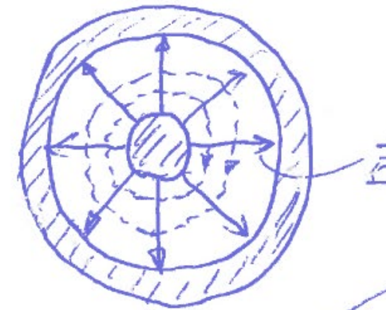
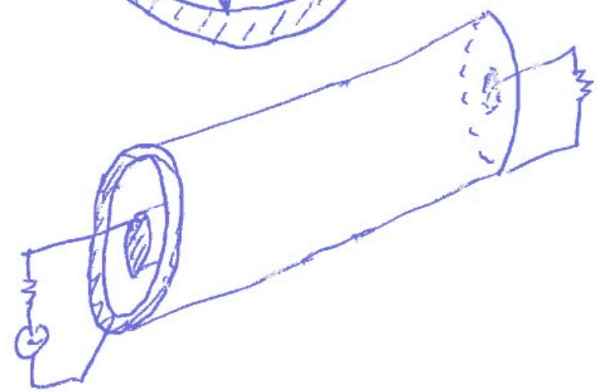
Electrostatics

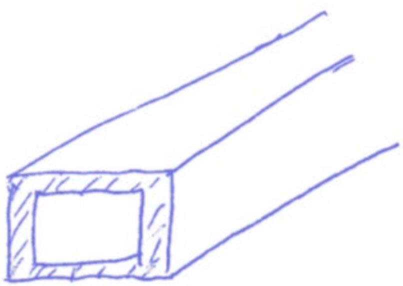
Now we have finished our discussion on transmission line theory.

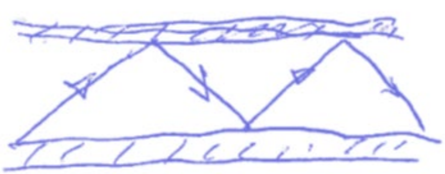
The transmission line theory deals with a special type of waveguides, where there are two conductors.

As such, you can define local voltages $v(z,t)$ and local currents $i(z,t)$. Such a distributive circuit theory is one step beyond the lumped element circuit theory.









But, there are other types of waveguides.

In general, you do not need two conductors to guide an EM wave.

A metal tube is a wave guide.

Here, you cannot define local voltages or currents. A “real” EM field theory is needed.

You may imagine a very coarse ray optics picture: metal walls are like mirrors. But this is not accurate. Ray optics breaks down when waveguide dimensions are comparable to the wavelength.

Besides waveguides, there are antennas and other things, where simple theories like lumped circuit theory and ray optics do not work.

We have to resort to the "real" EM field theory.

Electrostatics is the simplest field theory, about static electric fields.

Keep in mind that nothing is "static".

Keep in mind that fields are vectors.

One reason the transmission line theory is simpler is that voltages and currents are scalars.

To handle vectors, you need vector algebra and vector calculus.

Chapter 3 of the textbook is about the math needed.

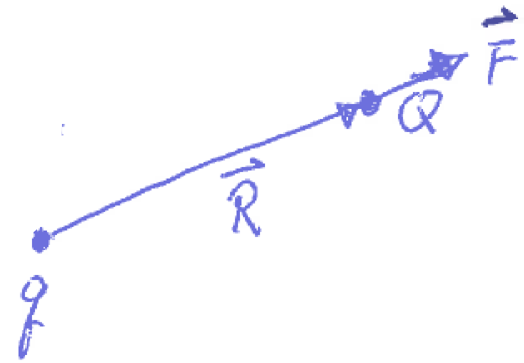
Read on your own. Individualized effort.

Homework 8 is about Chapter 3.

Coulomb's Law

First, let's review Coulomb's law in free space (vacuum).

How do you express the force **F** on the probe charge Q exerted by charge q, if the position of Q relative to q is represented by position vector **R**?



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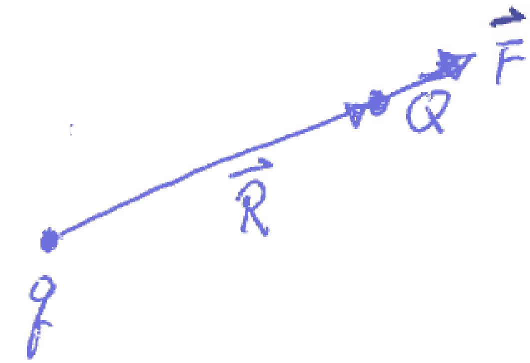

Pay attention to notations: vectors, scalars.

$$\vec{F} = \frac{qQ}{4\pi\varepsilon_0 R^2}\hat{R}$$

Define the unit vector:

$$\hat{R} = \frac{\vec{R}}{R}, \quad R = |\vec{R}|$$

$$\vec{R} = R\hat{R}$$

Take home messages:

- **F** is a vector. Its direction is the same as **R**.
- Its magnitude is proportional to R^2.
- Like charges repel each other. Opposite charges attract each other.
- Here we focus on free space (i.e. vacuum), where permittivity = ε_0. View that as just a proportional constant. Dielectrics will be discussed later.

A different way to write this is: $\vec{F} = \frac{qQ}{4\pi\varepsilon_0 R^3}\vec{R}$

Hand-written	Printed
$\vec{F} = \dfrac{qQ}{4\pi\varepsilon_0 R^2}\hat{R}$	$\mathbf{F} = \dfrac{qQ}{4\pi\varepsilon_0 R^2}\hat{\mathbf{R}}$
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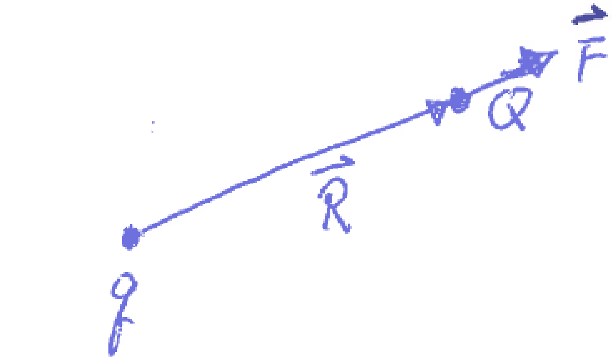
Vectors: bold; italic or not
Scalars: not bold; italic
Numbers: not bold; not italic



$$\vec{F} = \frac{qQ}{4\pi\varepsilon_0 R^2}\,\hat{R}$$

What if our probe charge is Q' at position $\mathbf{R}'$?

$$\vec{F}' = \frac{qQ'}{4\pi\varepsilon_0 R'^2}\,\hat{R}'$$

So, we can imagine something called the electric field

$$\vec{E} = \frac{q}{4\pi\varepsilon_0 R^2}\,\hat{R}$$

When you put probe charge Q into the the electric field $\mathbf{E}$ of charge q, the field exerts a force $\mathbf{F}$ on Q:

$$\vec{F} = Q\vec{E}$$

Hand-written	Printed
$\vec{F} = \frac{qQ}{4\pi\varepsilon_0 R^2}\hat{R}$	$\mathbf{F} = \frac{qQ}{4\pi\varepsilon_0 R^2}\hat{\mathbf{R}}$
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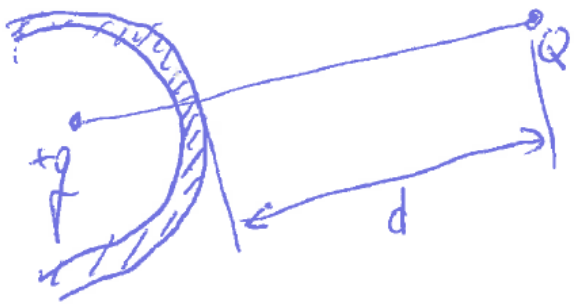
Long ago, people did not know whether the electric field is real or just a mathematical construct. Now we know it is real.



If you cover q with a metal lid, Q will feel the disappearance of the force after $t = d/c$, where c is speed of light.

In other words, the interaction cannot be instantaneous over distance.

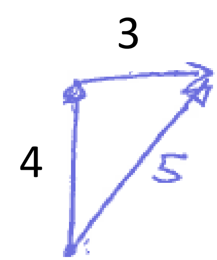
The field is the medium of the interaction.

Again, we emphasize that many quantities we deal with are vectors.

For vector quantities, keep in mind this very simple example:

You first walk 4 miles to the north, and then make a turn to the east and walk 3 miles. You are 5 miles away from where you started.

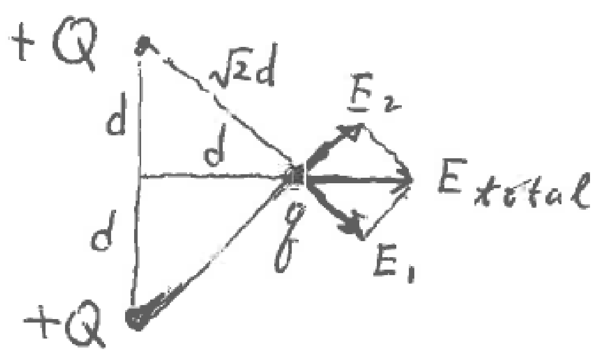
In the mathematical language, the sum of the two displacements is 5 miles (in the direction shown).



Total electric field of point charges

Example 1.1

Two point charges, each with a positive value $+Q$, are $2d$ apart. Find the total field at a point at distance d from the midpoint of the line segment connecting them. Find the force $\mathbf{F}$ on a point charge q, assumed to be positive.

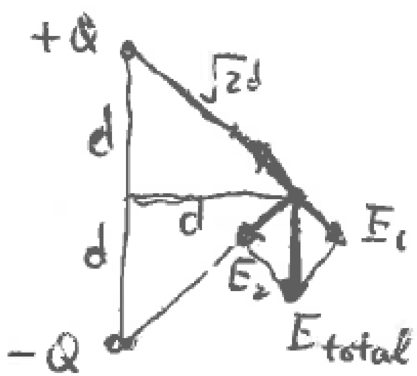


$$E_1 = E_2 = E = \frac{1}{4\pi\varepsilon_0}\frac{Q}{(\sqrt{2}d)^2} = \frac{1}{8\pi\varepsilon}\frac{Q}{d^2}$$

$$E_{total} = \sqrt{2}\,E = \frac{\sqrt{2}}{8\pi\varepsilon}\frac{Q}{d^2}$$

$$F = q E_{total}$$

Note: Here we use scalars since we know the directions of these vectors, which are shown in the figure.

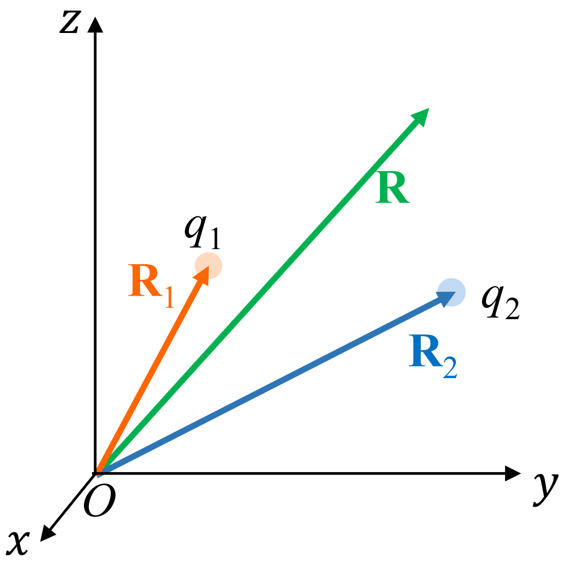


Example 1.2

Repeat the above for two point charges $+Q$ and $-Q$, assuming Q is positive.

Example 1.0: the general case

We use this generic example to get familiar with the notations.



Charges q_1 and q_2 are at positions $\mathbf{R}_1$ and $\mathbf{R}_2$. Find the total electric field a position $\mathbf{R}$.

$$\vec{E} = \vec{E}_1 + \vec{E}_2$$

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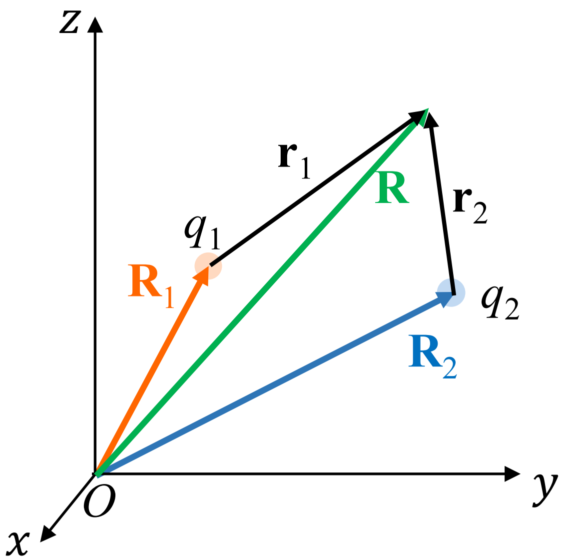

Charges q_1 and q_2 are at positions $\mathbf{R}_1$ and $\mathbf{R}_2$. Find the total electric field a position $\mathbf{R}$.

$$\vec{E} = \vec{E}_1 + \vec{E}_2 = \frac{1}{4\pi\varepsilon_0}\left(\frac{q_1}{r_1^2}\hat{r}_1 + \frac{q_2}{r_2^2}\hat{r}_2\right)$$

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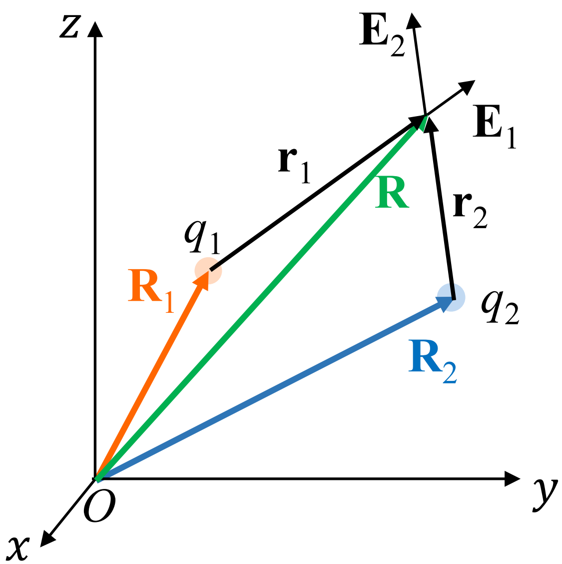

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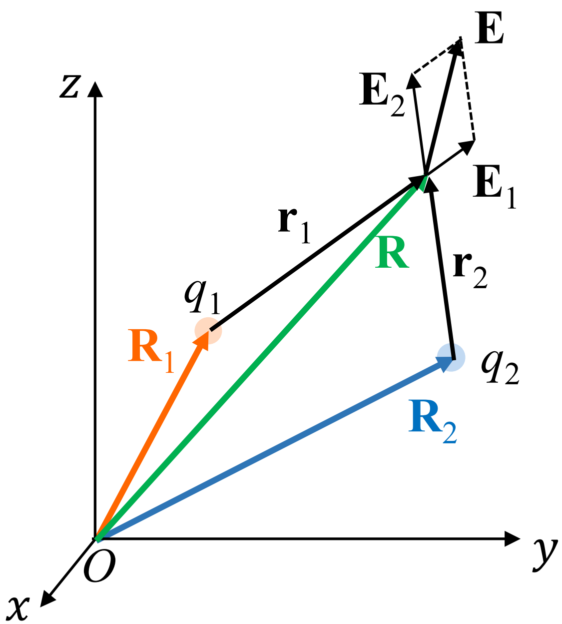

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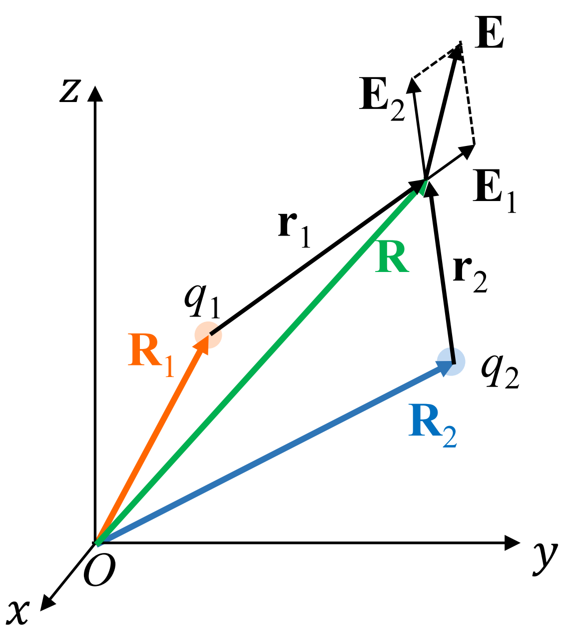

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$$= \frac{1}{4\pi\varepsilon_0}\left(\frac{q_1}{|\vec{R}-\vec{R}_1|^2}\frac{\vec{R}-\vec{R}_1}{|\vec{R}-\vec{R}_1|} + \frac{q_2}{|\vec{R}-\vec{R}_2|^2}\frac{\vec{R}-\vec{R}_2}{|R-\vec{R}_2|}\right)$$

where $\vec{r}_1 = \vec{R} - \vec{R}_1$ and $\vec{r}_2 = \vec{R} - \vec{R}_2$

Example 1.0: the general case

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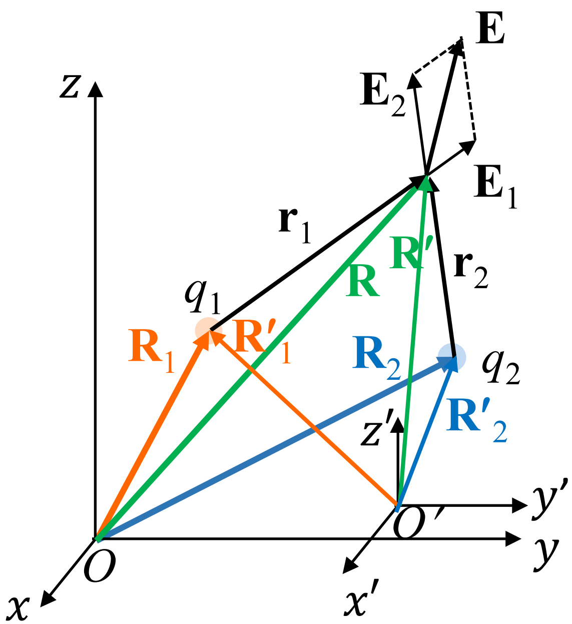

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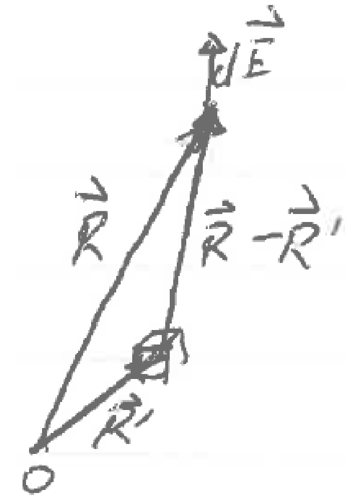
where $\vec{r}_1 = \vec{R} - \vec{R}_1$ and $\vec{r}_2 = \vec{R} - \vec{R}_2$

If we use a different set of coordinates *x'*, *y'*, and *z'*,

$\vec{r}_1 = \vec{R} - \vec{R}_1 = \vec{R}' - \vec{R}_1'$ and $\vec{r}_2 = \vec{R} - \vec{R}_2 = \vec{R}' - \vec{R}_2'$

The expression for **E** does not change. In other words, the expression is independent of the choice of coordinate system.

