Propagation of Pulses on Transmission Lines

So far we have been talking about single-frequency (harmonic) “signals”, which carry no information.

To transmit information, we must modulate the signal.

For example, we modulate a microwave carrier to send a bit down the line.

This envelope travels along the line at speed $v_g$, the “group velocity,” which is usually a little different from $v_p$, due to dispersion.
For simplicity, we ignore dispersion and assume $v_g = v_p$.

If $Z_L = Z_0$, this pulse is totally absorbed upon arrival at the load. This is what we want.

For a lossless line, $Z_0$ is real.

If $Z_L$ is purely resistive, this match is (assumed to be) frequency-independent.
If $Z_L \neq Z_0 \neq Z_g$, things become complicated.

We first look at the case, $\tau < \frac{l}{v_p}$:

Lots of echoes. Echoes die off.
First case, $\tau < l/v_p$:

Lots of echoes. Echoes die off. May corrupt other bits

Second case, $\tau > l/v_p$:

The bit is distorted and broadened.

$Z_L \neq Z_0 \neq Z_g$
Recall that we always have multiple reflections inside any matching network.

Does impedance matching really help us? Why?

\[ z_L = 0.5 - j \]

\[ y_L = 0.4 + 0.8j \]

\[ l = 0.09\lambda \]

\[ d = 0.063\lambda \]

\[ y(d) = 1 + 1.58j \]
Recall that we always have multiple reflections inside any matching network.

Does impedance matching really help us? Why?

Notice that we can always choose to have $d < \lambda/2$ and $l < \lambda/2$.

The time to travel $\lambda/2$ is $1/(2f)$. Thus the bit is broadened only by several $1/f$, at most.

Without matching, we have echoes.

Recall that we have two solutions for $d < \lambda/2$. We may want to choose the smaller $d$.

The modulated case is quite complicated. We now look into a simple case quantitatively.
\( z_L = 0.5 - j \quad \quad y_L = 0.4 + 0.8j \)

On this circle, \( g = 1 \)
\( y = 1 + jb \), i.e., \( Y = Y_0 + jB \)

\( d = 0.178 \lambda - 0.115 \lambda \)
\( = 0.063 \lambda \)
\( y(d) = 1 + 1.58j \)

When working with \( y(d) \), keep in mind that \( y(d) \leftrightarrow -\Gamma_d \).

When \( y(d) \) is on \( g = 1 \) circle of the \( y \)-chart, \( z(d) \) is on \( g = 1 \) circle of the \( z \)-chart, i.e. the \( r = 1 \) circle of the \( y \)-chart.
\[ z_L = 0.5 - j \quad y_L = 0.4 + 0.8j \]

On this circle, \( g = 1 \)

\( y = 1 + jb \), i.e.,

\( Y = Y_0 + jB \)

\[ d = 0.178\lambda - 0.115\lambda \]
\[ = 0.063\lambda \]

\( y(d) = 1 + 1.58j \)

When working with \( y(d) \), keep in mind that \( y(d) \leftrightarrow -\Gamma_d \).

When \( y(d) \) is on \( g = 1 \) circle of the \( y \)-chart, \( z(d) \) is on \( g = 1 \) circle of the \( z \)-chart, i.e. the \( r = 1 \) circle of the \( y \)-chart.
First, let’s list the basic assumptions to be used:

1. Lossless line. $Z_0$ is purely real.
3. Therefore, $\Gamma$ is frequency-independent.
   (If $R_L = Z_0$, impedance matched for all frequencies)
4. Dispersionless: $v_g = v_p$ for all frequencies.

Know the simplifying assumptions. Know the limitations.
Propagation of a voltage step on a transmission line

For $0 < t < T = l/v_p$

The “turn-on” event has not reached the load yet. It does not know about $R_L$.

The transmission line feels like infinitely long. In other words, no reflection yet.

What is the equivalent input impedance seen by the incident voltage step at $z = 0$?
Propagation of a voltage step on a transmission line

For $0 < t < T = l/v_p$, 

The “turn-on” event has not reached the load yet. It does not know about $R_L$. The transmission line feels like infinitely long. In other words, no reflection yet.

![Diagram of transmission line with voltage step](image)

The equivalent input impedance seen by the incident voltage step at $z = 0$ is $Z_0$. Not $Z_{\text{in}}$!

Subscript “1” means the first round trip. Superscript “+” means the incident direction.

\[ V_i^+ = \frac{V_g Z_0}{R_g + Z_0} \]

\[ I_i^+ = \frac{V_g}{R_g + Z_0} \]
Propagation of a voltage step on a transmission line

\[ V_i^+ = \frac{V_g Z_0}{R_g + Z_0} \]

\[ I_i^+ = \frac{V_g}{R_g + Z_0} \]

Snapshots at \( t = T/2 \)

Edge/front moving at \( v_p \) (actually \( v_g \))

The leading edge reaches the load at \( t = T \). Reflection.

\[ V_i^- = \Gamma_l \cdot V_i^+ \]

\[ I_i^- = \Gamma_l \cdot I_i^+ \]

What is the voltage at the load at \( t = T \)?
What is the voltage at the load at $t = T$? $V_i^+ + V_i^-$

Snapshots at $t = 3T/2$

At $t = 2T$, the front hits the source. Reflection.

Snapshots at $t = 3T/2$

What is the voltage at the load at $t = 2T$?

Assuming

\[ \Gamma_g > 0 \]
\[ \Gamma_g > 0 \]

\[ \Gamma_g = \frac{R_g - Z_0}{R_g + Z_0} \]

We paused here on Tue 10/11/2022.
What is the voltage at the load at $t = T$?

Snapshots at $t = 3T/2$

Snapshots at $t = 5T/2$

At $t = 2T$, the front hits the source. Reflection.

$V_2^+ = \Gamma_g V_1^- = \Gamma_g \Gamma_L V_1^+$

$I_2^+ = -\Gamma_g I_1^- = \Gamma_g \Gamma_L I_1^+$

Assuming

$\Gamma_L > 0$

$\Gamma_g > 0$

$\Gamma_g = \frac{R_g - Z_o}{R_g + Z_o}$

What is the voltage at the load at $t = 2T$?
At $t = 3T$, the front hits the load again.

$$V_2^+ + V_2^- = V_2^+ \left(1 + \Gamma_L \right)$$

$$I_2^+ + I_2^- = I_2^- \left(1 - \Gamma_L \right)$$

Again, notice the sign.

Again, notice that reflection happens instantaneously.

It goes on and on. For the $i$th round trip,

$$V_i^+ + V_i^- = V_i^+ \left(1 + \Gamma_L \right)$$

$$I_i^+ + I_i^- = I_i^- \left(1 - \Gamma_L \right)$$

$$V(t = \infty) = V_1^+ + V_1^- + V_2^+ + V_2^- + V_3^+ + V_3^- + \ldots$$

$$= \sum_{i=1}^{\infty} \left( V_i^+ + V_i^- \right)$$

$$= V_1^+ \left(1 + \Gamma_L \right) + V_2^+ \left(1 + \Gamma_L \right) + \ldots = \left(1 + \Gamma_L \right) \left[ V_1^+ + V_2^+ + \ldots \right]$$

$$= \left(1 + \Gamma_L \right) \sum_{i=1}^{\infty} V_i^+$$

Note: At the steady state, $v$ is the same at all $z$, therefore we do not specify $z$. 

$$V_2^- = \Gamma_L V_2^+$$

$$I_2^- = -\Gamma_L I_2^+$$
\[ u(x=\infty) = V_1^+ + V_1^- + V_2^+ + V_2^- + V_3^+ + V_3^- + \cdots \]
\[ = \sum_{i=1}^{\infty} (V_i^+ + V_i^-) \]
\[ = V_1^+ (1+\Gamma_\ell) + V_2^+ (1+\Gamma_\ell) + \cdots = (1+\Gamma_\ell) \left[ V_1^+ + V_2^+ + \cdots \right] \]
\[ = (1+\Gamma_\ell) \sum_{i=1}^{\infty} V_i^+ \]

\[ V_2^+ = \Gamma_g\Gamma_\ell V_1^+ \quad \quad V_{i+1}^+ = \Gamma_g\Gamma_\ell V_i^+ \]

\[ u(x=\infty) = V_1^+ (1+\Gamma_\ell) \left[ 1 + \Gamma_g\Gamma_\ell + (\Gamma_g\Gamma_\ell)^2 + \cdots \right] \]
\[ = V_1^+ (1+\Gamma_\ell) \sum_{i=0}^{\infty} (\Gamma_g\Gamma_\ell)^i \]

\[ u(x=\infty) = V_1^+ (1+\Gamma_\ell) \times ? \]
\( u(t = \infty) = V_i^+ + V_i^- + V_2^+ + V_2^- + V_3^+ + V_3^- + \ldots \)

\[
= \sum_{i=1}^{\infty} (V_i^+ + V_i^-)
\]

\[
= V_i^+ (1 + \Gamma_L) + V_2^+ (1 + \Gamma_L) + \ldots = (1 + \Gamma_L) \left[ V_i^+ + V_2^+ + \ldots \right]
\]

\[
= C (1 + \Gamma_L) \sum_{i=1}^{\infty} V_i^+
\]

\[
V_2^+ = \Gamma_g \Gamma_L V_1^+ \quad \text{and} \quad V_{i+1}^+ = \Gamma_g \Gamma_L V_i^+
\]

\[
u (t = \infty) = V_1^+ (1 + \Gamma_L) \left[ 1 + \Gamma_g \Gamma_L + (\Gamma_g \Gamma_L)^2 + \ldots \right]
\]

\[
= V_1^+ (1 + \Gamma_L) \sum_{i=0}^{\infty} (\Gamma_g \Gamma_L)^i
\]

Use

\[
l + x + x^2 + \ldots = \sum_{i=0}^{\infty} x^i = \frac{1}{1 - x}
\]

with \( x = \Gamma_g \Gamma_L \)

We get:

\[
u (t = \infty) = V_1^+ (1 + \Gamma_L) \cdot \frac{1}{1 - \Gamma_g \Gamma_L}
\]
\[
\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots + \frac{1}{2^i} + \cdots = \sum_{i=1}^{\infty} \frac{1}{2^i} = ?
\]
$$\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots + \frac{1}{2^i} + \cdots = \sum_{i=1}^{\infty} \frac{1}{2^i} = ?$$

$$1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots + \frac{1}{2^i} + \cdots = 1 + \sum_{i=1}^{\infty} \frac{1}{2^i} = 2$$

$$= \sum_{i=0}^{\infty} \frac{1}{2^i} = \frac{1}{1 - 1/2} = 2$$
\[ u(t = \infty) = V_i^+ (1 + \Gamma_i) \cdot \frac{1}{1 - \Gamma_g \Gamma_l} \]

\[ V_i^+ = \frac{V_g Z_0}{R_g + Z_0} \]

\[ \Gamma_g = \frac{R_g - Z_0}{R_g + Z_0} \]

\[ \Gamma_l = \frac{R_l - Z_0}{R_l + Z_0} \]

\[ u(t = \infty) = \frac{V_g R_L}{R_g + R_L} \]

Surprising?

Similarly,

\[ i(t = \infty) = I_i^+ (1 - \Gamma_i) \sum_{i=0}^{\infty} (\Gamma_g \Gamma_l)^i \]

\[ = I_i^+ \cdot \frac{1 - \Gamma_l}{1 - \Gamma_g \Gamma_l} \]

\[ = \frac{V_g}{R_g + R_L} \]

We have traced \( v(t) \) and \( i(t) \) all the way to \( t = \infty \).

That’s quite tedious.

We have a graphical tool to trace this bouncing back and forth.

It’s called the bounce diagram.

Review textbook Section 2-12 overview, Section 2-12.1

We went through this slide set and moved on to the next one on Thu 10/13/2022.