The Low Q Approximation

Given a second-order denominator polynomial, of the form

$$G(s) = \frac{1}{1 + a_1 s + a_2 s^2} \qquad \text{or} \qquad G(s) = \frac{1}{1 + \frac{s}{Q\omega_0} + \left(\frac{s}{\omega_0}\right)^2}$$

When the roots are real, i.e., when $Q < 0.5$, then we can factor the denominator, and construct the Bode diagram using the asymptotes for real poles. We would then use the following normalized form:

$$G(s) = \frac{1}{\left(1 + \frac{s}{\omega_1}\right)\left(1 + \frac{s}{\omega_2}\right)}$$

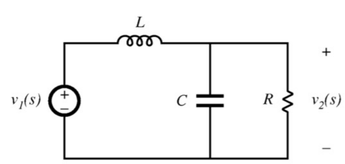
This is a particularly desirable approach when $Q << 0.5$, i.e., when the corner frequencies ω_1 and ω_2 are well separated.

L-C-R Example

A problem with this procedure is the complexity of the quadratic formula used to find the corner frequencies.

R-L-C network example:

$$G(s) = \frac{v_2(s)}{v_1(s)} = \frac{1}{1 + s\frac{L}{R} + s^2 LC}$$



Use quadratic formula to factor denominator. Corner frequencies are:

$$\omega_1, \omega_2 = \frac{L/R \pm \sqrt{\left(L/R\right)^2 - 4\,LC}}{2\,LC}$$

Factoring the Denominator

$$\omega_1, \omega_2 = \frac{L/R \pm \sqrt{(L/R)^2 - 4\,LC}}{2\,LC}$$

This complicated expression yields little insight into how the corner frequencies ω_1 and ω_2 depend on R, L, and C.

When the corner frequencies are well separated in value, it can be shown that they are given by the much simpler (approximate) expressions

$$\omega_1 \approx \frac{R}{L}, \qquad \omega_2 \approx \frac{1}{RC}$$

ω_1 is then independent of C, and ω_2 is independent of L.

These simpler expressions can be derived via the Low-Q Approximation.

Derivation of Low-*Q* Approximation

Given

$$G(s) = \frac{1}{1 + \frac{s}{Q\omega_0} + \left(\frac{s}{\omega_0}\right)^2}$$

Use quadratic formula to express corner frequencies ω_1 and ω_2 in terms of Q and ω_0 as:

$$\omega_1 = \frac{\omega_0}{Q}\,\frac{1 - \sqrt{1 - 4Q^2}}{2} \qquad\qquad \omega_2 = \frac{\omega_0}{Q}\,\frac{1 + \sqrt{1 - 4Q^2}}{2}$$

Corner Frequency ω_1

$$\omega_1 = \frac{\omega_0}{Q} \frac{1 - \sqrt{1 - 4Q^2}}{2}$$

can be written in the form

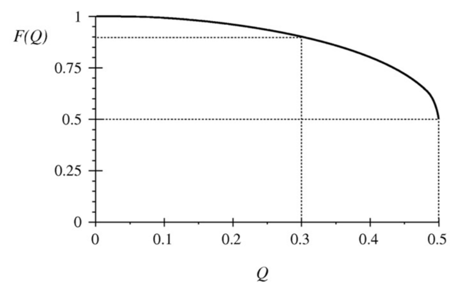
$$\omega_1 = \frac{Q\,\omega_0}{F(Q)}$$

where

$$F(Q) = \frac{1}{2}\left(1 + \sqrt{1 - 4Q^2}\right)$$

For small Q, $F(Q)$ tends to 1. We then obtain

$$\omega_1 \approx Q\,\omega_0 \quad \text{for } Q << \frac{1}{2}$$



For $Q < 0.3$, the approximation $F(Q) = 1$ is within 10% of the exact value.

Corner Frequency ω_2

$$\omega_2 = \frac{\omega_0}{Q} \frac{1 + \sqrt{1 - 4Q^2}}{2}$$

can be written in the form

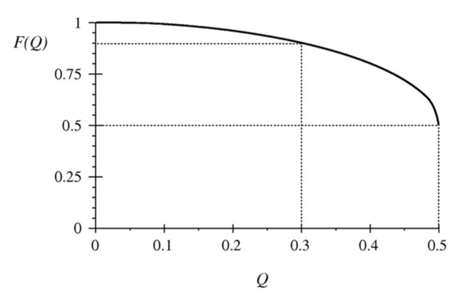
$$\omega_2 = \frac{\omega_0}{Q} F(Q)$$

where

$$F(Q) = \frac{1}{2}\left(1 + \sqrt{1 - 4Q^2}\right)$$

For small Q, $F(Q)$ tends to 1. We then obtain

$$\omega_2 \approx \frac{\omega_0}{Q} \quad \text{for } Q << \frac{1}{2}$$



For $Q < 0.3$, the approximation $F(Q) = 1$ is within 10% of the exact value.