Electric field of a continuous distribution of charge



The charge density ρ (or ρ_V in textbook) is a function of position.

The charge in a small volume dV around position $\mathbf{R}'$ is ρdV.

The field at position $\mathbf{R}$ due to the charge ρdV at $\mathbf{R}'$ is

$$d\vec{E} = \frac{\rho dV}{4\pi\varepsilon_0} \cdot \frac{1}{|\vec{R}-\vec{R}'|^2} \frac{\vec{R}-\vec{R}'}{|\vec{R}-\vec{R}'|}$$

The total field is $\vec{E} = \int d\vec{E} = \frac{1}{4\pi\varepsilon_0}\int \frac{\rho}{|\vec{R}-\vec{R}'|^2} \frac{\vec{R}-\vec{R}'}{|\vec{R}-\vec{R}'|} dV$

Notice that the integrant is a vector. (There will be a concrete example.)

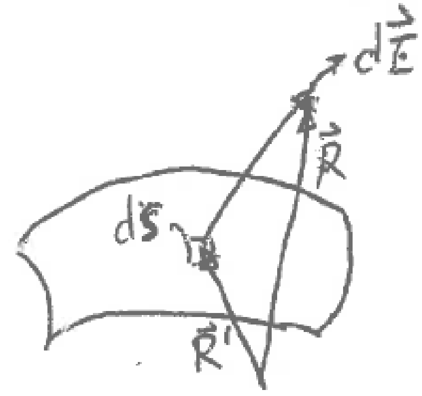
Of course, the total charge is $Q = \int \rho \, dV$

The charge could be confined in a surface (e.g. the surface of a conductor).

In this case we define the areal charge density ρ_S (or σ in some books) as a function of position. What's the unit of ρ_S?

The charge in a small area dS around position $\mathbf{R}'$ is $\rho_S dS$.

The field at position $\mathbf{R}$ is



$$\vec{E} = \int_S \frac{1}{4\pi\varepsilon_0} \frac{\rho_s}{|\vec{R}-\vec{R}'|^2} \frac{\vec{R}-\vec{R}'}{|\vec{R}-\vec{R}'|} dS$$

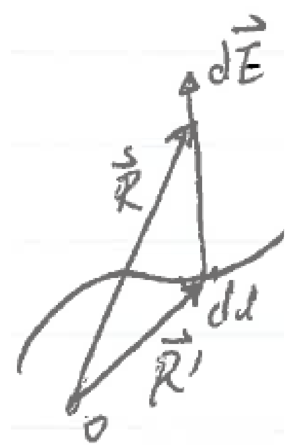
And, the total charge is $Q = \int_S \rho_s \, dS$

The charge could be confined along a line (not necessarily straight).

In this case we define the line charge density ρ_l as a function of position. Unit of ρ_l?

The charge in a small segment dl around position $\mathbf{R}'$ is $\rho_l dl$.

The field at position $\mathbf{R}$ is



$$\vec{E} = \frac{1}{4\pi\varepsilon_0} \int \frac{\rho_l}{|\vec{R}-\vec{R}'|^2} \frac{\vec{R}-\vec{R}'}{|\vec{R}-\vec{R}'|} dl$$

And, the total charge is $Q = \int \rho_l \, dl$

Finish Problems 1 through 4 of Homework 7.

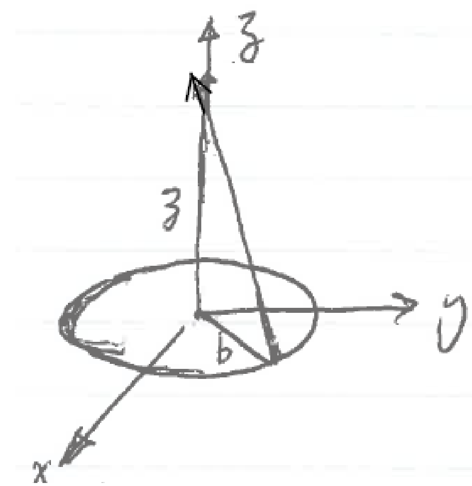
The above equations are general, therefore abstract.

To get more insight, let's look at some examples.

Example 2: Uniformly charged ring

We have a ring of charge, with a uniform line charge density ρ_l and radius b.

So, the total charge is $Q = \rho_l \cdot 2\pi b$



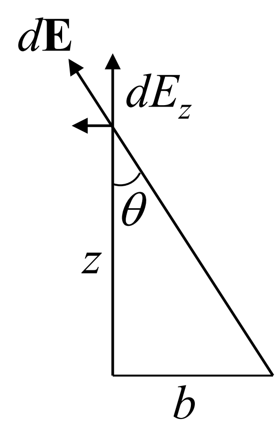
Let's find the field at any arbitrary point along the z axis.

For each small segment $bd\phi$ of the ring, the field

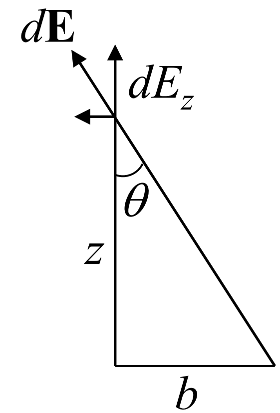
$$dE = \frac{1}{4\pi\varepsilon_0} \cdot \frac{1}{z^2+b^2} \cdot \rho_l \, b \, d\phi$$

Due to symmetry, we only need to sum up dE_z of all segments.

$$\left.\begin{aligned} dE_z &= dE\cos\theta \\ \cos\theta &= \frac{z}{\sqrt{b^2+z^2}} \end{aligned}\right\} \Rightarrow \begin{aligned} dE_z &= \frac{z}{\sqrt{b^2+z^2}} dE \\ &= \frac{1}{4\pi\varepsilon_0} \frac{\rho_l b d\phi}{b^2+z^2} \frac{z}{\sqrt{b^2+z^2}} \\ &= \frac{1}{4\pi\varepsilon_0} \frac{\rho_l b z}{(b^2+z^2)^{3/2}} d\phi \end{aligned}$$



$$
\begin{aligned}
dE_z &= \frac{z}{\sqrt{b^2+z^2}}\, dE \\
&= \frac{1}{4\pi\varepsilon_0} \frac{\rho_l b\, d\phi}{b^2+z^2} \frac{z}{\sqrt{b^2+z^2}} \\
&= \frac{1}{4\pi\varepsilon_0} \frac{\rho_l b z}{(b^2+z^2)^{3/2}}\, d\phi
\end{aligned}
$$

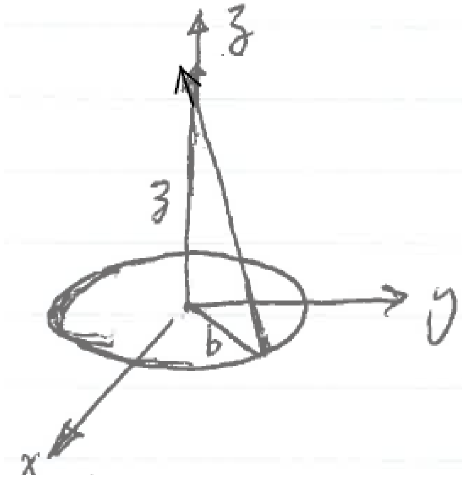


$$
\left.
\begin{aligned}
&E = E_z = \int_0^{2\pi} \frac{1}{4\pi\varepsilon_0} \frac{\rho_l b z}{(b^2+z^2)^{3/2}}\, d\phi = \frac{1}{4\pi\varepsilon_0} (2\pi) \frac{\rho_l b z}{(b^2+z^2)^{3/2}} \\
&Q = 2\pi\rho_l b
\end{aligned}
\right\} \Rightarrow
$$

$$
\boxed{E = \frac{1}{4\pi\varepsilon_0} \frac{z}{(b^2+z^2)^{3/2}} Q}
$$

Compare this method with the one used in the textbook.
This is much simpler and faster.
Why? We made use of symmetry.

$$E = \boxed{\frac{1}{4\pi\varepsilon_0}\frac{z}{(b^2+z^2)^{3/2}}Q}$$



Now, let's do a sanity check.

$$E(z=0)=0$$

$$E(z\to\infty) = \frac{z}{4\pi\varepsilon_0 (z^2)^{\frac{3}{2}}} Q = \frac{Q}{4\pi\varepsilon_0 z^2}$$

Same as a point charge.

Passed!

Now, we can use the above result to find the field along the axis of a charged disk.

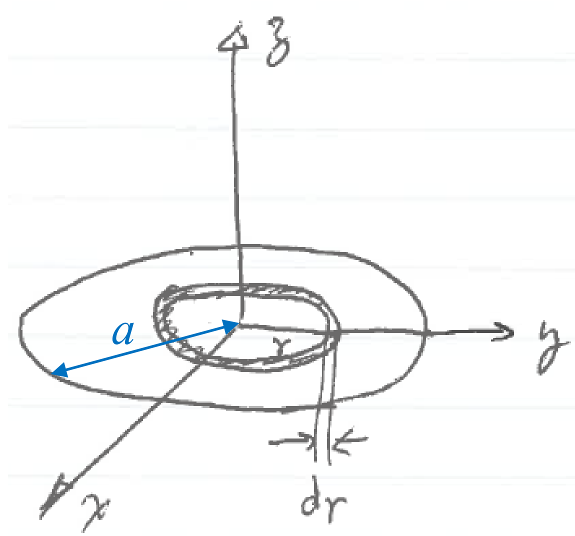
Uniform areal charge density ρ_S, radius a.

The charge of each ring of radius r and width dr is

$$dQ = 2\pi r \cdot dr \cdot \rho_s$$

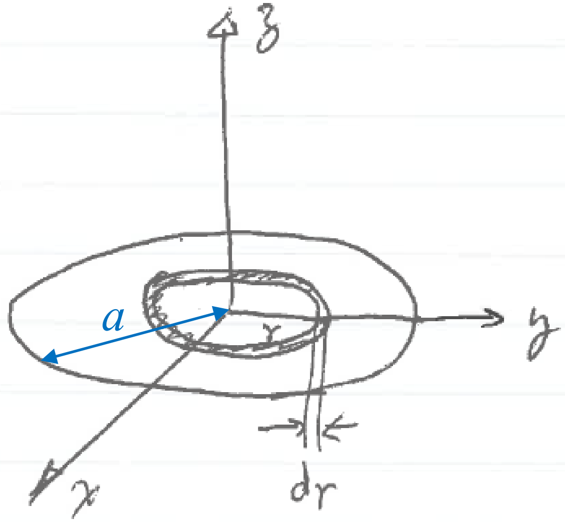
Using the above result with $dQ \to Q$ and $dE \to E$, we have the field due to such a ring:

$$dE = \frac{z}{4\pi\varepsilon_0 (r^2+z^2)^{\frac{3}{2}}} 2\pi\rho_s r dr$$



The field due to such a ring:

$$dE = \frac{z}{4\pi\varepsilon_0 (r^2 + z^2)^{\frac{3}{2}}} \quad 2\pi \rho_s r \, dr$$



Now, sum up all the rings:

$$E = \frac{z}{\cancel{4}\pi\varepsilon_0} \cancel{2}\pi \rho_s \int_0^a \frac{r\,dr}{(r^2 + z^2)^{\frac{3}{2}}}$$

$$\quad\;\, 2$$

This integral is a bit of math exercise:

$$\int_0^a \frac{r\,dr}{(r^2 + z^2)^{\frac{3}{2}}} = \frac{1}{2} \int_0^a \frac{d(r^2)}{(r^2 + z^2)^{\frac{3}{2}}}$$

$r^2 \rightarrow x$, $\quad \displaystyle\int \frac{dx}{(x + x_0)^{\frac{3}{2}}} = \int \frac{d(x + x_0)}{(x + x_0)^{\frac{3}{2}}}$

Recall that $\displaystyle\int x^n \, dx = \frac{x^{n+1}}{n+1}$. With $n = -3/2$,

$$\int \frac{dx}{(x + x_0)^{\frac{3}{2}}} = \int \frac{d(x + x_0)}{(x + x_0)^{\frac{3}{2}}} = \frac{1}{-\frac{3}{2} + 1} (x + x_0)^{-\frac{3}{2} + 1}$$

$$= -2 \, (x + x_0)^{-\frac{1}{2}} = -2 \frac{1}{\sqrt{x + x_0}}$$

Using $\int \frac{dx}{(x+x_0)^{\frac{3}{2}}} = -2\,\frac{1}{\sqrt{x+x_0}}$, we have

$$\int_0^a \frac{r\,dr}{(r^2+z^2)^{\frac{3}{2}}} = \frac{1}{2}\int_0^a \frac{d(r^2)}{(r^2+z^2)^{\frac{3}{2}}} = \frac{1}{2}(-2)\,\frac{1}{\sqrt{r^2+z^2}}\Bigg|_0^a$$

$$= \frac{1}{\sqrt{r^2+z^2}}\Bigg|_a^0$$

$$= \frac{1}{z} - \frac{1}{\sqrt{a^2+z^2}}$$

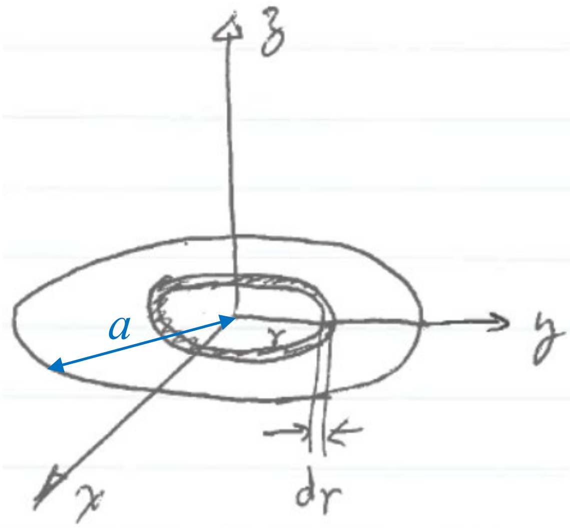
$(r^2 \leftarrow x,\ z^2 \leftarrow x_0)$

Therefore

$$E = \frac{z}{\underset{2}{\cancel{4\pi}}\varepsilon_0}\,2\cancel{\pi}\rho_s \int_0^a \frac{r\,dr}{(r^2+z^2)^{\frac{3}{2}}} = \frac{z\rho_s}{2\varepsilon_0}\left(\frac{1}{z} - \frac{1}{\sqrt{a^2+z^2}}\right) = \frac{\rho_s}{2\varepsilon_0}\left(1 - \frac{z}{\sqrt{a^2+z^2}}\right)$$

$$\boxed{E = \frac{z\rho_s}{2\varepsilon_0}\left(\frac{1}{z} - \frac{1}{\sqrt{a^2+z^2}}\right) = \frac{\rho_s}{2\varepsilon_0}\left(1 - \frac{z}{\sqrt{a^2+z^2}}\right) \text{ for } z>0.}$$

$$E = \frac{3\rho_s}{2\varepsilon_0}\left(\frac{1}{z} - \frac{1}{\sqrt{a^2+z^2}}\right) = \frac{\rho_s}{2\varepsilon_0}\left(1 - \frac{z}{\sqrt{a^2+z^2}}\right)$$



$$\rho_s = \frac{Q}{\pi a^2}$$

Therefore

$$E = \frac{1}{2\varepsilon_0}\frac{Q}{\pi a^2}\left(1 - \frac{z}{\sqrt{a^2+z^2}}\right)$$

