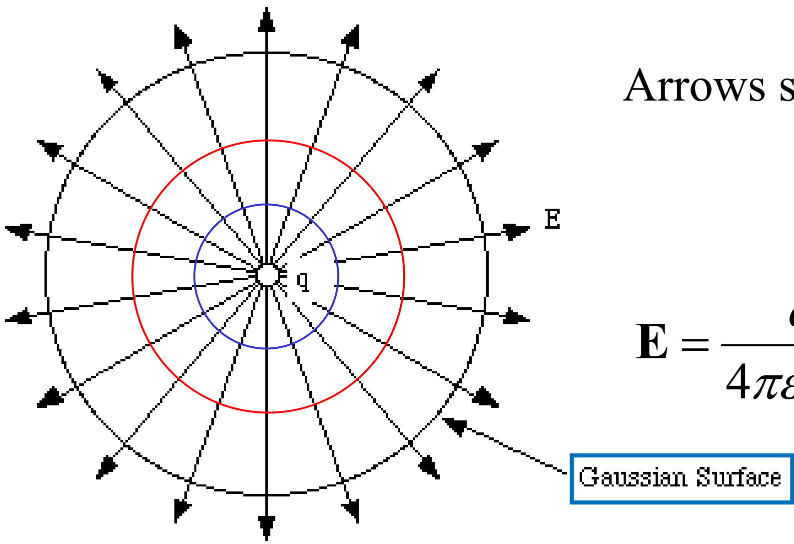
Gauss's Law

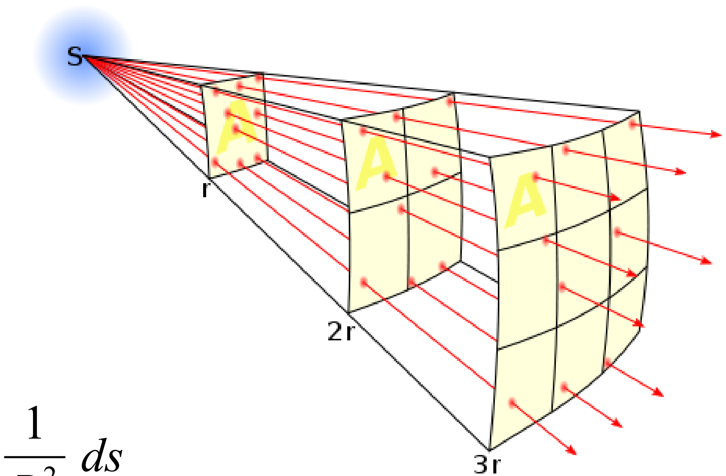
The beauty of $1/R^2$



Remember the electric field stream lines?

Arrows signify directions, density of lines the strength.

$$\mathbf{E} = \frac{q}{4\pi\varepsilon_0 R^2}\hat{\mathbf{R}}$$



For sphere, $\oint \mathbf{E}\cdot d\mathbf{s} = \oint \frac{q}{4\pi\varepsilon_0}\frac{1}{R^2}\hat{\mathbf{R}}\cdot d\mathbf{s} = \oint \frac{q}{4\pi\varepsilon_0}\frac{1}{R^2}\,ds$

$$= \frac{q}{4\pi\varepsilon_0}\frac{1}{R^2}4\pi R^2 = \frac{q}{\varepsilon_0}$$

This is why we put a factor 4π in Coulomb's law.

Since $\mathbf{R} \perp \mathrm{d}\mathbf{s}$, $\hat{\mathbf{R}}\cdot d\mathbf{s} = ds$.

The spheres centered at the point charge are Gaussian surfaces.
A constant flux through all these closed surfaces.

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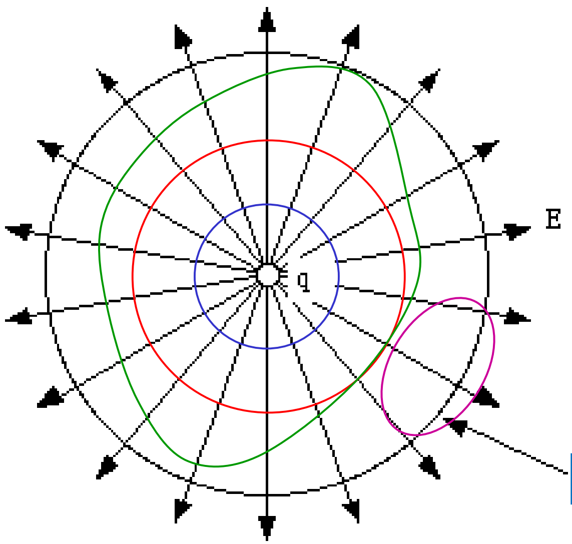
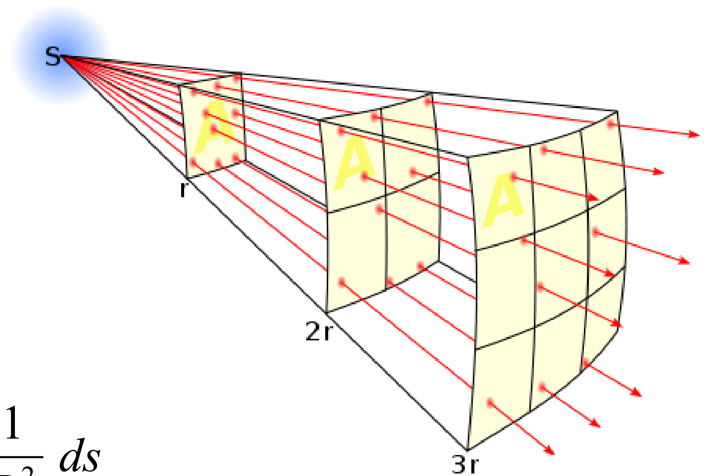
The beauty of $1/R^2$



For a single point charge:
The density of field lines signifies field strength, $\propto 1/R^2$.
Surface area $\propto 1/R^2$.
Constant flux of field lines through spheres, regardless of R.

$$\mathbf{E} = \frac{q}{4\pi\varepsilon_0 R^2}\hat{\mathbf{R}}$$



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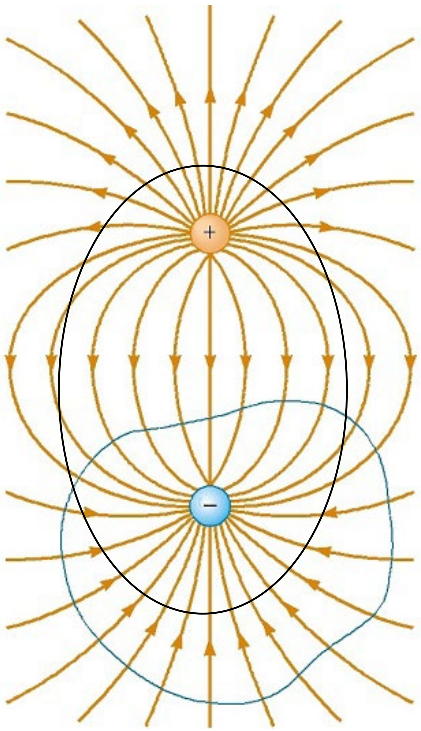
$$= \frac{q}{4\pi\varepsilon_0}\frac{1}{R^2}4\pi R^2 = \frac{q}{\varepsilon_0}$$

This is why we put a factor 4π in Coulomb's law.

Since $\mathbf{R} \perp$ ds, $\hat{\mathbf{R}}\cdot d\mathbf{s} = ds$.

The Gaussian surfaces do not have to be spheres. Constant flux through any closed surface.

"From our derivation you see that Gauss' law follows from the fact that the exponent in Coulomb's law is exactly two. A $1/r^3$ field, or any $1/r^n$ field with $n \neq 2$, would not give Gauss' law. So Gauss' law is just an expression, in a different form, of the Coulomb law..." -- Richard Feynman

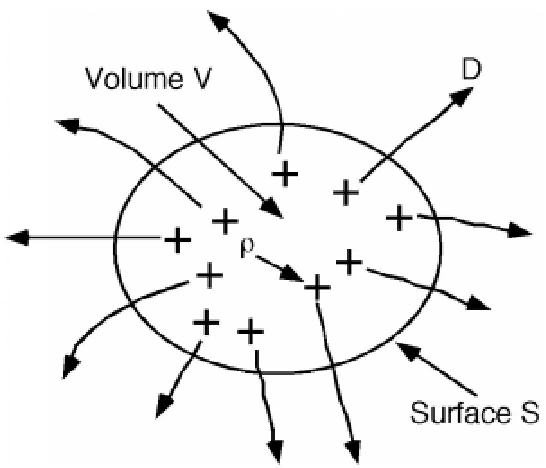


For multiple point charges:
The flux of field stream lines is proportional to the net charge enclosed by a Gaussian surface, due to superposition.

A field line comes out of a positive charge, and go into a negative charge.

For a continuous chunk of charge,

$$\oint \mathbf{E} \cdot d\mathbf{s} = \int \frac{\rho}{\varepsilon_0} dV = \frac{Q}{\varepsilon_0}$$

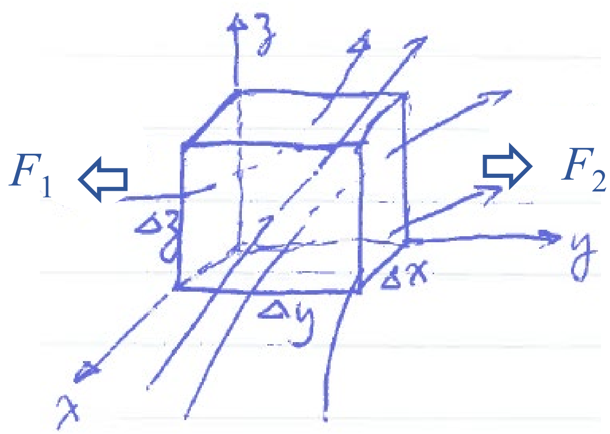


This is the integral form of Gauss's law.
We may call it the "big picture" of Gauss's law.

Next, we look at the differential form of Gauss's law.
We may call that the "small picture" of the law.

The differential form (or "small picture") of Gauss's law

To understand the physics (Gauss's law), we first talk about the math (Gauss's theorem).



Flux out of the cube through the left face

$$F_1 = -E_y \, \Delta x \Delta z$$

(Flux out of a closed surface is defined as positive)

Flux out of the cube through the right face

$$F_2 = \left(E_y + \frac{\partial E_y}{\partial y}\Delta y\right)\Delta x \Delta z$$

The net flux of these two faces: $F_1 + F_2 = \frac{\partial E_y}{\partial y}\Delta y \, \Delta x \Delta z$

Considering the other two pairs of faces, the net flux out of the cube:

$$\left(\frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} + \frac{\partial E_z}{\partial z}\right)\Delta x \Delta y \Delta z$$

defined as the divergence of **E** , indicating how much flux comes out of a small volume ΔV around a point

Notice that the divergence is a scalar.

$$\Delta x \Delta y \Delta z = \Delta V$$

$$\left(\frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} + \frac{\partial E_z}{\partial z}\right)\Delta x \Delta y \Delta z \qquad \Delta x \Delta y \Delta z = \Delta V$$

defined as the divergence of **E** , indicating how much flux comes out of a small volume *ΔV* around a point

Define vector operator $\nabla = \hat{x}\frac{\partial}{\partial x} + \hat{y}\frac{\partial}{\partial y} + \hat{z}\frac{\partial}{\partial z}$

thus
$$\nabla \cdot \vec{E} = \left(\hat{x}\frac{\partial}{\partial x} + \hat{y}\frac{\partial}{\partial y} + \hat{z}\frac{\partial}{\partial z}\right)\cdot\left(\hat{x}E_x + \hat{y}E_y + \hat{z}E_z\right)$$
$$= \frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} + \frac{\partial E_z}{\partial z}$$

This is just notation.

For a small volume *ΔV*

$$\oint_{\Delta S} \vec{E}\cdot d\vec{S} = \left(\frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} + \frac{\partial E_z}{\partial z}\right)\Delta V$$
$$= (\nabla \cdot \vec{E})\,\Delta V$$

ΔS is the closed surface enclosing *ΔV* of any arbitrary shape.
ΔV could be a small cube and *ΔS* then includes its 6 faces.

$$\therefore \quad \nabla \cdot \vec{E} = \lim_{\Delta V \to 0} \frac{\oint_{\Delta S} \vec{E}\cdot d\vec{S}}{\Delta V}$$

Up to here, just math. "Gauss's theorem."
The **E** here does not have to be an electric field.

$$\oint_{\Delta S} \vec{E}\cdot d\vec{S} = \left(\frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} + \frac{\partial E_z}{\partial z}\right)\Delta V$$

$$= (\nabla\cdot\vec{E})\,\Delta V$$

$$\therefore\quad \nabla\cdot\vec{E} = \lim_{\Delta V\to 0} \frac{\oint_{\Delta S}\vec{E}\cdot d\vec{S}}{\Delta V}$$

Gauss's theorem in math.
It relates the integral form ("big picture") and the differential form ("small picture") of Gauss's law in physics.