Low-Q Approximation Results

For the previous example:

$$G(s) = \frac{v_2(s)}{v_1(s)} = \frac{1}{1 + s\frac{L}{R} + s^2 LC}$$

$$f_0 = \frac{\omega_0}{2\pi} = \frac{1}{2\pi\sqrt{LC}}$$

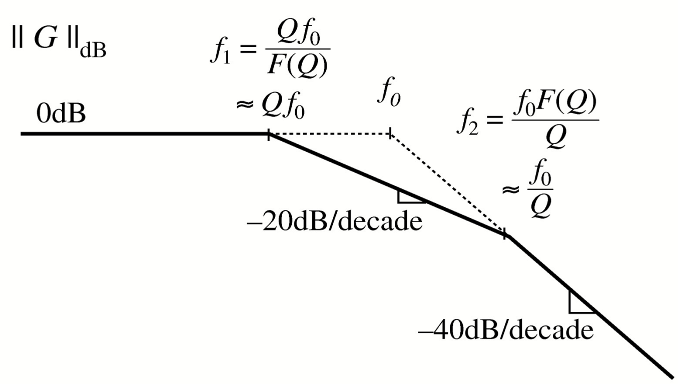
$$Q = R\sqrt{\frac{C}{L}}$$

Use of the Low-Q Approximation leads to

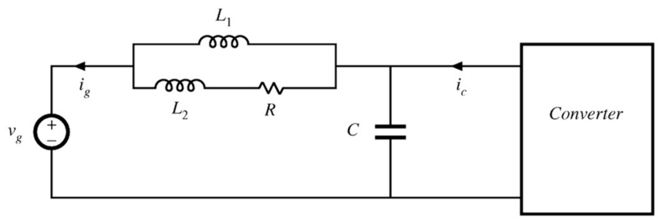
$$\omega_1 \approx Q\,\omega_0 = R\sqrt{\frac{C}{L}}\,\frac{1}{\sqrt{LC}} = \frac{R}{L}$$

$$\omega_2 \approx \frac{\omega_0}{Q} = \frac{1}{\sqrt{LC}}\,\frac{1}{R\sqrt{\frac{C}{L}}} = \frac{1}{RC}$$

The Low-Q Approximation



Example: Damped Input EMI Filter



$$G(s) = \frac{i_g(s)}{i_c(s)} = \frac{1 + s\,\dfrac{L_1 + L_2}{R}}{1 + s\,\dfrac{L_1 + L_2}{R} + s^2 L_1 C + s^3\,\dfrac{L_1 L_2 C}{R}}$$

8.1.8: Approximate Roots of a Polynomial

Generalize the low-Q approximation to obtain approximate factorization of the n^{th}-order polynomial

$$P(s) = 1 + a_1 s + a_2 s^2 + \cdots + a_n s^n$$

It is desired to factor this polynomial in the form

$$P(s) = \left(1 + \tau_1 s\right)\left(1 + \tau_2 s\right) \cdots \left(1 + \tau_n s\right)$$

When the roots are real and well separated in value, then approximate analytical expressions for the time constants τ_1, τ_2, ... τ_n can be found, that typically are simple functions of the circuit element values.

Objective: find a general method for deriving such expressions. Include the case of complex root pairs.

Derivation of the Approximation

Multiply out factored form of polynomial, then equate to original form (equate like powers of s):

$$\begin{aligned} a_1 &= \tau_1 + \tau_2 + \cdots + \tau_n \\ a_2 &= \tau_1\left(\tau_2 + \cdots + \tau_n\right) + \tau_2\left(\tau_3 + \cdots + \tau_n\right) + \cdots \\ a_3 &= \tau_1\tau_2\left(\tau_3 + \cdots + \tau_n\right) + \tau_2\tau_3\left(\tau_4 + \cdots + \tau_n\right) + \cdots \\ &\vdots \\ a_n &= \tau_1\tau_2\tau_3\cdots\tau_n \end{aligned}$$

- Exact system of equations relating roots to original coefficients
- Exact general solution is hopeless
- Under what conditions can solution for time constants be easily approximated?