Now, sanity check for $E(z \to \infty)$ and $E(z \to 0)$

Obviously, $E(z \to \infty) = 0$. But this is not too useful. We want to see the trend.

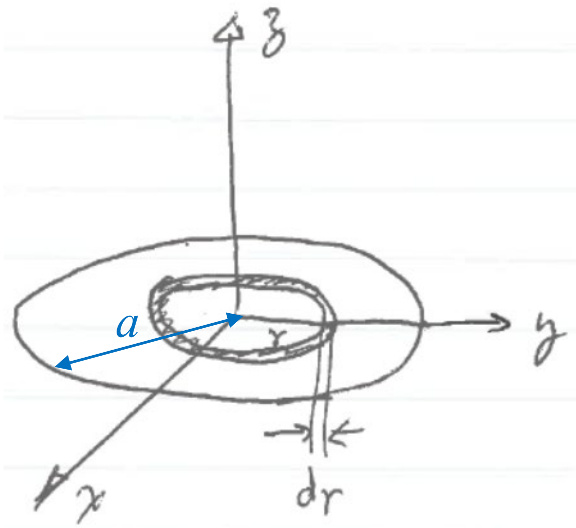
$$E = \frac{1}{2\varepsilon_0}\frac{Q}{\pi a^2}\left(1 - \frac{z}{\sqrt{a^2+z^2}}\right) = \frac{1}{2\varepsilon_0}\frac{Q}{\pi a^2}\left[1 - \frac{1}{\sqrt{1+\left(\frac{a}{z}\right)^2}}\right]$$

$$E(z \to \infty) = \frac{1}{2\varepsilon}\frac{Q}{\pi a^2}\left[1 - \left(1 - \frac{1}{2}\left(\frac{a}{z}\right)^2\right)\right] = \frac{1}{2\varepsilon_0}\cdot\frac{Q}{\pi \cancel{a^2}}\cdot\frac{1}{2}\frac{\cancel{a^2}}{z^2} = \frac{Q}{4\pi\varepsilon_0 z^2}$$

Same as a point charge. Passed!

You see, what's important is the ratio z/a. Everything is relative.

Along z axis, $$E = \frac{1}{2\varepsilon_0} \frac{Q}{\pi a^2} \left(1 - \frac{z}{\sqrt{a^2+z^2}}\right)$$



$$E(z \to 0) = \frac{1}{2\varepsilon_0} \cdot \frac{Q}{\pi a^2} \neq 0 \, !$$

Anything wrong? Donut vs pie?

Again, what's important is the ratio z/a. Everything is relative.

$$z \to 0 \iff a \to \infty \qquad \left(\frac{z}{a}\right) \to 0$$

$$E\left(\frac{z}{a} \to 0\right) = \frac{1}{2\varepsilon_0} \cdot \frac{Q}{\pi a^2} = \frac{\rho_s}{2\varepsilon_0}$$

Charge density

$$\rho_s = \frac{Q}{\pi a^2}$$

This is the field of an infinitely large sheet of charge.

From another point of view, this is the field at the center of a finite sheet:
The donut is different from the pie no matter how small the hole is!

We will get back to these points after we talk about Gauss's law.

See next page about this

Consider a "donut" of outer radius a and inner (hole) radius b.

Take the integral from b to a instead of 0 to a.

$$E = \frac{z}{\cancel{4\pi}_{2}\,\varepsilon_0}\,\cancel{2\pi}\,\rho_s \boxed{\int_0^a \frac{r\,dr}{(r^2+z^2)^{\frac{3}{2}}}} \qquad \boxed{\int_0^a \frac{r\,dr}{(r^2+z^2)^{\frac{3}{2}}} = \left.\frac{1}{\sqrt{r^2+z^2}}\right|_a^0}$$

b

Another way: The field due to the "pie" of radius a minus that due to a pie of radius b.

Do it on your own: Write the expression for the field at z along the axis for such a pie.

Either way, you'll see, for the "donut" with a hole radius b, $E = 0$ at the center at $z = 0$, as long as the hole is there, no matter how small b is!

Now, you can use the "charge pie" result to calculate the field of a charged cylinder along its axis. How?

Reminder: Finish Problems 1 through 4 of Homework 7. Read Sections 4-1, 4-2.1, & 4-3 of textbook. Continue working on Chapter 3 (vectorers analysis).

We finished this slide set on Tue 10/18/2022.