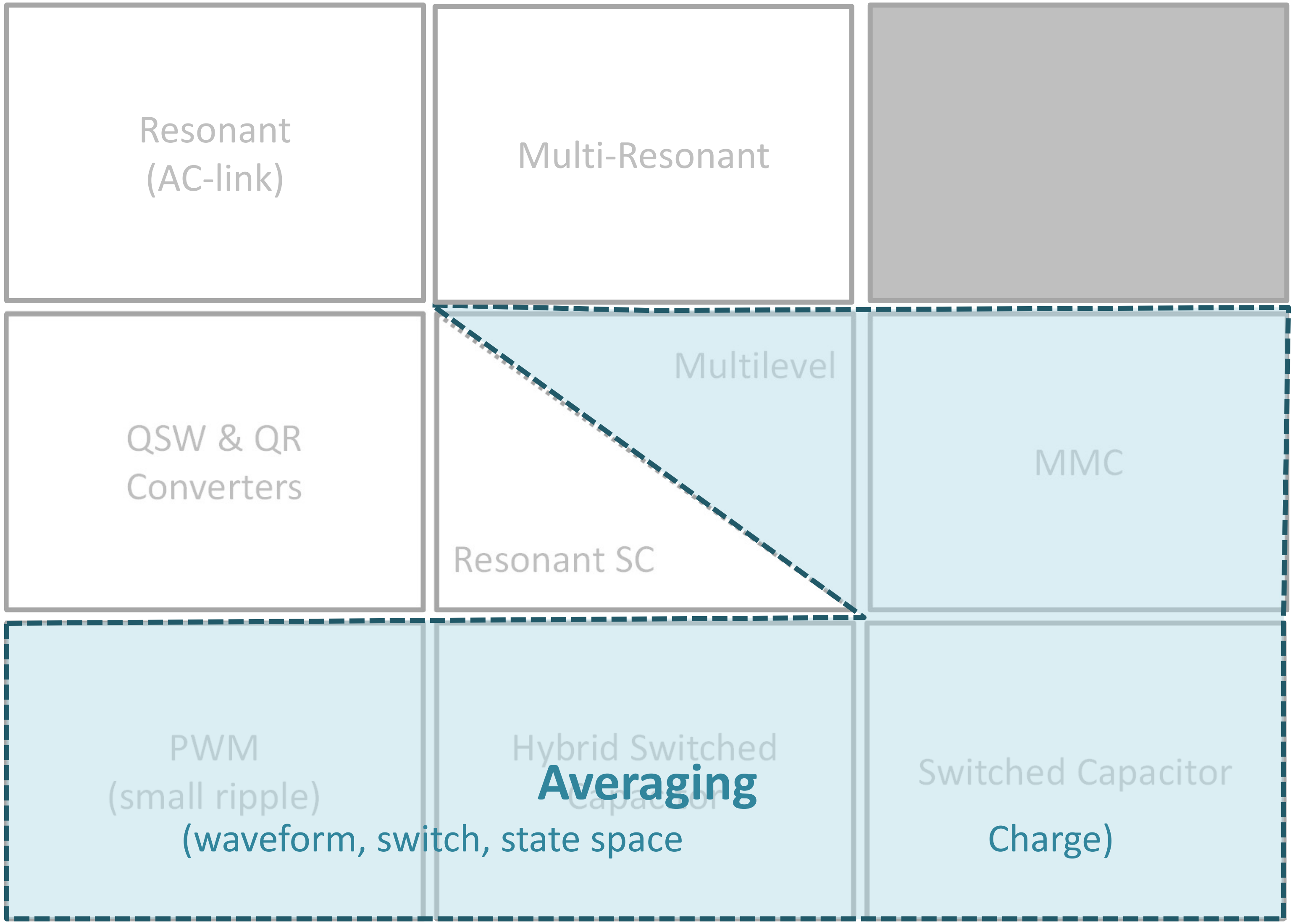
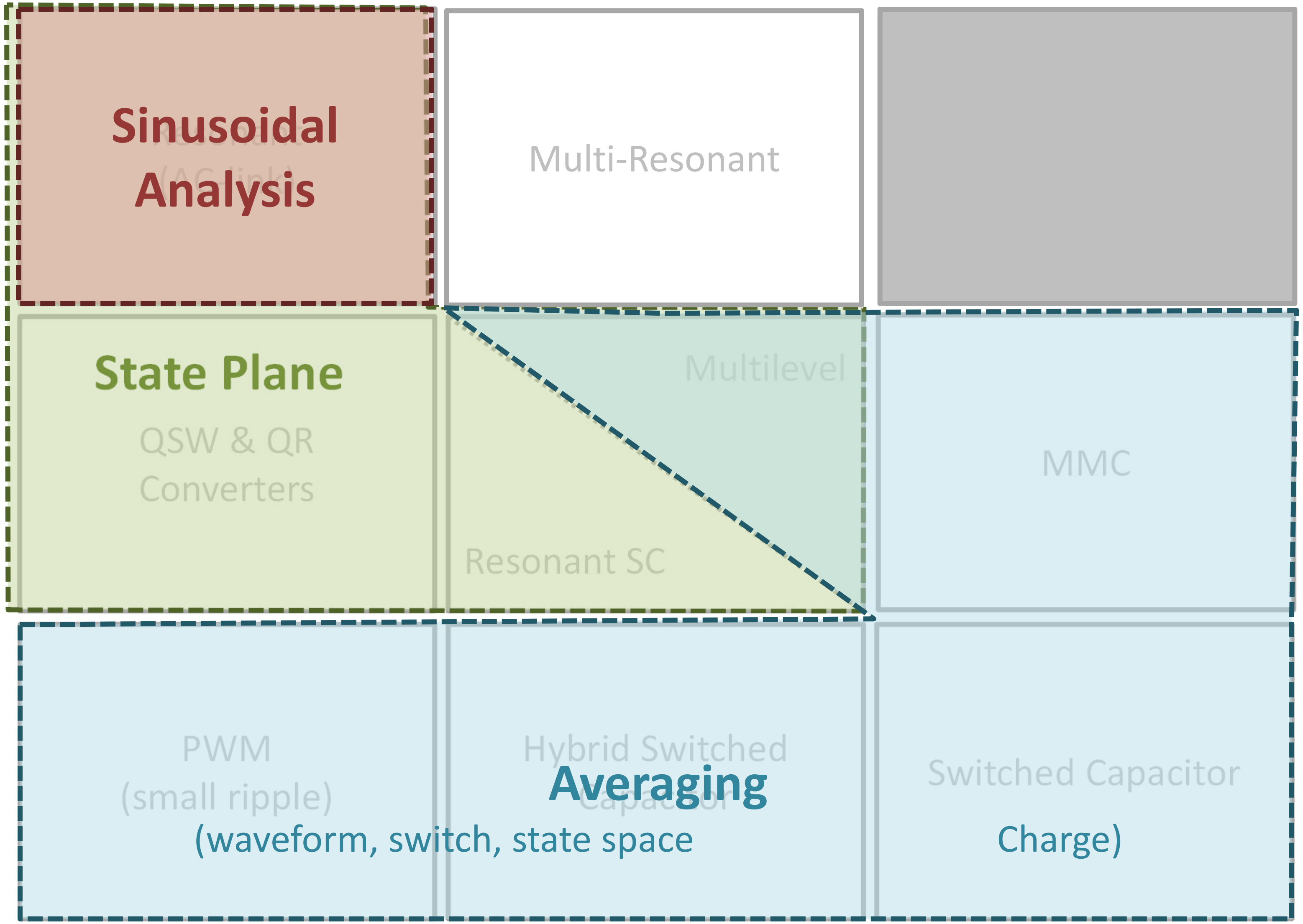
Converter Analysis

Resonant (AC-link)	Multi-Resonant	Z-source, matrix, others
QSW & QR Converters	Multilevel / Resonant SC	MMC
PWM (small ripple)	Hybrid Switched Capacitor	Switched Capacitor