Case When All Roots Separate

System of equations:

(from previous slide)

$$\begin{aligned} a_1 &= \tau_1 + \tau_2 + \cdots + \tau_n \\ a_2 &= \tau_1\left(\tau_2 + \cdots + \tau_n\right) + \tau_2\left(\tau_3 + \cdots + \tau_n\right) + \cdots \\ a_3 &= \tau_1\tau_2\left(\tau_3 + \cdots + \tau_n\right) + \tau_2\tau_3\left(\tau_4 + \cdots + \tau_n\right) + \cdots \\ &\vdots \\ a_n &= \tau_1\tau_2\tau_3\cdots\tau_n \end{aligned}$$

Suppose that roots are real and well-separated, and are arranged in decreasing order of magnitude:

$$\left|\tau_1\right| >> \left|\tau_2\right| >> \cdots >> \left|\tau_n\right|$$

Then the first term of each equation is dominant

$\Rightarrow$ Neglect second and following terms in each equation above

Approximation When Roots are Well Separated

System of equations:

(only first term in each equation is included)

$$
\begin{aligned}
a_1 &\approx \tau_1 \\
a_2 &\approx \tau_1\tau_2 \\
a_3 &\approx \tau_1\tau_2\tau_3 \\
&\vdots \\
a_n &= \tau_1\tau_2\tau_3\cdots\tau_n
\end{aligned}
$$

Solve for the time constants:

$$
\begin{aligned}
\tau_1 &\approx a_1 \\
\tau_2 &\approx \frac{a_2}{a_1} \\
\tau_3 &\approx \frac{a_3}{a_2} \\
&\vdots \\
\tau_n &\approx \frac{a_n}{a_{n-1}}
\end{aligned}
$$

Results

If the following inequalities are satisfied

$$\left| a_1 \right| >> \left| \frac{a_2}{a_1} \right| >> \left| \frac{a_3}{a_2} \right| >> \cdots >> \left| \frac{a_n}{a_{n-1}} \right|$$

Then the polynomial $P(s)$ has the following approximate factorization

$$P(s) \approx \left(1 + a_1\, s\right)\left(1 + \frac{a_2}{a_1}\, s\right)\left(1 + \frac{a_3}{a_2}\, s\right)\cdots\left(1 + \frac{a_n}{a_{n-1}}\, s\right)$$

- If the a_n coefficients are simple analytical functions of the element values L, C, etc., then the roots are similar simple analytical functions of L, C, etc.
- Numerical values are used to justify the approximation, but analytical expressions for the roots are obtained

Quadratic Roots: Not Well Separated

Suppose inequality k is not satisfied:

$$\left|a_1\right| >> \left|\frac{a_2}{a_1}\right| >> \cdots >> \left|\frac{a_k}{a_{k-1}}\right| \quad \underset{\substack{\uparrow \\ \text{not} \\ \text{satisfied}}}{\not>>} \quad \left|\frac{a_{k+1}}{a_k}\right| >> \cdots >> \left|\frac{a_n}{a_{n-1}}\right|$$

Then leave the terms corresponding to roots k and $(k+1)$ in quadratic form, as follows:

$$P(s) \approx \left(1 + a_1 s\right)\left(1 + \frac{a_2}{a_1} s\right) \cdots \left(1 + \frac{a_k}{a_{k-1}} s + \frac{a_{k+1}}{a_{k-1}} s^2\right) \cdots \left(1 + \frac{a_n}{a_{n-1}} s\right)$$

This approximation is accurate provided

$$\left|a_1\right| >> \left|\frac{a_2}{a_1}\right| >> \cdots >> \left|\frac{a_k}{a_{k-1}}\right| >> \left|\frac{a_{k-2}\, a_{k+1}}{a_{k-1}^2}\right| >> \left|\frac{a_{k+2}}{a_{k+1}}\right| >> \cdots >> \left|\frac{a_n}{a_{n-1}}\right|$$

First Inequality Violated

When inequality 1 is not satisfied:

$$\left|a_1\right| \quad \underset{\substack{\uparrow \\ \text{not} \\ \text{satisfied}}}{\not>>} \quad \left|\frac{a_2}{a_1}\right| >> \left|\frac{a_3}{a_2}\right| >> \cdots >> \left|\frac{a_n}{a_{n-1}}\right|$$

Then leave the first two roots in quadratic form, as follows:

$$P(s) \approx \left(1 + a_1 s + a_2 s^2\right)\left(1 + \frac{a_3}{a_2} s\right) \cdots \left(1 + \frac{a_n}{a_{n-1}} s\right)$$

This approximation is justified provided

$$\left|\frac{a_2^2}{a_3}\right| >> \left|a_1\right| >> \left|\frac{a_3}{a_2}\right| >> \left|\frac{a_4}{a_3}\right| >> \cdots >> \left|\frac{a_n}{a_{n-1}}\right|$$

Other Cases

- Several nonadjacent inequalities violated
  - Apply same process multiple times
- Multiple adjacent inequalities violated
  - More than two roots close in value
  - Must use 3^{rd} order or higher polynomial

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8.2 Analysis of Converter TFs

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Example: Buck-Boost

Small-signal ac model of the buck-boost converter, derived in Chapter 7: