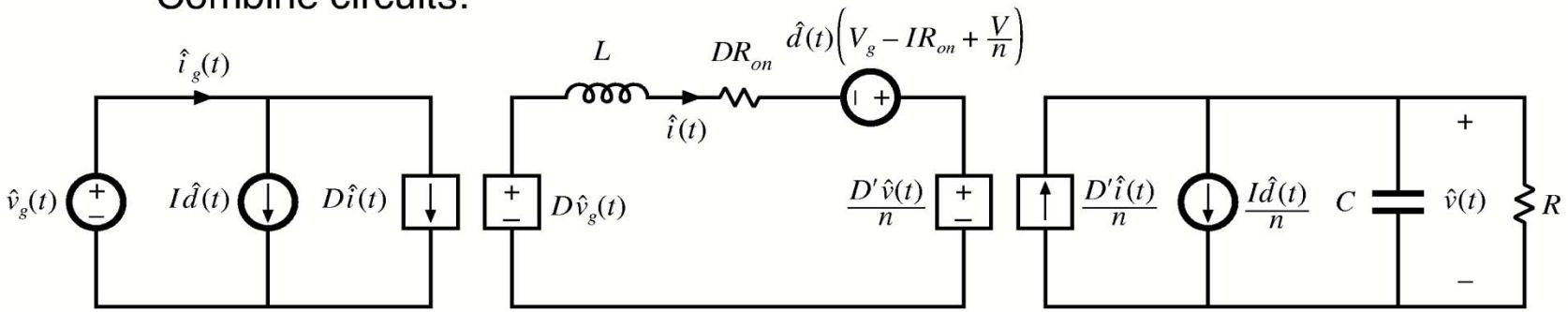
Small-signal ac equivalent circuit: converter input node

$$\hat{i}_g(t) = D\hat{i}(t) + I\hat{d}(t)$$



Small-signal ac model, nonideal flyback converter example

Combine circuits:



Replace dependent sources with ideal transformers: