The Averaged System

This equation is now the model of a new, equivalent linear system

$$\dot{\boldsymbol{x}}(t) = \boldsymbol{A}\boldsymbol{x}(t) + \boldsymbol{B}u(t)$$

where

$$\boldsymbol{A} = D\boldsymbol{A}_1 + D'\boldsymbol{A}_2$$
$$\boldsymbol{B} = D\boldsymbol{B}_1 + D'\boldsymbol{B}_2$$

which has averaged behavior over one switching period

This approximation is *perhaps* valid, if

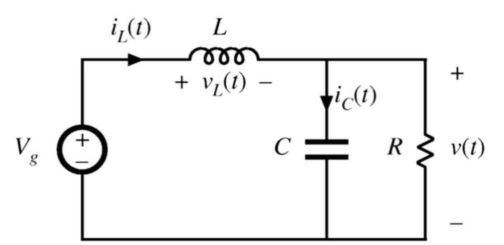
- State waveforms are dominantly linear
- Dynamics of interest are at $f_{bw} \ll f_s$

Buck State Space Averaging

In switch position 1

$$\dot{\boldsymbol{x}}(t) = \boldsymbol{A}_1\boldsymbol{x}(t) + \boldsymbol{B}_1u(t)$$

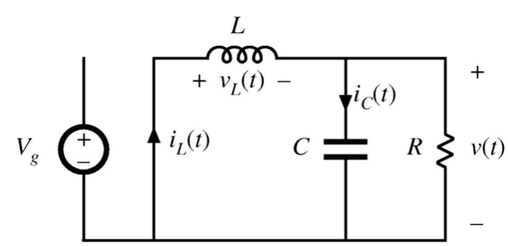
$$\frac{d}{dt}\begin{bmatrix} i_L(t) \\ v_c(t) \end{bmatrix} = \begin{bmatrix} 0 & \frac{-1}{L} \\ \frac{1}{C} & \frac{-1}{RC} \end{bmatrix} \cdot \begin{bmatrix} i_L(t) \\ v_c(t) \end{bmatrix} + \begin{bmatrix} 1/L \\ 0 \end{bmatrix} v_g(t)$$



In switch position 2

$$\dot{\boldsymbol{x}}(t) = \boldsymbol{A}_2\boldsymbol{x}(t) + \boldsymbol{B}_2u(t)$$

$$\frac{d}{dt}\begin{bmatrix} i_L(t) \\ v_c(t) \end{bmatrix} = \begin{bmatrix} 0 & \frac{-1}{L} \\ \frac{1}{C} & \frac{-1}{RC} \end{bmatrix} \cdot \begin{bmatrix} i_L(t) \\ v_c(t) \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \end{bmatrix} v_g(t)$$



Buck Averaged Model

So, our average model is

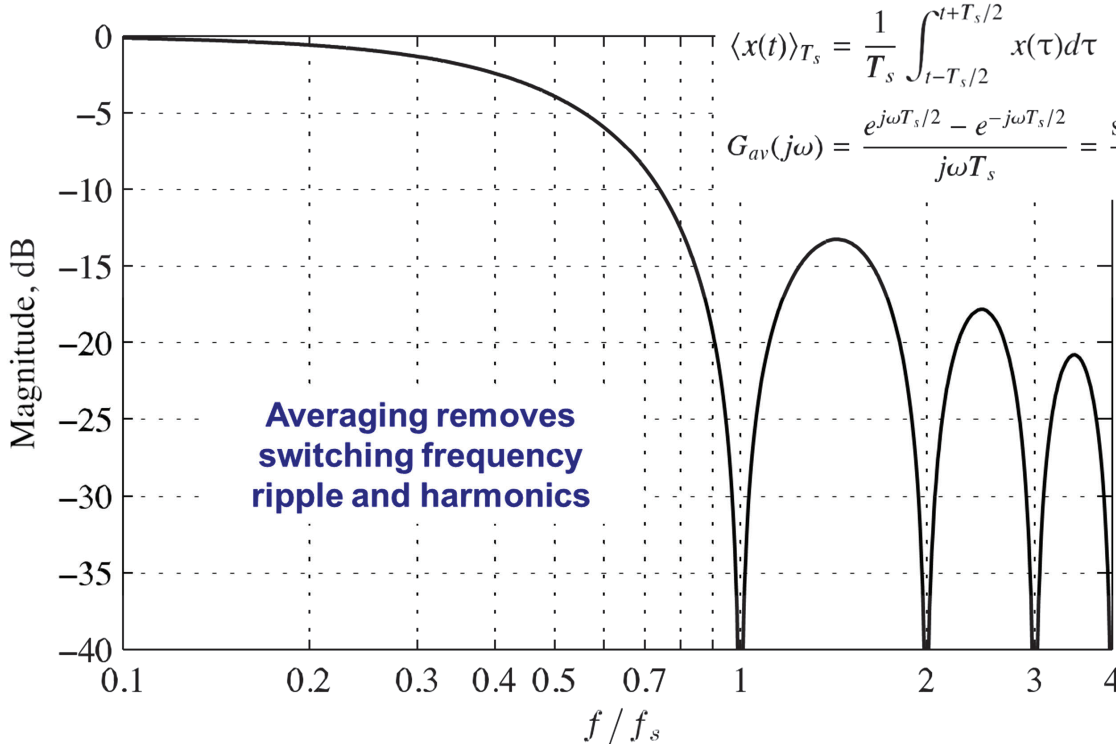
$$\langle \dot{\boldsymbol{x}}(t)\rangle = (D\boldsymbol{A_1} + D'\boldsymbol{A_2})\langle \boldsymbol{x}(t)\rangle + (D\boldsymbol{B_1} + D'\boldsymbol{B_2})\langle u(t)\rangle$$