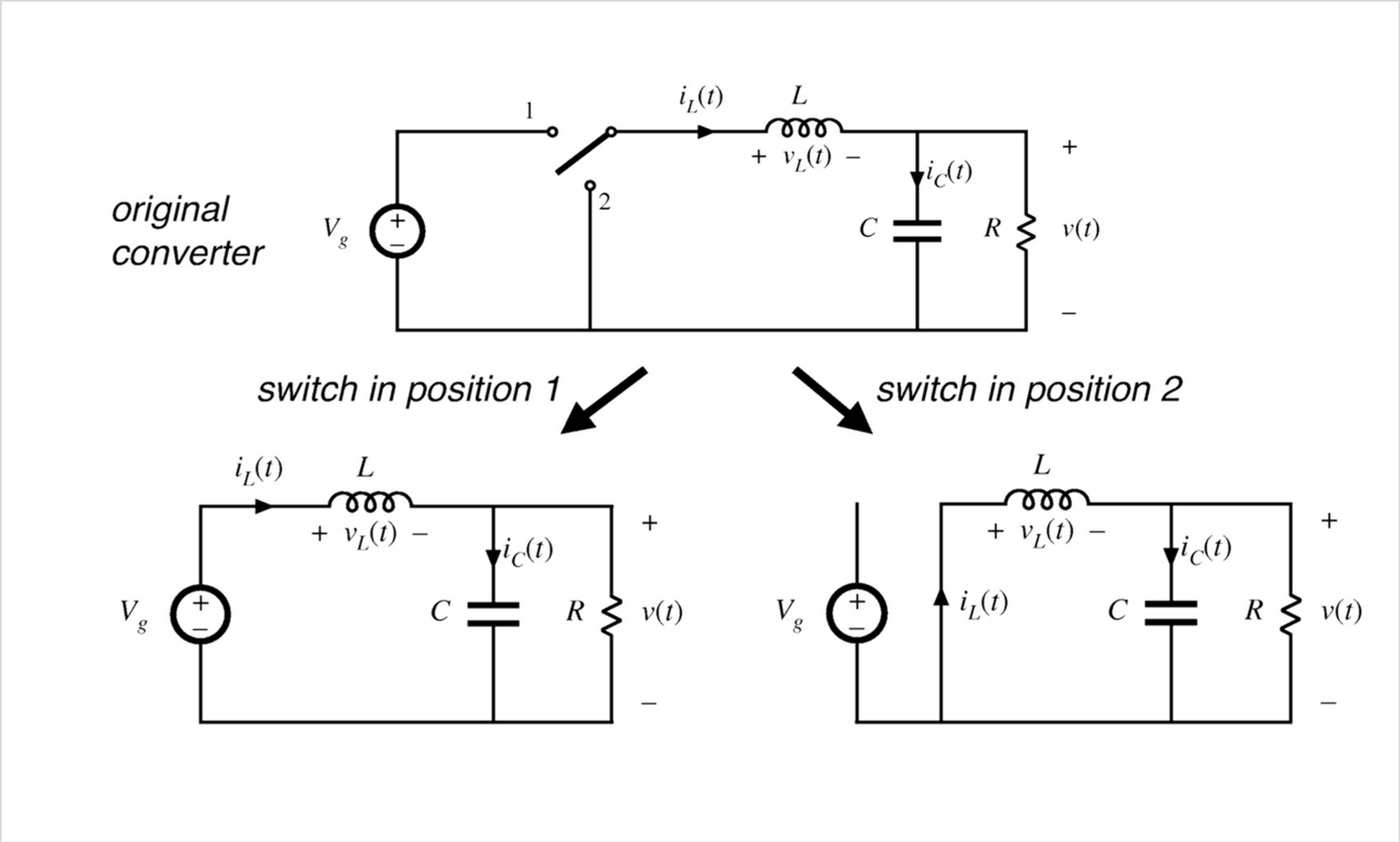
Converter Analysis



Converter Analysis



Analysis of Switched Systems



- Every switching subcircuit approximated as a passive, *linear* circuit
  - Piecewise linear models, if necessary

Historical Perspective

Robert D Middlebrook
PhD, Standford, 1955
CalTech Professor, 1955-1998

Slobodan Cúk
PhD CalTech, 1976
CalTech Prof, 1977-1999

Modelling, analysis, and design of switching converters

Model a switched system as an averaged, time-invariant system with

$$\dot{\boldsymbol{x}}(t) = \boldsymbol{A}\boldsymbol{x}(t) + \boldsymbol{B}u(t)$$

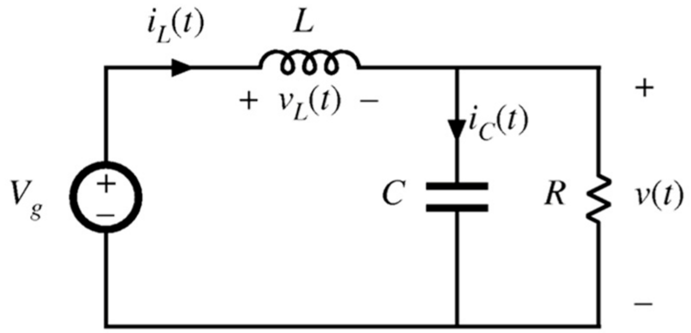
where

$$\boldsymbol{A} = D\boldsymbol{A}_1 + D'\boldsymbol{A}_2$$
$$\boldsymbol{B} = D\boldsymbol{B}_1 + D'\boldsymbol{B}_2$$

Linear Circuit Modeling Using State Space

In switch position 1

$$\begin{cases} v_g(t) - v_c(t) = L\dfrac{di_L(t)}{dt} \\ i_L(t) - \dfrac{v_c(t)}{R} = C\dfrac{dv_c(t)}{dt} \end{cases}$$



Which can be written, in state space, form as

$$\frac{d}{dt}\begin{bmatrix} i_L(t) \\ v_c(t) \end{bmatrix} = \begin{bmatrix} 0 & \dfrac{-1}{L} \\ \dfrac{1}{C} & \dfrac{-1}{RC} \end{bmatrix} \cdot \begin{bmatrix} i_L(t) \\ v_c(t) \end{bmatrix} + \begin{bmatrix} 1/L \\ 0 \end{bmatrix} v_g(t)$$

Or, generally,

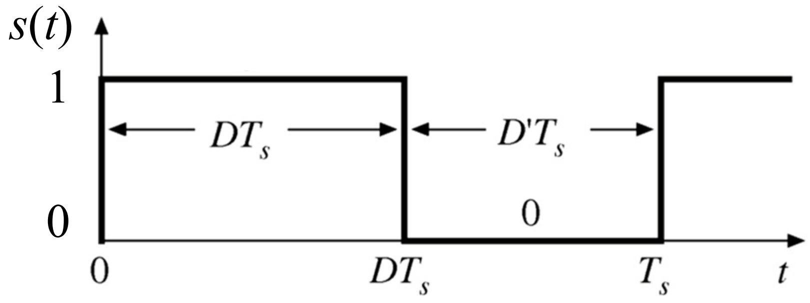
$$\dot{\boldsymbol{x}}(t) = \boldsymbol{A}_1\boldsymbol{x}(t) + \boldsymbol{B}_1u(t)$$

In the second switch position, we will have a new (linear) circuit with

$$\dot{\boldsymbol{x}}(t) = \boldsymbol{A}_2\boldsymbol{x}(t) + \boldsymbol{B}_2u(t)$$

Switching Signal

In a PWM converter with two switch positions, the two linear circuits combine according to a switching function $s(t)$



$$\dot{\boldsymbol{x}}(t) = [\boldsymbol{A}_1 s(t) + \boldsymbol{A}_2 s'(t)]\boldsymbol{x}(t) + [\boldsymbol{B}_1 s(t) + \boldsymbol{B}_2 s'(t)]u(t)$$

where

$$s(t) = \begin{cases} 1, & \text{if } \; nT_s < t < (n+D)T_s \\ 0, & \text{if } \; (n+D)T_s < t < (n+1)T_s \end{cases}$$

$$s'(t) = 1 - s(t)$$

SMPS State Space

In traditional state space modeling of linear systems

$$\dot{\boldsymbol{x}}(t) = \boldsymbol{A}\boldsymbol{x}(t) + \boldsymbol{B}u(t)$$

with $u(t)$ containing a control input. When $\boldsymbol{A}$ and $\boldsymbol{B}$ are constant, this is a linear system. However, we have

$$\dot{\boldsymbol{x}}(t) = [\boldsymbol{A}_1 s(t) + \boldsymbol{A}_2 s'(t)]\boldsymbol{x}(t) + [\boldsymbol{B}_1 s(t) + \boldsymbol{B}_2 s'(t)]u(t)$$

or, equivalently

$$\dot{\boldsymbol{x}}(t) = \boldsymbol{A}(t)\boldsymbol{x}(t) + \boldsymbol{B}(t)u(t)$$

which is nonlinear: how do we deal with it?

Converting to Linear System

Assume that our system model

$$\dot{\boldsymbol{x}}(t) = [\boldsymbol{A}_1 s(t) + \boldsymbol{A}_2 s'(t)]\boldsymbol{x}(t) + [\boldsymbol{B}_1 s(t) + \boldsymbol{B}_2 s'(t)]u(t)$$

can be approximated by some linear system

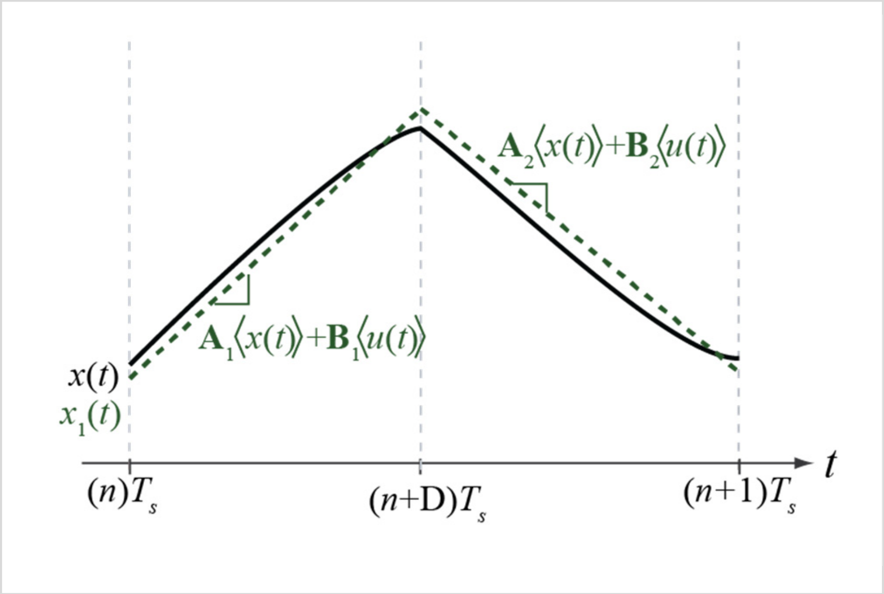
$$\dot{\boldsymbol{x}}(t) = \boldsymbol{A}\boldsymbol{x}(t) + \boldsymbol{B}u(t)$$

which removes the nonlinearity of the system

– Nonlinearities came from switching
– Expect that switching dynamics will be lost

Note: This system is now linear in $\boldsymbol{x}(t)$ and $u(t)$, but not in our control signal, $s(t)$

Average Modeling



Approximate *waveforms* as piecewise linear (PWL)

$$\dot{\boldsymbol{x}}(t) = \begin{cases} \boldsymbol{A_1}\langle \boldsymbol{x}(t)\rangle + \boldsymbol{B_1}\langle \boldsymbol{u}(t)\rangle, & \text{if } \ nT_s < t < (n+D)T_s \\ \boldsymbol{A_2}\langle \boldsymbol{x}(t)\rangle + \boldsymbol{B_2}\langle \boldsymbol{u}(t)\rangle, & \text{if } \ (n+D)T_s < t < (n+1)T_s \end{cases}$$

where

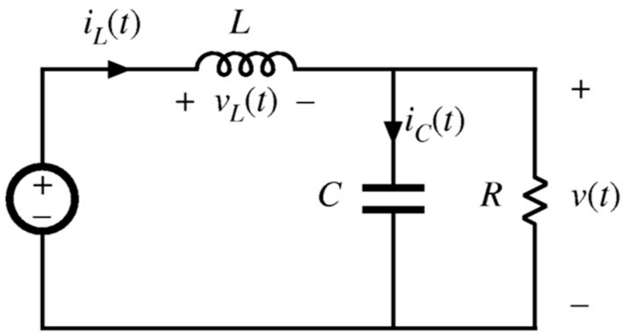
$$\langle \boldsymbol{x}(t)\rangle = \frac{1}{T_s}\int_0^{T_s} \boldsymbol{x}(t)dt = \boldsymbol{X}$$

so the average slope is

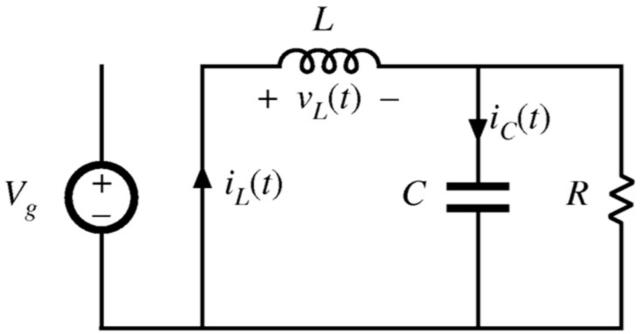
$$\langle \dot{\boldsymbol{x}}(t)\rangle = (\boldsymbol{D}\boldsymbol{A_1} + \boldsymbol{D}'\boldsymbol{A_2})\langle \boldsymbol{x}(t)\rangle + (\boldsymbol{D}\boldsymbol{B_1} + \boldsymbol{D}'\boldsymbol{B_2})\langle \boldsymbol{u}(t)\rangle$$

This equation is now the model of a new, equivalent LTI system

$$\langle \dot{\boldsymbol{x}}(t)\rangle = \boldsymbol{A}\langle \boldsymbol{x}(t)\rangle + \boldsymbol{B}\langle \boldsymbol{u}(t)\rangle$$



$$\dot{\boldsymbol{x}}(t) = \boldsymbol{A}_1\boldsymbol{x}(t) + \boldsymbol{B}_1\boldsymbol{u}(t)$$



$$\dot{\boldsymbol{x}}(t) = \boldsymbol{A}_2\boldsymbol{x}(t) + \boldsymbol{B}_2\boldsymbol{u}(t)$$

The Averaged System

This equation is now the model of a new, equivalent linear system

$$\dot{\boldsymbol{x}}(t) = \boldsymbol{A}\boldsymbol{x}(t) + \boldsymbol{B}u(t)$$

where

$$\boldsymbol{A} = D\boldsymbol{A_1} + D'\boldsymbol{A_2}$$
$$\boldsymbol{B} = D\boldsymbol{B_1} + D'\boldsymbol{B_2}$$

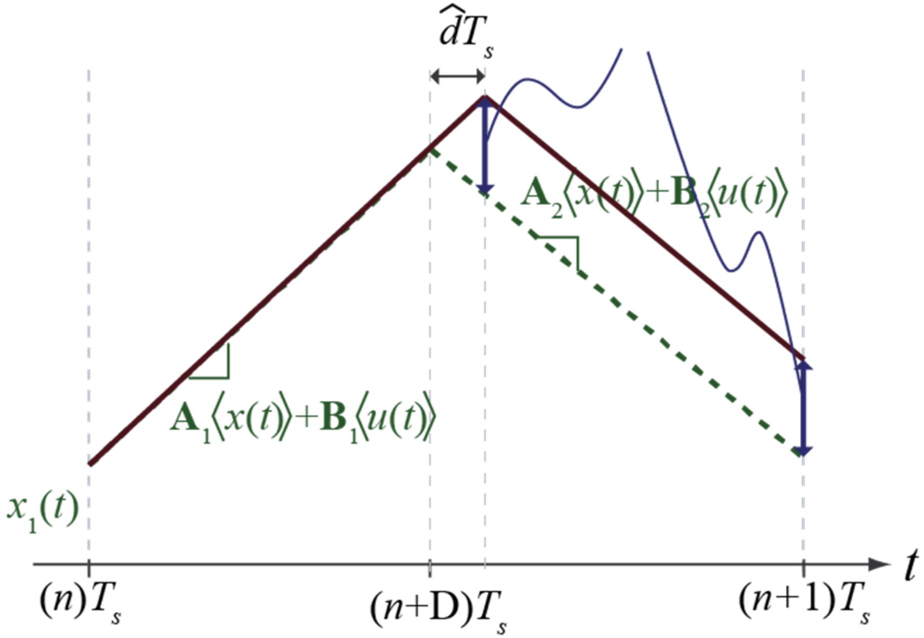
which has averaged behavior over one switching period

This approximation is *perhaps* valid, if

- State waveforms are dominantly linear
- Dynamics of interest are at $f_{bw} \ll f_s$

Average Control Response

$$(A_1\langle \boldsymbol{x}(t)\rangle + B_1\langle u(t)\rangle)\hat{d}T_s - (A_2\langle \boldsymbol{x}(t)\rangle + B_2\langle u(t)\rangle)\hat{d}T_s$$



$$\langle \dot{\boldsymbol{x}}(t)\rangle = \boldsymbol{A}\langle \boldsymbol{x}(t)\rangle + \boldsymbol{B}\langle \boldsymbol{u}(t)\rangle$$

System is LTI with respect to inputs U

Still, the control input (D) is hidden in the state matrices
Find dynamic model through small-signal linearization

So, the complete small signal system is

$$\dot{\hat{\boldsymbol{x}}}(t) = \boldsymbol{A}\hat{\boldsymbol{x}}(t) + \boldsymbol{B}\hat{\boldsymbol{u}}(t) + \boldsymbol{F}\hat{d}(t)$$

with

$$\boldsymbol{F} = \left((\boldsymbol{A_1} - \boldsymbol{A_2})\boldsymbol{X} + (\boldsymbol{B_1} - \boldsymbol{B_2})\boldsymbol{U}\right)$$

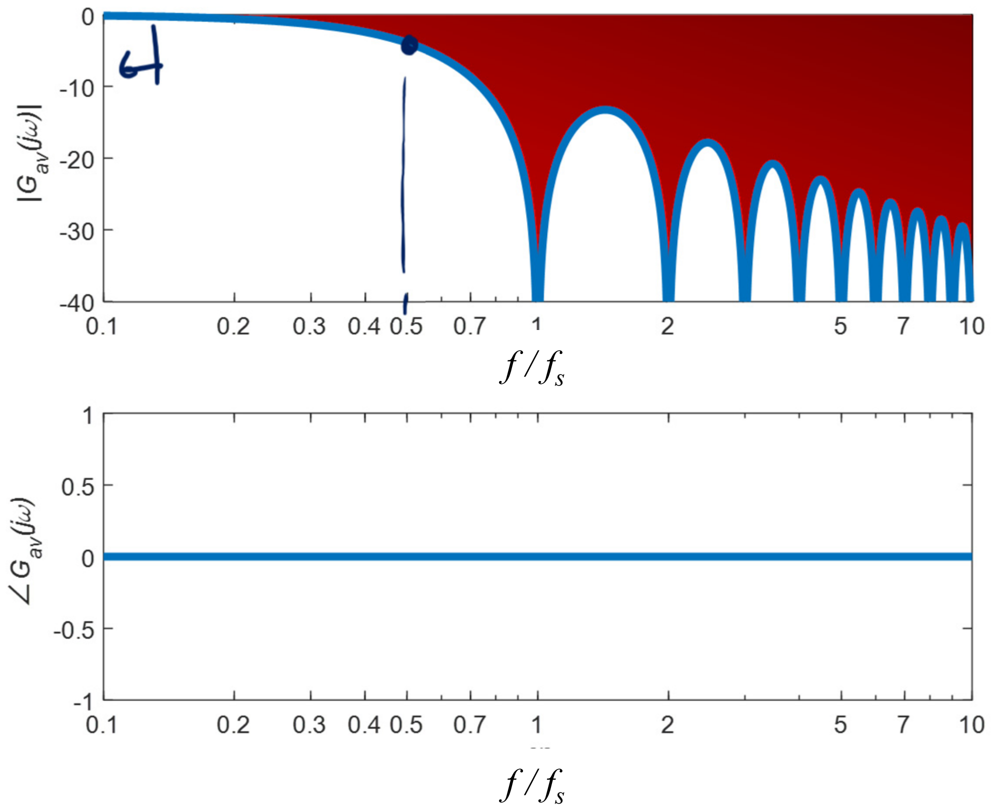
where $\boldsymbol{X}$ and $\boldsymbol{U}$ are defined at the large-signal steady-state operating point

$$\langle \dot{\boldsymbol{x}}(t)\rangle = \boldsymbol{0} = \boldsymbol{AX} + \boldsymbol{BU}$$

$$\boldsymbol{X} = \boldsymbol{A}^{-1}(-\boldsymbol{BU})$$

Impacts of Averaging

- Moving average filter completely attenuates f_s and harmonics
- Artificial modification of true circuit dynamics
  - Model is incorrect at frequencies approaching f_s
- Minimal impact on systems with only switching ripple at high frequency



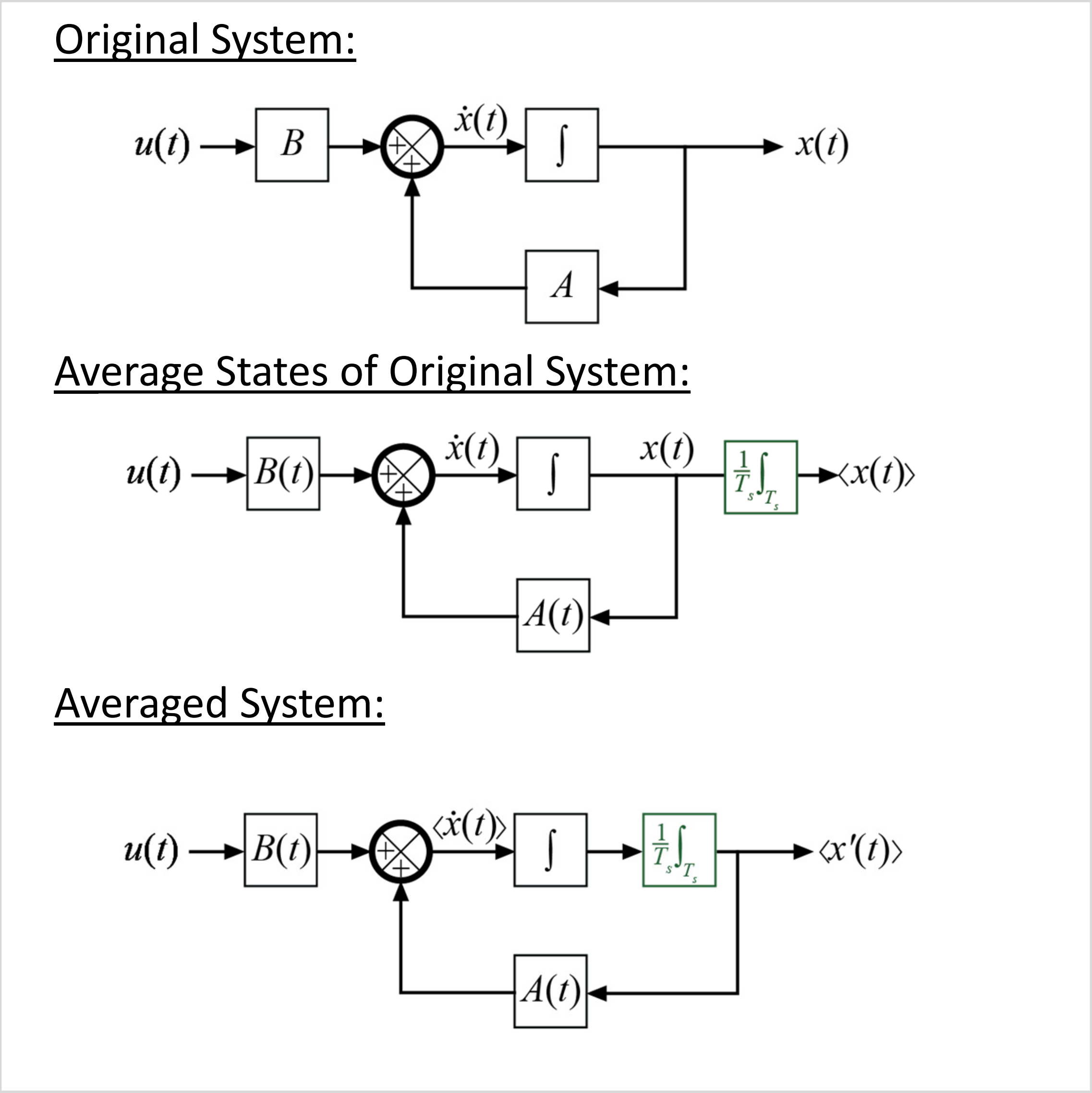
$$\mathcal{F}\left\{\frac{1}{T_s}\int_{t-\frac{T_s}{2}}^{t+\frac{T_s}{2}} x(\tau)\,d\tau\right\} = \frac{e^{j\omega\frac{T_s}{2}}X(j\omega) - e^{-j\omega\frac{T_s}{2}}X(j\omega)}{j\omega T_s}$$

$$= \text{sinc}\left(\omega\frac{T_s}{2}\right)\cdot X(j\omega)$$

$$= G_{av}(j\omega)\cdot X(j\omega)$$

Limitations of Average Modeling

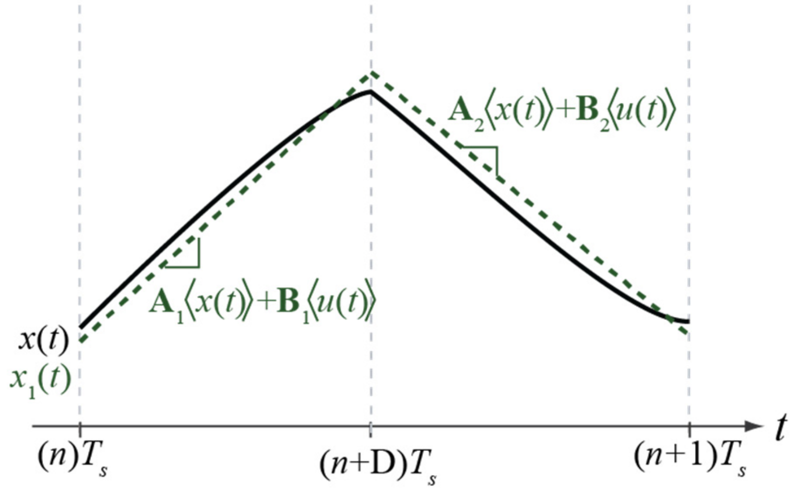
- Average modeling only meant to model low-frequency dynamics
- Introduction of the averaging function *destroys* some internal system dynamics
  - \+ Switching ripple on dc output ports
  - − Resonant and soft switching dynamics
  - − ac waveforms
- Inherent sampling behaviors of PWM modulators unmodeled



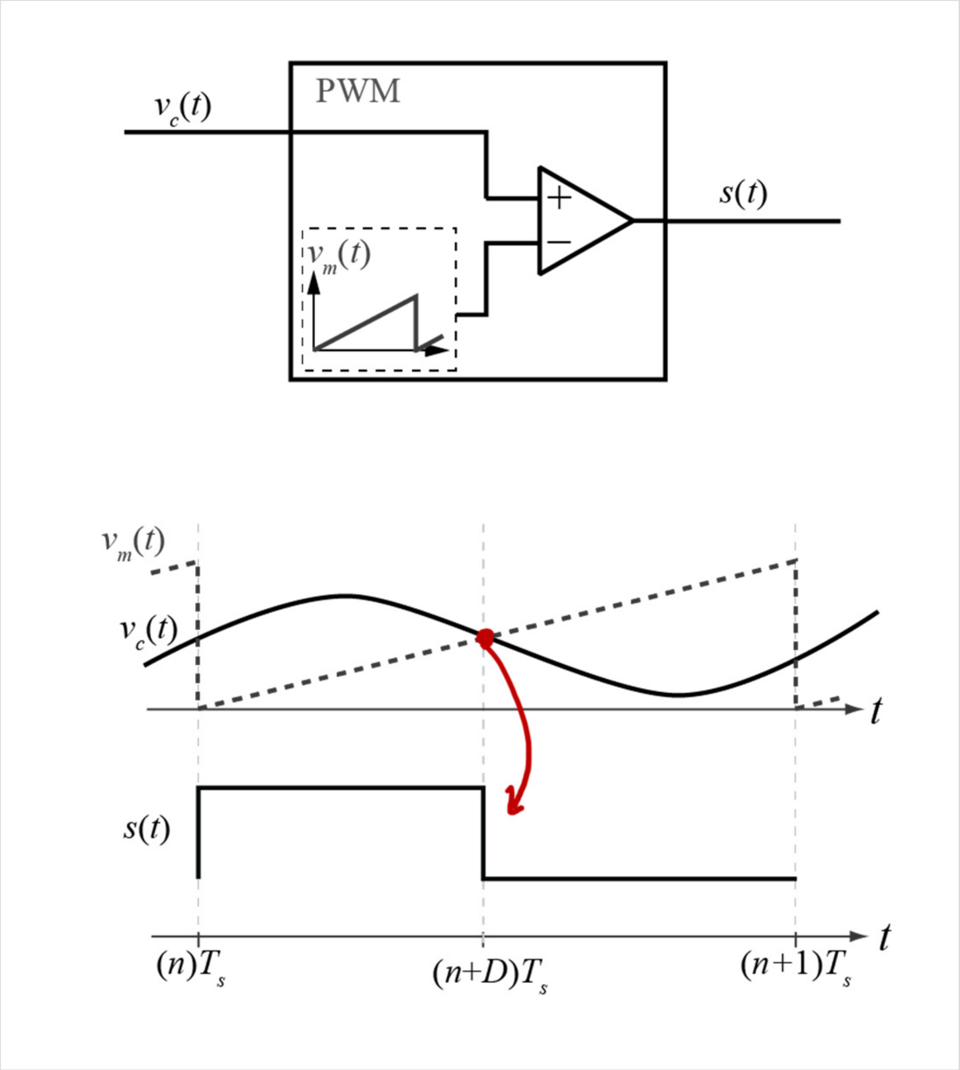
Average Modeling

- Model Error
- Neglected Effects
  - Sampling Effects
  - Delay
  - Quantization

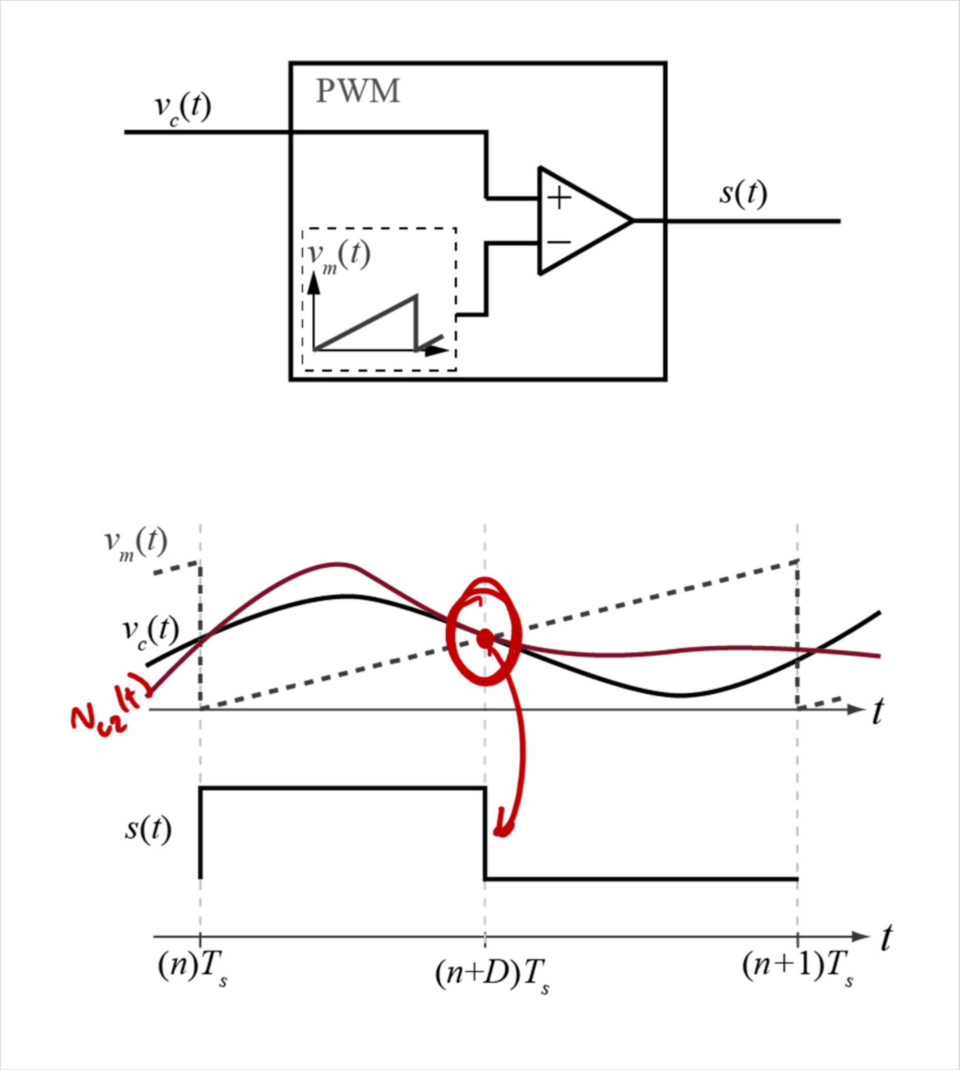
Design $f_c < f_s/10$
Add HF pole to attenuate switching ripple



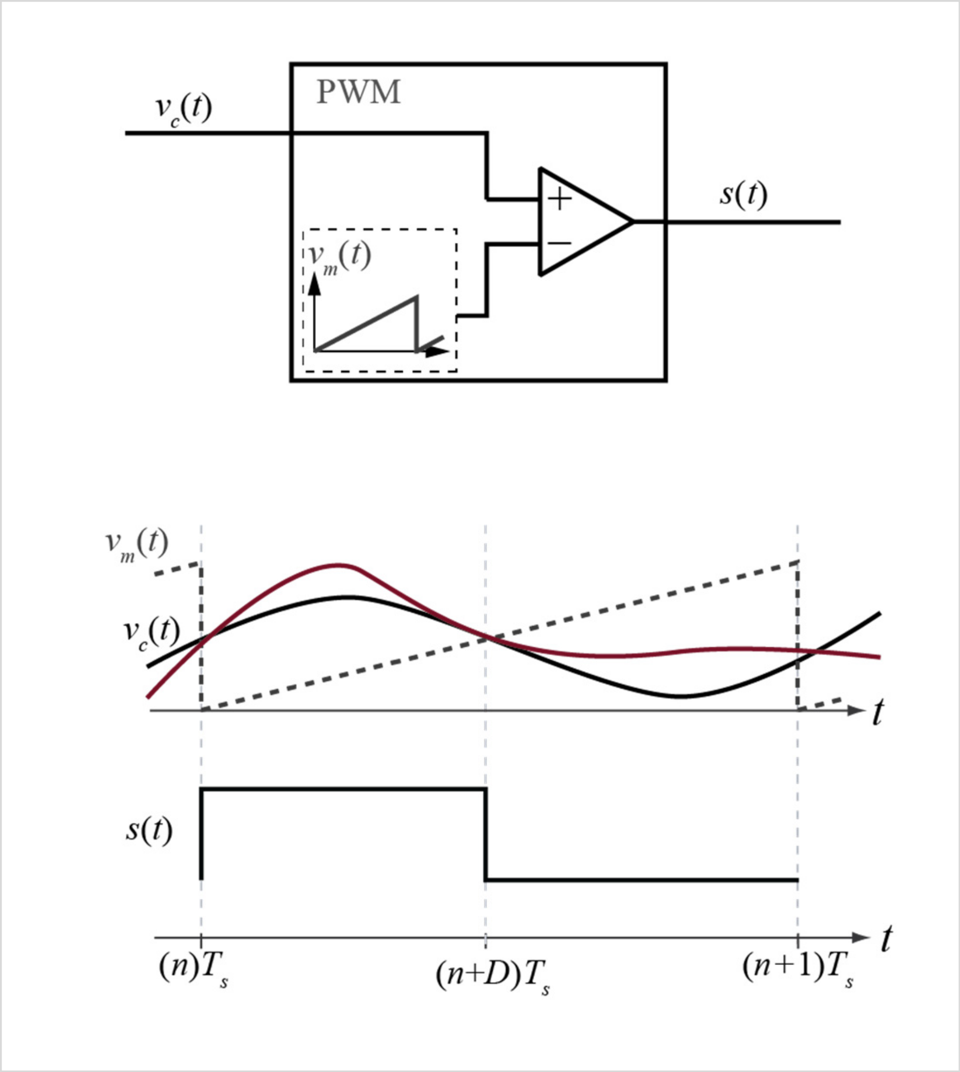
Discrete Time Nature of PWM

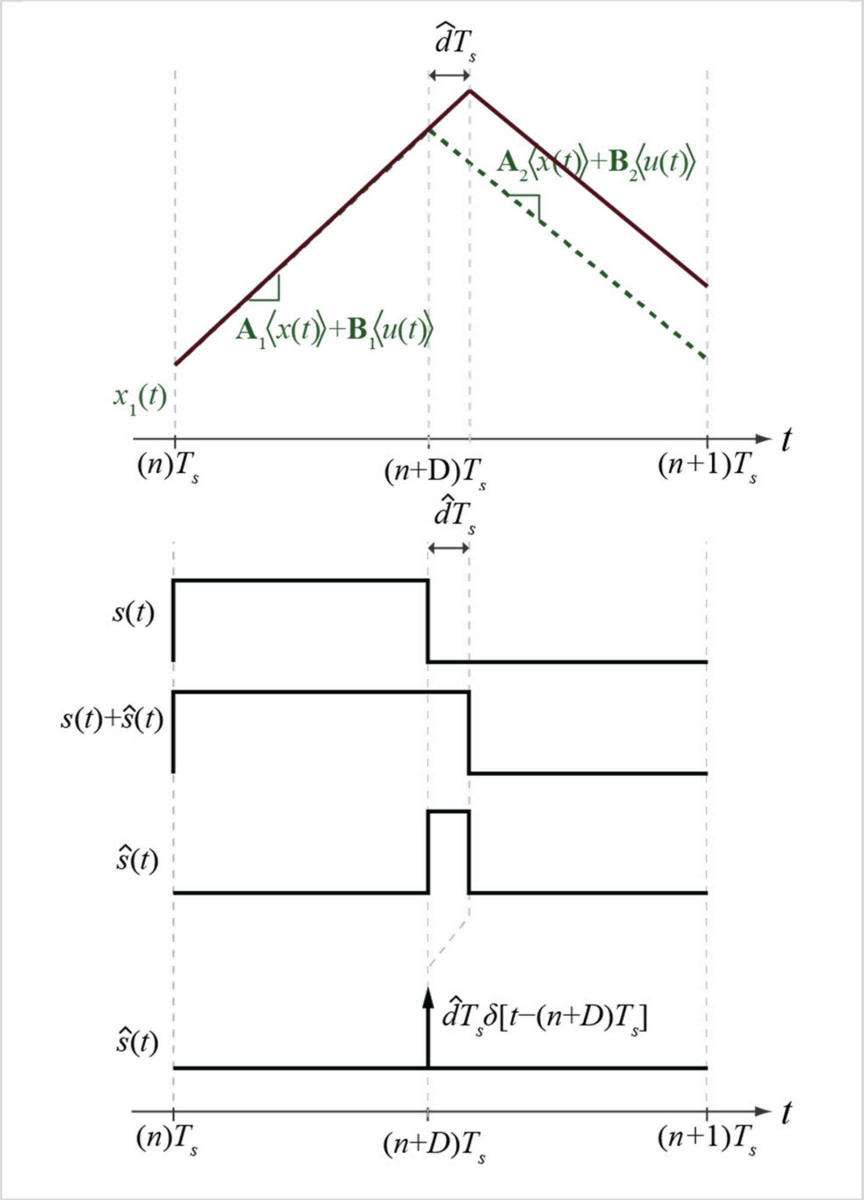


Discrete Time Nature of PWM



Discrete Time Nature of PWM





Sampling Phenomena

- PWM, itself is a sampler
  - Analog PWM is naturally sampled at the comparison point
  - Digital PWMs often uniformly sampled
    - Leading edge, trailing edge, dual edge
    - Shadow registers for T_s hold
- Additional sampling action from ADC in digitally-controlled converters

Historical Perspective

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PhD, Standford, 1955
CalTech Professor, 1955-1998

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where

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$$\boldsymbol{B} = D\boldsymbol{B}_1 + D'\boldsymbol{B}_2$$

Dennis John Packard
PhD, CalTech 1976

Discrete modeling and analysis of switching regulators

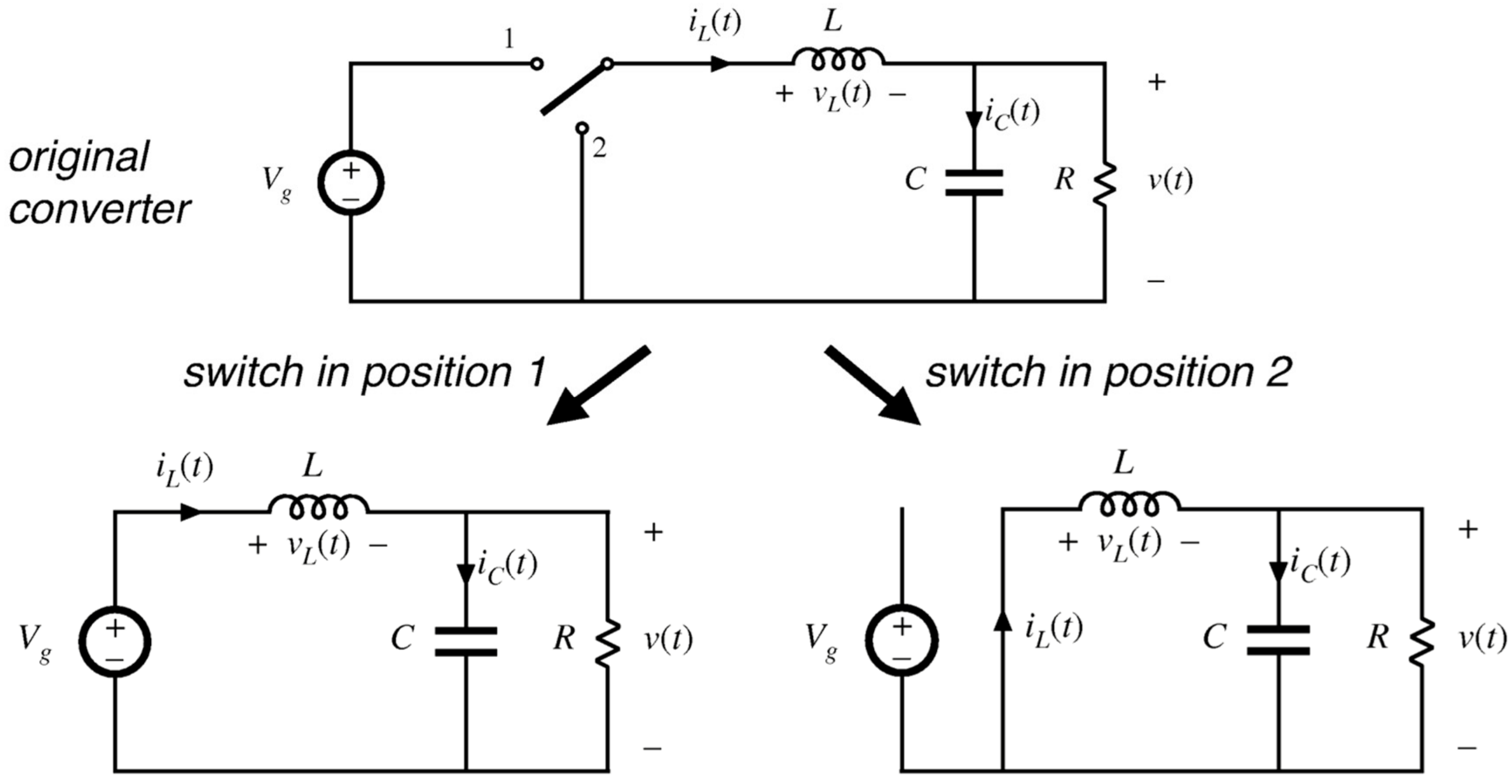
Model a switched system as a discrete-time system with

$$\boldsymbol{x}[n+1] = \boldsymbol{\Phi}\boldsymbol{x}[n] + \boldsymbol{\Psi}U[n]$$

where

$$\boldsymbol{\Phi} = \left(\prod_{i=n_{sw}}^{1} e^{\boldsymbol{A}_i t_i}\right)$$
$$\boldsymbol{\Psi} = \sum_{i=1}^{n_{sw}} \left\{ \left(\prod_{k=n_{sw}}^{i+1} e^{\boldsymbol{A}_k t_k}\right) \boldsymbol{A}_i^{-1} \left(e^{\boldsymbol{A}_i t_i} - \boldsymbol{I}\right) \boldsymbol{B}_i \right\}$$

Large Signal Modeling of SMPS



Discrete Time Modeling

- Every subcircuit is a passive, linear circuit
- Passive, linear circuits can be solved in closed-form
  - Can model states at discrete times without averaging
- Only assumption required
  - Independent inputs are DC or slowly varying

Solution to State Space Equation

Closed form solution to state space equation

$$\dot{\boldsymbol{x}}(t) = \boldsymbol{A}\boldsymbol{x}(t) + \boldsymbol{B}u(t)$$

Multiply both sides by $e^{-\boldsymbol{A}t}$

$$e^{-\boldsymbol{A}t}\dot{\boldsymbol{x}}(t) - e^{-\boldsymbol{A}t}\boldsymbol{A}\boldsymbol{x}(t) = e^{-\boldsymbol{A}t}\boldsymbol{B}u(t)$$

Left-hand side is

$$\frac{d}{dt}\left(e^{-\boldsymbol{A}t}\boldsymbol{x}(t)\right) = e^{-\boldsymbol{A}t}\boldsymbol{B}u(t)$$

Solution to State Space Equation

$$\frac{d}{dt}\left(e^{-At}\boldsymbol{x}(t)\right) = e^{-At}\boldsymbol{B}u(t)$$

Can now be solved by direct integration

$$e^{-At}\boldsymbol{x}(t) - \boldsymbol{x}(0) = \int_0^t e^{-A\tau}\boldsymbol{B}u(\tau)\,d\tau$$

Rearranging

$$\boldsymbol{x}(t) = e^{At}\boldsymbol{x}(0) + \int_0^t e^{-A(t-\tau)}\boldsymbol{B}u(\tau)\,d\tau$$

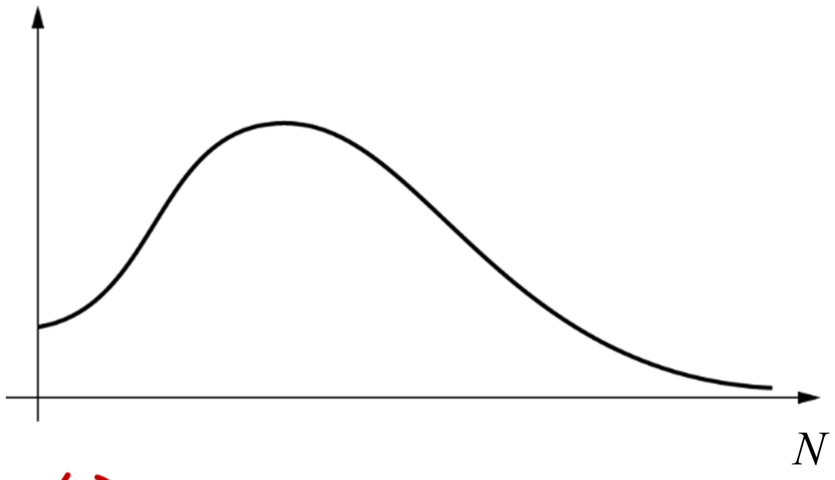
Matrix Exponential

Matrix exponential defined by Taylor series expansion

$$e^{\boldsymbol{A}t} = I + \boldsymbol{A}t + \frac{(\boldsymbol{A}t)^2}{2!} + \cdots + \frac{(\boldsymbol{A}t)^N}{N!} = \sum_{k=0}^{N} \frac{(\boldsymbol{A}t)^k}{k!}$$

Well-known issue with convergence in many cases

$$\left| e^{\boldsymbol{A}t} - \sum_{k=0}^{N} \frac{(\boldsymbol{A}t)^k}{k!} \right|$$



Matlab $e^A \rightarrow$ expm(·)

C. Moler and C. V. Loan, “Nineteen dubious ways to compute the exponential of a matrix,” SIAM Review, vol. 20, pp. 801–836, 1978.

Properties of the Matrix Exponential

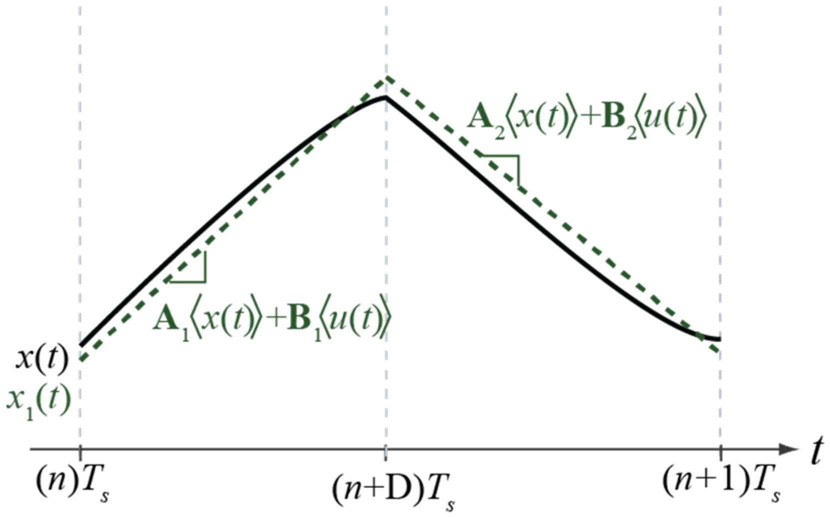
- Matrix exponential always exists
  - i.e. summation will always converge
- Exponential of any matrix is always invertible, with

$$e^{\boldsymbol{A}}e^{-\boldsymbol{A}} = \boldsymbol{I}$$

First Order Taylor Series Expansion

Linear ripple approximation

$$e^{At} \approx I + At$$



Valid only if switching frequency much faster than system modes

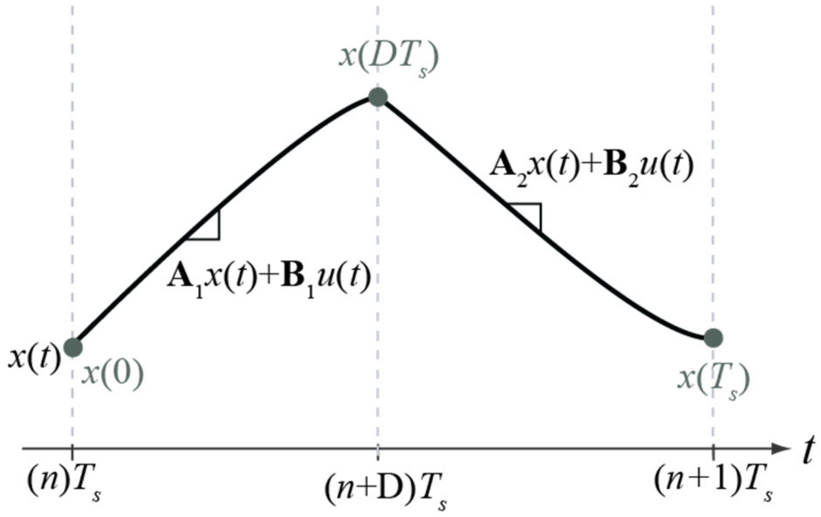
Simplification for Slow-Varying Inputs

$$\boldsymbol{x}(t) = e^{At}\boldsymbol{x}(0) + \int_0^t e^{-A(t-\tau)}\boldsymbol{B}u(\tau)\,d\tau$$

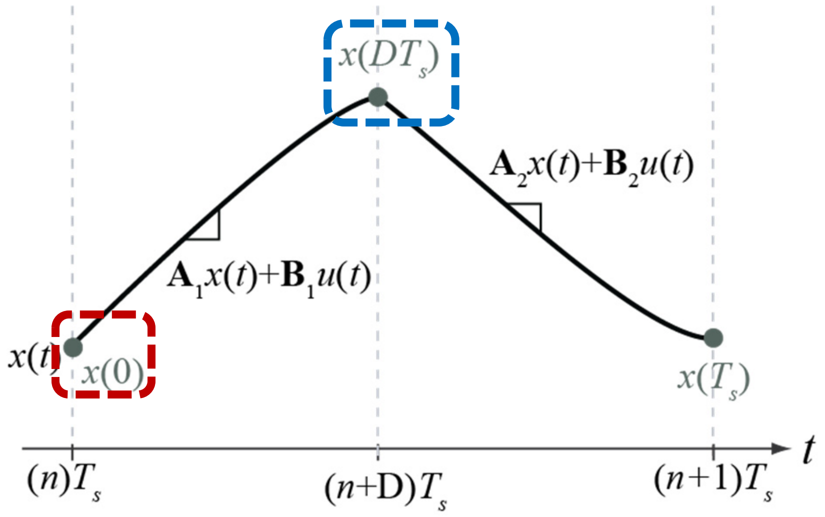
If A is invertible and $u(\tau) \approx U$

$$\boldsymbol{x}(t) = e^{At}\boldsymbol{x}(0) + \boldsymbol{A}^{-1}(e^{At} - \boldsymbol{I})\boldsymbol{B}U$$

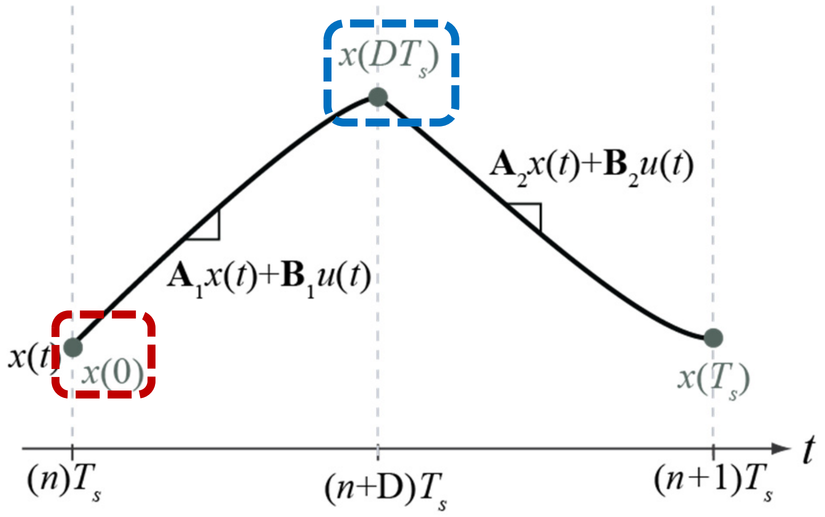
Application to Switching Converter



Application to Switching Converter

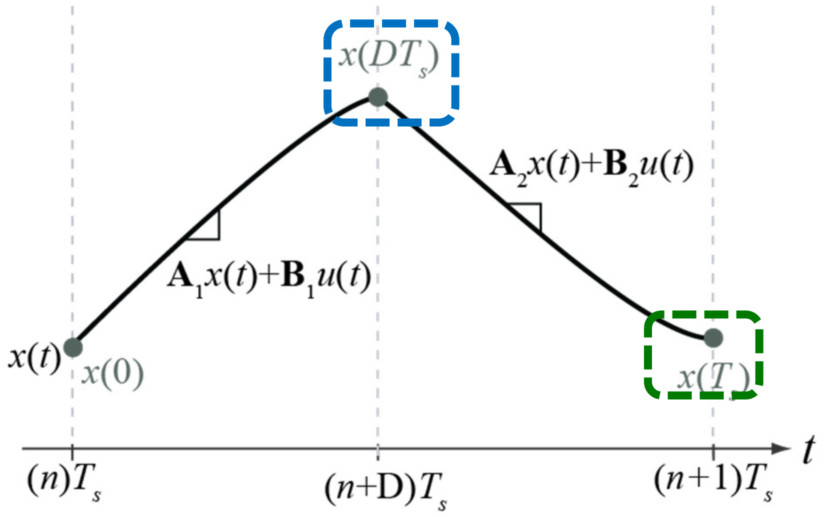


Application to Switching Converter



$$\boldsymbol{x}(DT_s) = e^{\boldsymbol{A}_1 DT_s}\boldsymbol{x}(0) + \boldsymbol{A}_1{}^{-1}(e^{\boldsymbol{A}_1 DT_s} - \boldsymbol{I})\boldsymbol{B}_1 U$$

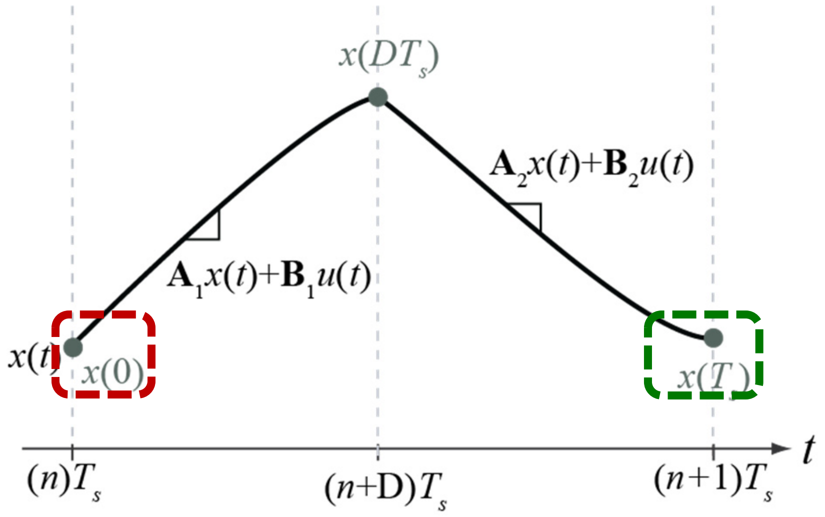
Application to Switching Converter



$$\boldsymbol{x}(DT_s) = e^{\boldsymbol{A}_1 DT_s}\boldsymbol{x}(0) + \boldsymbol{A}_1{}^{-1}(e^{\boldsymbol{A}_1 DT_s} - \boldsymbol{I})\boldsymbol{B}_1 U$$

$$\boldsymbol{x}(T_s) = e^{\boldsymbol{A}_2 D' T_s}\boldsymbol{x}(DT_s) + \boldsymbol{A}_2{}^{-1}(e^{\boldsymbol{A}_2 D' T_s} - \boldsymbol{I})\boldsymbol{B}_2 U$$

Application to Switching Converter



$$\boldsymbol{x}(DT_s) = e^{\boldsymbol{A}_1 DT_s}\boldsymbol{x}(0) + \boldsymbol{A}_1{}^{-1}(e^{\boldsymbol{A}_1 DT_s} - \boldsymbol{I})\boldsymbol{B}_1 U$$

$$\boldsymbol{x}(T_s) = e^{\boldsymbol{A}_2 D' T_s}\boldsymbol{x}(DT_s) + \boldsymbol{A}_2{}^{-1}(e^{\boldsymbol{A}_2 D' T_s} - \boldsymbol{I})\boldsymbol{B}_2 U$$

$$\boldsymbol{x}(T_s) = e^{\boldsymbol{A}_2 D' T_s} e^{\boldsymbol{A}_1 DT_s}\boldsymbol{x}(0) + \boldsymbol{A}_2{}^{-1}\left(e^{\boldsymbol{A}_2 D' T_s} - \boldsymbol{I}\right)\boldsymbol{B}_2 U + e^{\boldsymbol{A}_2 D' T_s}\boldsymbol{A}_1{}^{-1}(e^{\boldsymbol{A}_1 DT_s} - \boldsymbol{I})\boldsymbol{B}_1 U$$

General Form

Generally, for n_{sw} separate switching positions

$$\boldsymbol{x}(T_s) = \left(\prod_{i=n_{sw}}^{1} e^{A_i t_i}\right)\boldsymbol{x}(0) + \sum_{i=1}^{n_{sw}}\left\{\left(\prod_{k=n_{sw}}^{i+1} e^{A_k t_k}\right)\boldsymbol{A}_i{}^{-1}\left(e^{A_i t_i} - \boldsymbol{I}\right)\boldsymbol{B}_i\right\}U$$

Equation is in the form of a discrete-time system with

$$\boldsymbol{x}[n+1] = \boldsymbol{\Phi}\boldsymbol{x}[n] + \boldsymbol{\Psi}U[n]$$

Again, the effect of changing modulation (i.e. t_i) is hidden in nonlinear terms

$$\hat{\boldsymbol{x}}[n+1] = \boldsymbol{\Phi}\hat{\boldsymbol{x}}[n] + \boldsymbol{\Psi}\hat{u}[n] + \boldsymbol{\Gamma}\hat{d}[n]$$

Find $\boldsymbol{\Gamma}$ by small-signal modeling

Steady-State Large-Signal Analysis

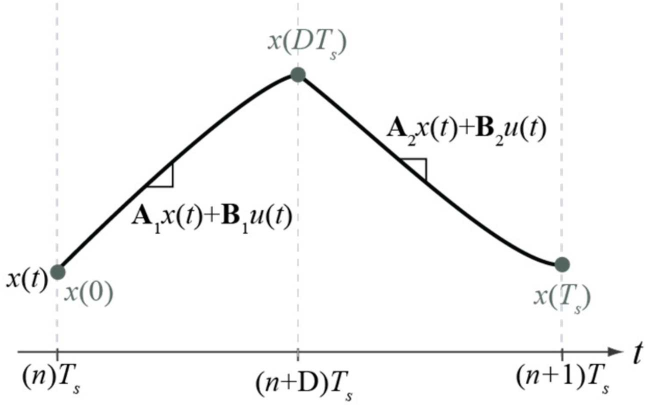
$$\boldsymbol{x}(T_s) = \left(\prod_{i=n_{sw}}^{1} e^{\boldsymbol{A}_i t_i}\right)\boldsymbol{x}(0) + \sum_{i=1}^{n_{sw}}\left\{\left(\prod_{k=n_{sw}}^{i+1} e^{\boldsymbol{A}_k t_k}\right)\boldsymbol{A}_i^{-1}\left(e^{\boldsymbol{A}_i t_i} - \boldsymbol{I}\right)\boldsymbol{B}_i\right\}U$$

In steady-state, $\boldsymbol{x}(T_s) = \boldsymbol{x}(0)$

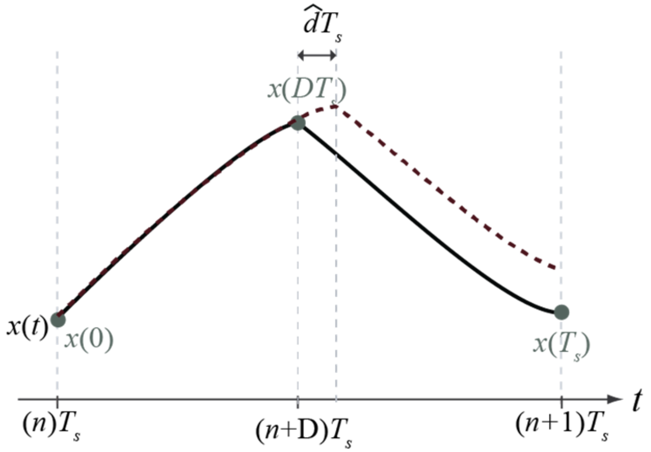
$$\boldsymbol{x}(T_s) = \left(I - \prod_{i=n_{sw}}^{1} e^{\boldsymbol{A}_i t_i}\right)^{-1} \sum_{i=1}^{n_{sw}}\left\{\left(\prod_{k=n_{sw}}^{i+1} e^{\boldsymbol{A}_k t_k}\right)\boldsymbol{A}_i^{-1}\left(e^{\boldsymbol{A}_i t_i} - \boldsymbol{I}\right)\boldsymbol{B}_i\right\}U$$

Gives explicit solution for steady-state operation of any switching circuit

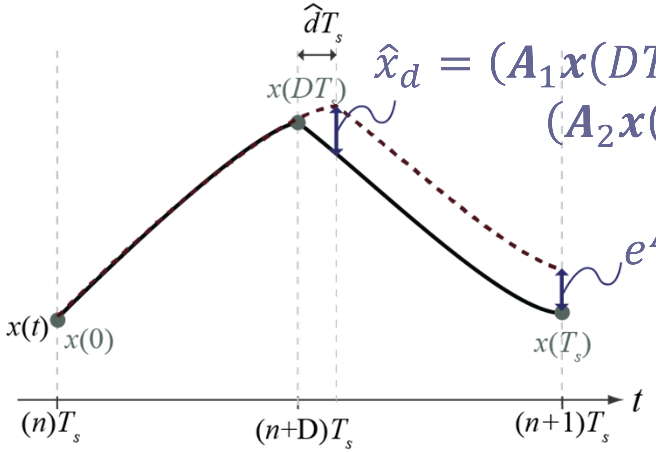
Small Signal Modeling



Small Signal Modeling



Small Signal Modeling



$$\hat{x}_d = (\boldsymbol{A}_1\boldsymbol{x}(DT_s) + \boldsymbol{B}_1 U)\hat{d}T_s - (\boldsymbol{A}_2\boldsymbol{x}(DT_s) + \boldsymbol{B}_2 U)\hat{d}T_s$$

$$e^{\boldsymbol{A}_2 D' T_s}\hat{x}_d$$

Complete Small Signal Model

This completes the small-signal model

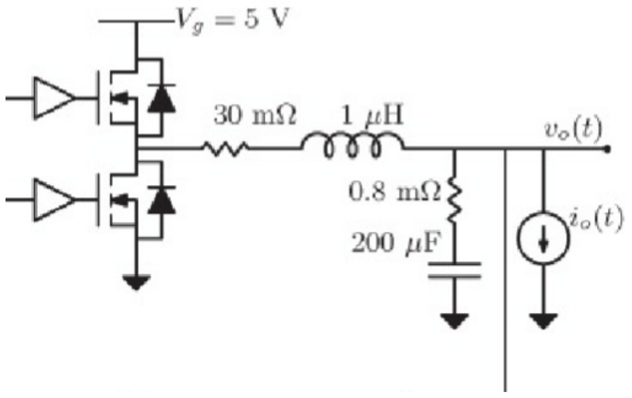
$$\hat{\boldsymbol{x}}[n+1] = \boldsymbol{\Phi}\hat{\boldsymbol{x}}[n] + \boldsymbol{\Psi}\hat{u}[n] + \boldsymbol{\Gamma}\hat{d}[n]$$

where

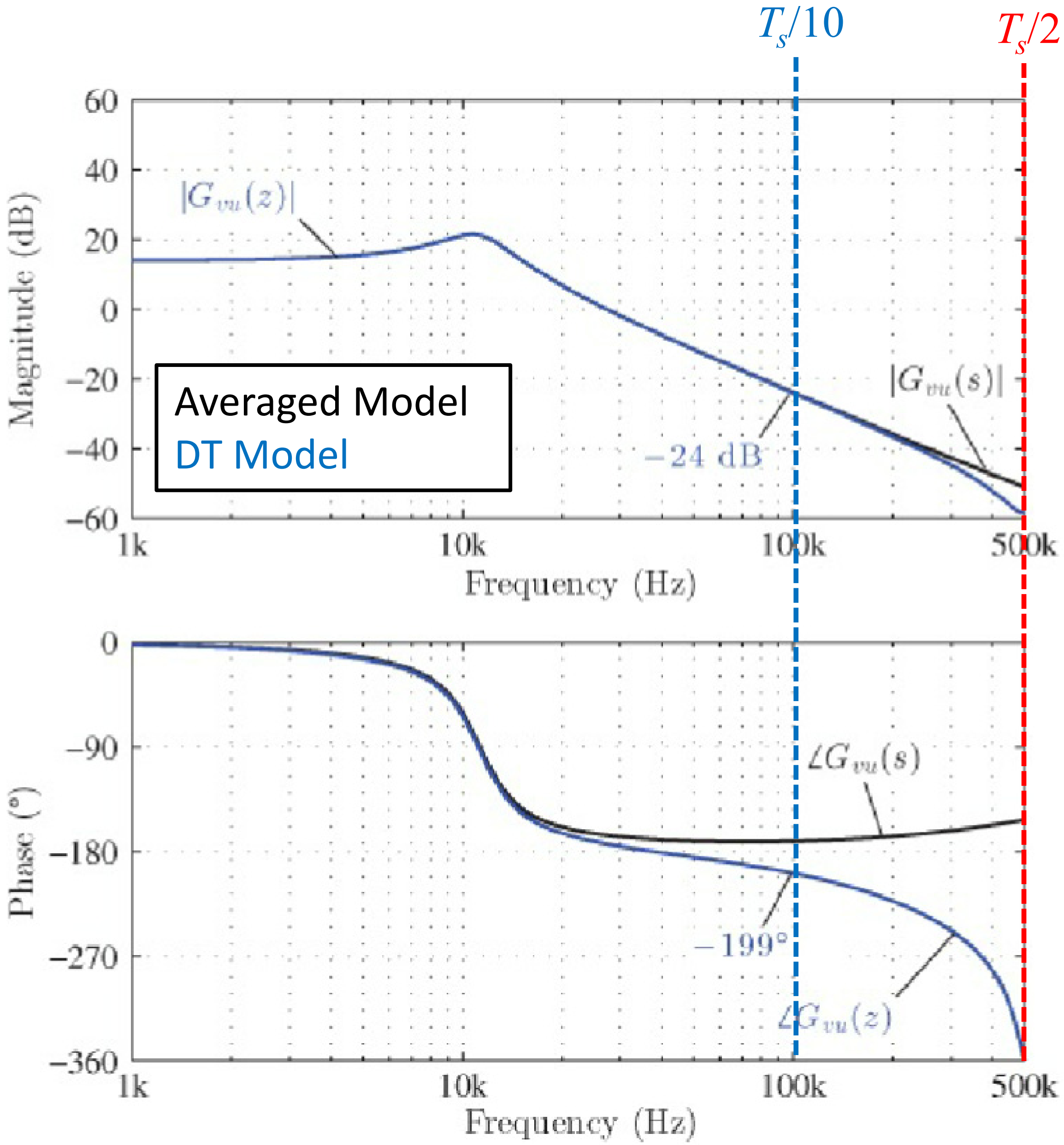
$$\boldsymbol{\Gamma} = e^{\boldsymbol{A}_2 D' T_s}\left((\boldsymbol{A}_1 - \boldsymbol{A}_2)\boldsymbol{X}_D + (\boldsymbol{B}_1 - \boldsymbol{B}_2)U\right)T_s$$

with $\boldsymbol{X}_D = \boldsymbol{x}(DT_s)$ in steady-state

Example Results



* Includes t_d=760ns of delay in feedback loop



ECE 692: Discrete Time Power Electronics

- Taught in alternative Fall semesters with ECE 581
- Covers discrete time modeling for small and large-signal converter analysis and design