$$\langle \dot{\boldsymbol{x}}(t)\rangle = \left(D\begin{bmatrix} 0 & \frac{-1}{L} \\ \frac{1}{C} & \frac{-1}{RC} \end{bmatrix} + D'\begin{bmatrix} 0 & \frac{-1}{L} \\ \frac{1}{C} & \frac{-1}{RC} \end{bmatrix} \right)\langle \boldsymbol{x}(t)\rangle + \left(D\begin{bmatrix} 1/L \\ 0 \end{bmatrix} + D'\begin{bmatrix} 0 \\ 0 \end{bmatrix} \right) V_g$$

$$\langle \dot{\boldsymbol{x}}(t)\rangle = \begin{bmatrix} 0 & \frac{-1}{L} \\ \frac{1}{C} & \frac{-1}{RC} \end{bmatrix}\langle \boldsymbol{x}(t)\rangle + \begin{bmatrix} D/L \\ 0 \end{bmatrix} V_g$$

$$\begin{cases} DV_g - \langle v_c(t)\rangle = L\dfrac{d\langle i_L(t)\rangle}{dt} \\ \langle i_L(t)\rangle - \dfrac{\langle v_c(t)\rangle}{R} = C\dfrac{d\langle v_c(t)\rangle}{dt} \end{cases}$$

Averaging: Discussion



$$\langle x(t)\rangle_{T_s} = \frac{1}{T_s}\int_{t-T_s/2}^{t+T_s/2} x(\tau)d\tau$$

$$G_{av}(j\omega) = \frac{e^{j\omega T_s/2} - e^{-j\omega T_s/2}}{j\omega T_s} = \frac{\sin(\omega T_s/2)}{\omega T_s/2}$$

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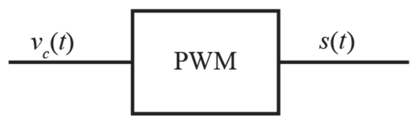
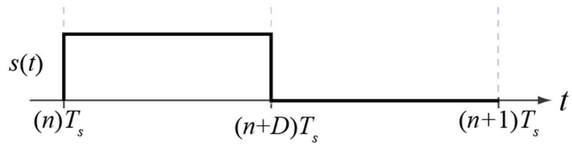
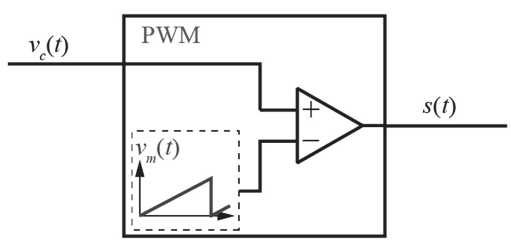
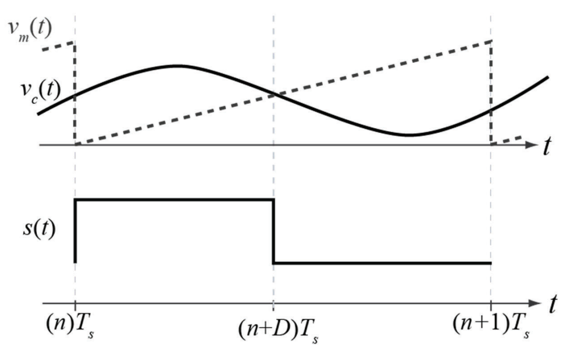
Discrete Time Nature of PWM



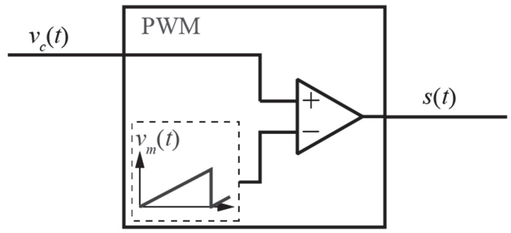


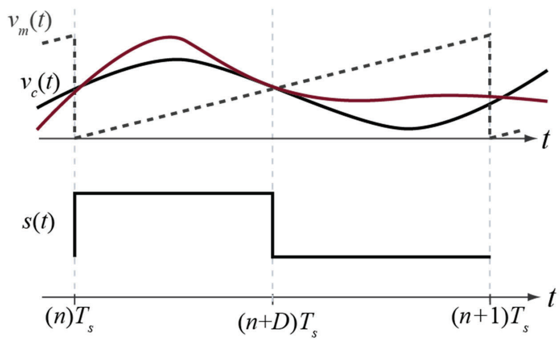
Discrete Time Nature of PWM





Discrete Time Nature of PWM





Historical Perspective

Robert D Middlebrook
PhD, Standford, 1955
CalTech Professor, 1955-1998

Slobodan Cúk
PhD CalTech, 1976
CalTech Prof, 1977-1999

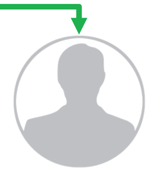
Modelling, analysis, and design of switching converters

Model a switched system as an averaged, time-invariant system with

$$\dot{\boldsymbol{x}}(t) = \boldsymbol{A}\boldsymbol{x}(t) + \boldsymbol{B}u(t)$$

where

$$\boldsymbol{A} = D\boldsymbol{A}_1 + D'\boldsymbol{A}_2$$
$$\boldsymbol{B} = D\boldsymbol{B}_1 + D'\boldsymbol{B}_2$$



Dennis John Packard
PhD, CalTech 1976

Discrete modeling and analysis of switching regulators

Model a switched system as a discrete-time system with

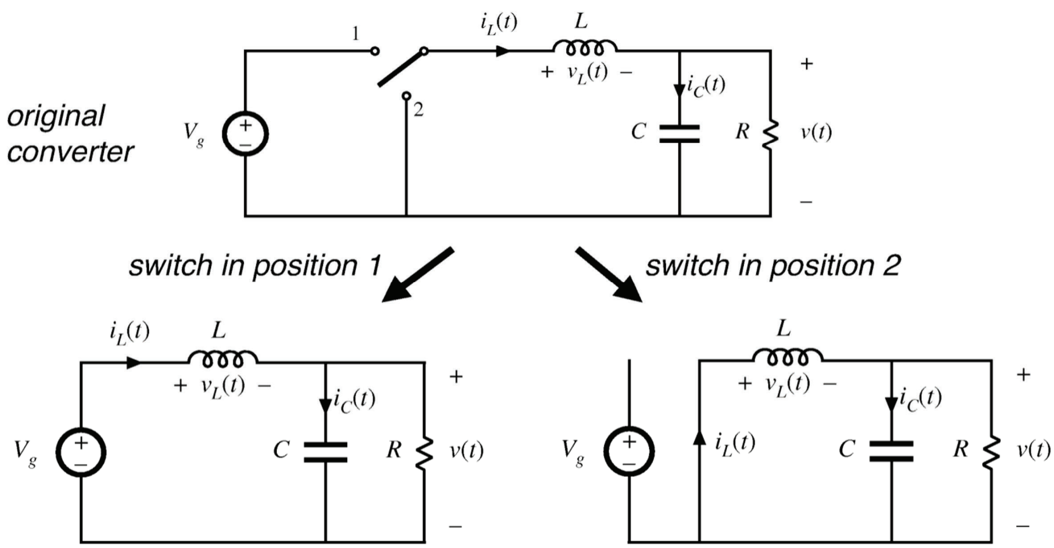
$$\boldsymbol{x}[n+1] = \boldsymbol{\Phi}\boldsymbol{x}[n] + \boldsymbol{\Psi}U[n]$$

where

$$\boldsymbol{\Phi} = \left(\prod_{i=n_{sw}}^{1} e^{\boldsymbol{A}_i t_i}\right)$$
$$\boldsymbol{\Psi} = \sum_{i=1}^{n_{sw}}\left\{\left(\prod_{k=n_{sw}}^{i+1} e^{\boldsymbol{A}_k t_k}\right)\boldsymbol{A}_i^{-1}\left(e^{\boldsymbol{A}_i t_i} - \boldsymbol{I}\right)\boldsymbol{B}_i\right\}$$

A. R. Brown and R. D. Middlebrook, “Sampled-data Modeling of Switching Regulators” PESC 1981

Large Signal Modeling of SMPS



Discrete Time Modeling

- Every subcircuit is a passive, linear circuit
- Passive, linear circuits can be solved in closed-form
  - Can model states at discrete times without averaging
- Only assumptions required
  - Independent inputs are DC or slowly varying

Solution to State Space Equation

Closed form solution to state space equation

$$\dot{\boldsymbol{x}}(t) = \boldsymbol{A}\boldsymbol{x}(t) + \boldsymbol{B}u(t)$$

Multiply both sides by $e^{-\boldsymbol{A}t}$

$$e^{-\boldsymbol{A}t}\dot{\boldsymbol{x}}(t) - e^{-\boldsymbol{A}t}\boldsymbol{A}\boldsymbol{x}(t) = e^{-\boldsymbol{A}t}\boldsymbol{B}u(t)$$

Left-hand side is

$$\frac{d}{dt}\left(e^{-\boldsymbol{A}t}\boldsymbol{x}(t)\right) = e^{-\boldsymbol{A}t}\boldsymbol{B}u(t)$$

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Solution to State Space Equation

$$\frac{d}{dt}\left(e^{-\boldsymbol{A}t}\boldsymbol{x}(t)\right) = e^{-\boldsymbol{A}t}\boldsymbol{B}u(t)$$

Can now be solved by direct integration

$$e^{-\boldsymbol{A}t}\boldsymbol{x}(t) - \boldsymbol{x}(0) = \int_0^t e^{-\boldsymbol{A}\tau}\boldsymbol{B}u(\tau)\,d\tau$$

Rearranging

$$\boldsymbol{x}(t) = e^{\boldsymbol{A}t}\boldsymbol{x}(0) + \int_0^t e^{-\boldsymbol{A}(t-\tau)}\boldsymbol{B}u(\tau)\,d\tau$$

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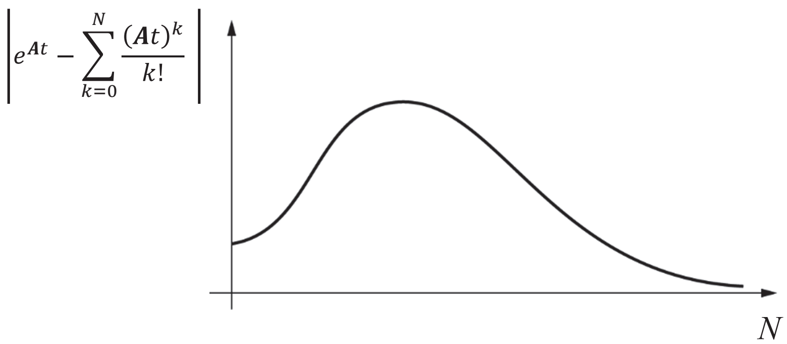
Matrix Exponential

Matrix exponential defined by Taylor series expansion

$$e^{\boldsymbol{A}t} = I + \boldsymbol{A}t + \frac{(\boldsymbol{A}t)^2}{2!} + \cdots + \frac{(\boldsymbol{A}t)^N}{N!} = \sum_{k=0}^{N} \frac{(\boldsymbol{A}t)^k}{k!}$$

Well-known issue with convergence in many cases



C. Moler and C. V. Loan, "Nineteen dubious ways to compute the exponential of a matrix," SIAM Review, vol. 20, pp. 801–836, 1978.

Properties of the Matrix Exponential

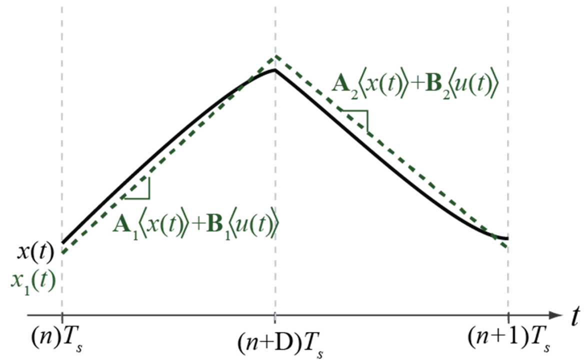
- Matrix exponential always exists
  - i.e. summation will always converge
- Exponential of any matrix is always invertible, with

$$e^{\boldsymbol{A}} e^{-\boldsymbol{A}} = \boldsymbol{I}$$

First Order Taylor Series Expansion

Linear ripple approximation

$$e^{At} \approx I + \boldsymbol{A}t$$



Valid only if switching frequency much faster than system modes

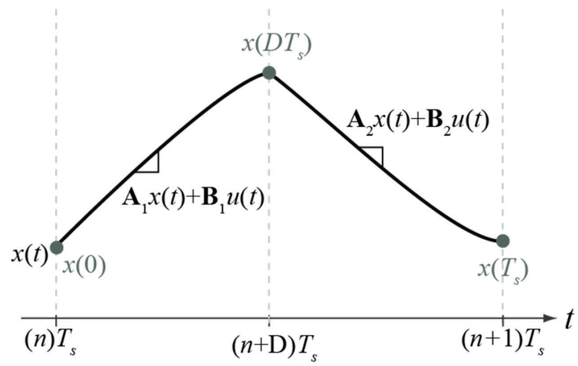
Simplification for Slow-Varying Inputs

$$\boldsymbol{x}(t) = e^{At}\boldsymbol{x}(0) + \int_0^t e^{-A(t-\tau)}\boldsymbol{B}u(\tau)\,d\tau$$

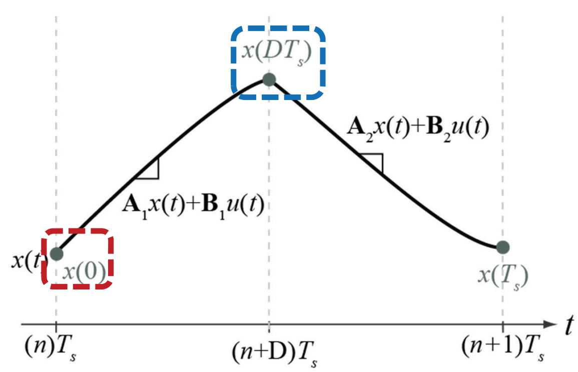
If A is invertible and $u(\tau) \approx U$

$$\boldsymbol{x}(t) = e^{At}\boldsymbol{x}(0) + \boldsymbol{A}^{-1}(e^{At} - \boldsymbol{I})\boldsymbol{B}U$$

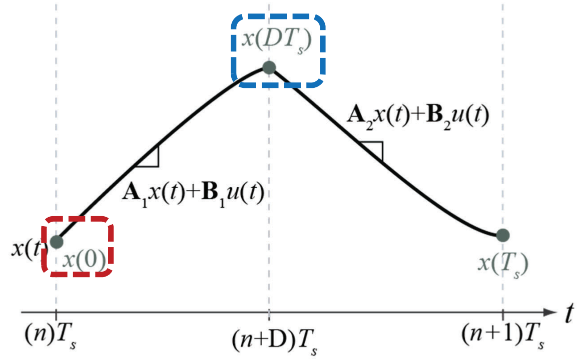
Application to Switching Converter



Application to Switching Converter

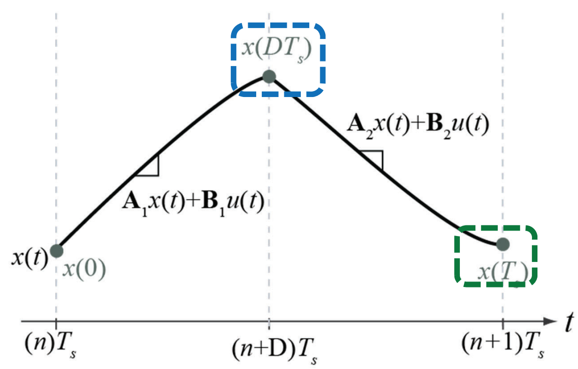


Application to Switching Converter



$$\boxed{\boldsymbol{x}(DT_s)} = e^{A_1 DT_s}\boxed{\boldsymbol{x}(0)} + \boldsymbol{A_1}^{-1}(e^{A_1 DT_s} - \boldsymbol{I})\boldsymbol{B}_1 U$$

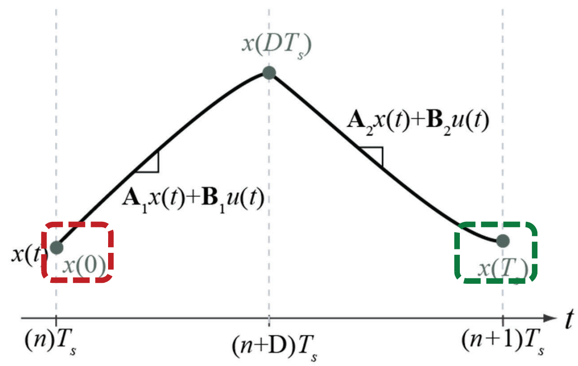
Application to Switching Converter



$$\boldsymbol{x}(DT_s) = e^{A_1 DT_s}\boldsymbol{x}(0) + \boldsymbol{A_1}^{-1}(e^{A_1 DT_s} - \boldsymbol{I})\boldsymbol{B}_1 U$$

$$\boxed{\boldsymbol{x}(T_s)} = e^{A_2 D' T_s}\boxed{\boldsymbol{x}(DT_s)} + \boldsymbol{A_2}^{-1}(e^{A_2 D' T_s} - \boldsymbol{I})\boldsymbol{B}_2 U$$

Application to Switching Converter



$$\boldsymbol{x}(DT_s) = e^{\boldsymbol{A}_1 DT_s}\boldsymbol{x}(0) + \boldsymbol{A}_1{}^{-1}(e^{\boldsymbol{A}_1 DT_s} - \boldsymbol{I})\boldsymbol{B}_1 U$$

$$\boldsymbol{x}(T_s) = e^{\boldsymbol{A}_2 D' T_s}\boldsymbol{x}(DT_s) + \boldsymbol{A}_2{}^{-1}(e^{\boldsymbol{A}_2 D' T_s} - \boldsymbol{I})\boldsymbol{B}_2 U$$

$$\boxed{\boldsymbol{x}(T_s)} = e^{\boldsymbol{A}_2 D' T_s} e^{\boldsymbol{A}_1 DT_s}\boxed{\boldsymbol{x}(0)} + \boldsymbol{A}_2{}^{-1}(e^{\boldsymbol{A}_2 D' T_s} - \boldsymbol{I})\boldsymbol{B}_2 U + e^{\boldsymbol{A}_2 D' T_s}\boldsymbol{A}_1{}^{-1}(e^{\boldsymbol{A}_1 DT_s} - \boldsymbol{I})\boldsymbol{B}_1 U$$

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General Form

Generally, for n_{sw} separate switching positions

$$\boldsymbol{x}(T_s) = \left(\prod_{i=n_{sw}}^{1} e^{\boldsymbol{A}_i t_i}\right)\boldsymbol{x}(0) + \sum_{i=1}^{n_{sw}}\left\{\left(\prod_{k=n_{sw}}^{i+1} e^{\boldsymbol{A}_k t_k}\right)\boldsymbol{A}_i{}^{-1}(e^{\boldsymbol{A}_i t_i} - \boldsymbol{I})\boldsymbol{B}_i\right\}U$$

Equation is in the form of a discrete-time system with

$$\boldsymbol{x}[n+1] = \boldsymbol{\Phi}\boldsymbol{x}[n] + \boldsymbol{\Psi}U[n]$$

Again, the effect of changing modulation (i.e. t_i) is hidden in nonlinear terms

$$\widehat{\boldsymbol{x}}[n+1] = \boldsymbol{\Phi}\widehat{\boldsymbol{x}}[n] + \boldsymbol{\Psi}\hat{u}[n] + \boldsymbol{\Gamma}\hat{d}[n]$$

Find $\boldsymbol{\Gamma}$ by small-signal modeling

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Aside: Comparison to Averaged Modeling

$$\boldsymbol{x}(T_s) = \left(\prod_{i=n_{sw}}^{1} e^{\boldsymbol{A}_i t_i}\right)\boldsymbol{x}(0) + \sum_{i=1}^{n_{sw}}\left\{\left(\prod_{k=n_{sw}}^{i+1} e^{\boldsymbol{A}_k t_k}\right)\boldsymbol{A}_i{}^{-1}(e^{\boldsymbol{A}_i t_i} - \boldsymbol{I})\boldsymbol{B}_i\right\}U$$

Approximate with straight-line waveforms, $e^{\boldsymbol{A}t} \approx I + \boldsymbol{A}t$

$$\boldsymbol{x}(T_s) = \left(\boldsymbol{I} + \sum_{i=n_{sw}}^{1} \boldsymbol{A}_i t_i + \cdots\right)\boldsymbol{x}(0) + \sum_{i=1}^{n_{sw}}\left\{\left(\boldsymbol{I} + \sum_{k=n_{sw}}^{i+1} \boldsymbol{A}_k t_k + \cdots\right) t_i \boldsymbol{B}_i\right\}U$$

Neglect all terms with product of two ore more t_i

$$\boldsymbol{x}(T_s) = \left(\boldsymbol{I} + \sum_{i=1}^{n_{sw}} \boldsymbol{A}_i t_i\right)\boldsymbol{x}(0) + \sum_{i=1}^{n_{sw}} (t_i \boldsymbol{B}_i)\, U$$

Continuous time conversion

$$\dot{\boldsymbol{x}}_{DT}(t) = \frac{\boldsymbol{x}(T_s) - \boldsymbol{x}(0)}{T_s} = \sum_{i=1}^{n_{sw}}\left(\boldsymbol{A}_i \frac{t_i}{T_s}\right)\boldsymbol{x} + \sum_{i=1}^{n_{sw}}\left(\boldsymbol{B}_i \frac{t_i}{T_s}\right)U$$

Aside: Discrete vs Averaged Modeling

So, averaged and discrete time formulations are equivalent if

- Ripple in states is
  1. Dominantly straight-line, so $e^{\boldsymbol{A}_i t_i} \approx (\boldsymbol{I} + \boldsymbol{A}_i t_i)$
  2. Low frequency, such that $t_i t_j \ll \left\|\boldsymbol{A}_i \boldsymbol{A}_j\right\|$

Steady-State Large-Signal Analysis

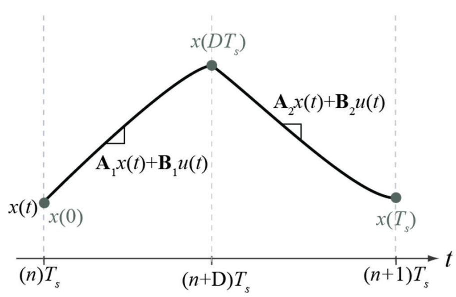
$$\boldsymbol{x}(T_s) = \left(\prod_{i=n_{sw}}^{1} e^{\boldsymbol{A}_i t_i}\right)\boldsymbol{x}(0) + \sum_{i=1}^{n_{sw}}\left\{\left(\prod_{k=n_{sw}}^{i+1} e^{\boldsymbol{A}_k t_k}\right)\boldsymbol{A}_i^{-1}\left(e^{\boldsymbol{A}_i t_i} - \boldsymbol{I}\right)\boldsymbol{B}_i\right\}U$$

In steady-state, $\boldsymbol{x}(T_s) = \boldsymbol{x}(0)$

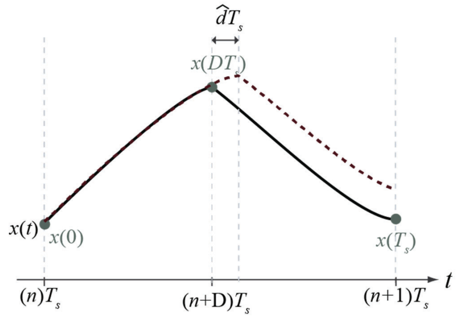
$$\boldsymbol{x}(T_s) = \left(I - \prod_{i=n_{sw}}^{1} e^{A_i t_i}\right)^{-1} \sum_{i=1}^{n_{sw}}\left\{\left(\prod_{k=n_{sw}}^{i+1} e^{\boldsymbol{A}_k t_k}\right)\boldsymbol{A}_i^{-1}\left(e^{\boldsymbol{A}_i t_i} - \boldsymbol{I}\right)\boldsymbol{B}_i\right\}U$$

Gives explicit solution for steady-state operation of any switching circuit

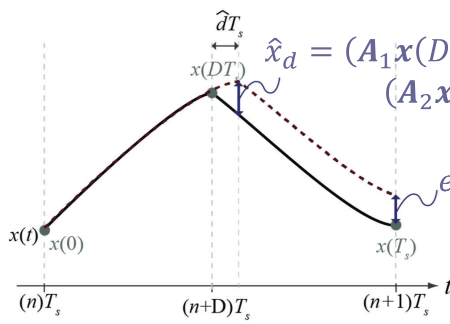
Small Signal Modeling



Small Signal Modeling



Small Signal Modeling



$$\hat{x}_d = (\boldsymbol{A}_1 \boldsymbol{x}(DT_s) + \boldsymbol{B}_1 U)\hat{d}T_s - (\boldsymbol{A}_2 \boldsymbol{x}(DT_s) + \boldsymbol{B}_2 U)\hat{d}T_s$$

$$e^{\boldsymbol{A}_2 D' T_s} \hat{x}_d$$

Complete Small Signal Model

This completes the small-signal model

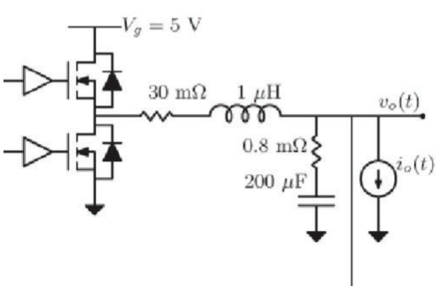
$$\hat{\boldsymbol{x}}[n+1] = \boldsymbol{\Phi}\hat{\boldsymbol{x}}[n] + \boldsymbol{\Psi}\hat{u}[n] + \boldsymbol{\Gamma}\hat{d}[n]$$

where

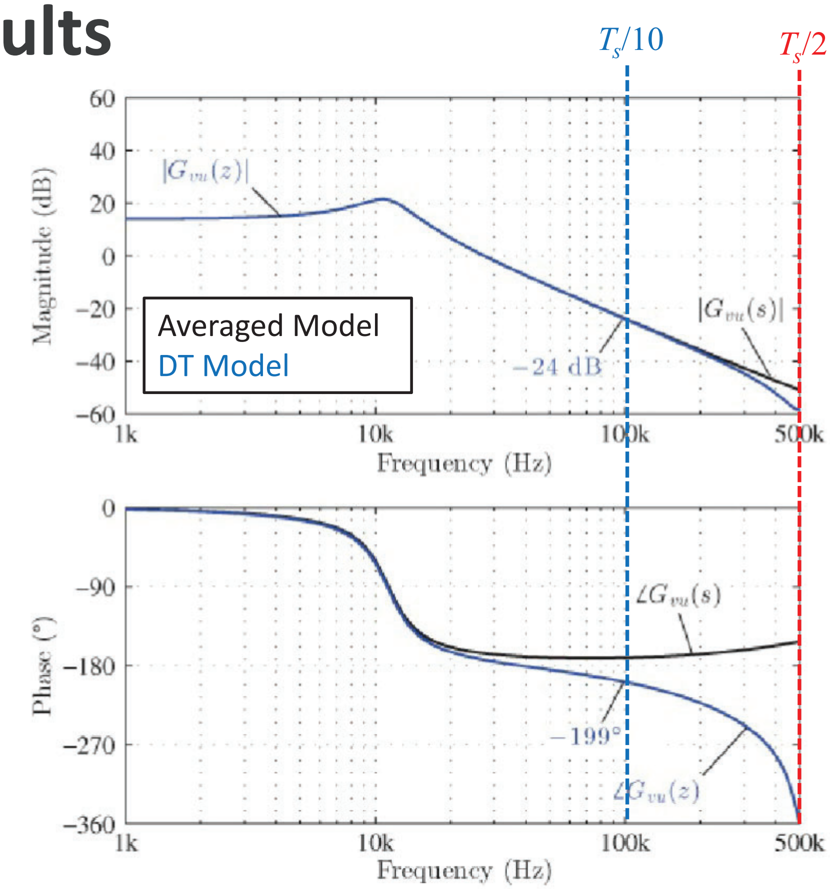
$$\boldsymbol{\Gamma} = e^{A_2 D' T_s}\big((\boldsymbol{A}_1 - \boldsymbol{A}_2)\boldsymbol{X}_D + (\boldsymbol{B}_1 - \boldsymbol{B}_2\,)U\big)T_s$$

with $\boldsymbol{X}_D = \boldsymbol{x}(DT_s)$ in steady-state

Example Results



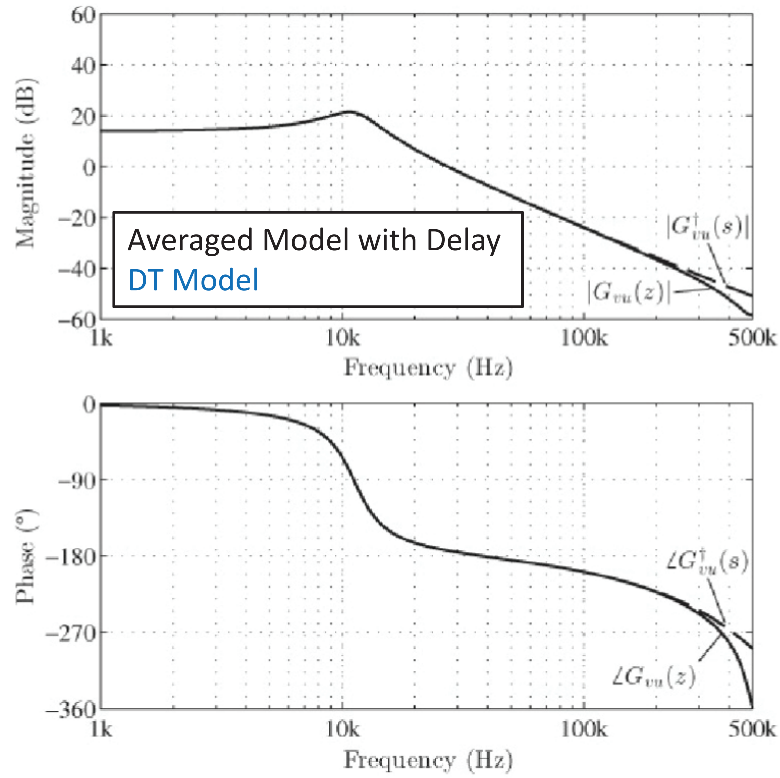
* Includes t_d=760ns of delay in feedback loop



L. Corradini et. al. *Digital Control of High Frequency Switched-Mode Power Converters,* Section 3.2

Inclusion of Delay

$$G_{vu}^{\dagger}(s) = G_{vu}(s)e^{-st_d}$$



Current Control