Equivalently, $\oint_S \vec{E}\cdot d\vec{S} = \int_V (\nabla\cdot\vec{E})\,dV$ holds for any arbitrary S

$$\oint_{\Delta S} \vec{E}\cdot d\vec{S} = \left(\frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} + \frac{\partial E_z}{\partial z}\right)\Delta V$$

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Equivalently, $\oint_S \vec{E}\cdot d\vec{S} = \int_V (\nabla\cdot\vec{E})\, dV = \frac{1}{\varepsilon_0}\int_V \rho\, dV$ holds for any arbitrary *S*

(by recalling that $\varepsilon_0\oint_S \vec{E}\cdot d\vec{S} = \int_V \rho\, dV = Q$)

Here the physics (Gauss’s law) kicks in.

$$\oint_{\Delta S} \vec{E}\cdot d\vec{S} = \left(\frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} + \frac{\partial E_z}{\partial z}\right)\Delta V$$

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$$\therefore\quad \nabla\cdot\vec{E} = \frac{\rho}{\varepsilon_0}$$

Differential form (“small picture”) of Gauss’s law:
The divergence of electric field at each point is proportional to the local charge density.

$$\oint_{\Delta S} \vec{E}\cdot d\vec{S} = \left(\frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} + \frac{\partial E_z}{\partial z}\right)\Delta V$$

$$= (\nabla\cdot\vec{E})\,\Delta V$$

$$\therefore\ \nabla\cdot\vec{E} = \lim_{\Delta V\to 0}\frac{\oint_{\Delta S}\vec{E}\cdot d\vec{S}}{\Delta V}$$

Gauss's theorem in math.
It relates the integral form ("big picture") and the differential form ("small picture") of Gauss's law in physics.

Equivalently, $\oint_S \vec{E}\cdot d\vec{S} = \int (\nabla\cdot\vec{E})\, dV = \frac{1}{\varepsilon_0}\int_V \rho\, dV$ holds for any arbitrary *S*

(by recalling that $\varepsilon_0\oint_S \vec{E}\cdot d\vec{S} = \int_V \rho\, dV = Q$)

Here the physics (Gauss's law) kicks in.

$$\therefore\ \nabla\cdot\vec{E} = \frac{\rho}{\varepsilon_0}$$

$\vec{D} \equiv \varepsilon_0\vec{E}$, thus $\nabla\cdot\vec{D} = \rho$

for any point in space.

Differential form ("small picture") of Gauss's law:
The divergence of electric field at each point is proportional to the local charge density.

$$\varepsilon_0\oint_S E\cdot d\vec{S} = \oint_S \vec{D}\cdot d\vec{S} = \int_V \rho\, dV = Q$$

for any arbitrary closed surface *S* enclosing volume *V*.

Integral form ("big picture") of Gauss's law:
The flux of electric field out of a closed surface is proportional to the charge it encloses.

$$\nabla \cdot \vec{E} = \frac{\rho}{\varepsilon_0}$$

$\vec{D} \equiv \varepsilon_0 \vec{E}$, thus

$$\nabla \cdot \vec{D} = \rho$$

for any point in space.

Differential form ("small picture") of Gauss's law:
The divergence of electric field at each point is proportional to the local charge density.

$$\varepsilon_0 \oint_S E \cdot d\vec{S} = \oint_S \vec{D} \cdot d\vec{S} = \int_V \rho \, dV = Q$$

for any arbitrary closed surface *S* enclosing volume *V*.

Integral form ("big picture") of Gauss's law:
The flux of electric field out of a closed surface is proportional to the charge it encloses.

The above is Gauss's law in free space (vacuum).
For a dielectric, just replace ε_0 with $\varepsilon = \varepsilon_r \varepsilon_0$, for now.
We will talk about what the dielectric constant ε_r really means.
Before that, let's look at some examples in free space (vacuum).

Finish Homework 7. Read Section 4-4 and 3-5 of textbook.
Continue working on Chapter 3.

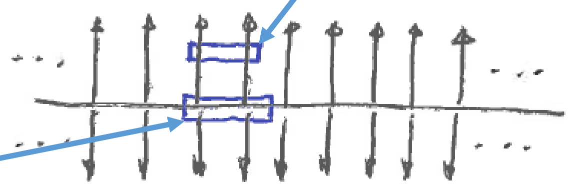
Example 1: find the field of an infinitely large charge plane

Find the electric field due to an infinitely large sheet of charge with an areal charge density ρ_S. It is a 2D sheet, with a zero thickness.

By symmetry, the **E** fields on the two sides of the sheet must be equal & opposite, and must be perpendicular to the sheet.

Imagine a cylinder (pie) with area A and zero height (thickness).

If the cylinder is at the sheet,



$$2\varepsilon_0 E A = \rho_s A \quad \Longrightarrow \quad E = \frac{\rho_s}{2\varepsilon_0}$$

Treat **E** as a scalar, since we already know the direction.

Recall our result for the charged disk: $E(z \to 0) = \frac{1}{2\varepsilon_0} \frac{Q}{\pi a^2} = \frac{\rho_s}{2\varepsilon_0}$

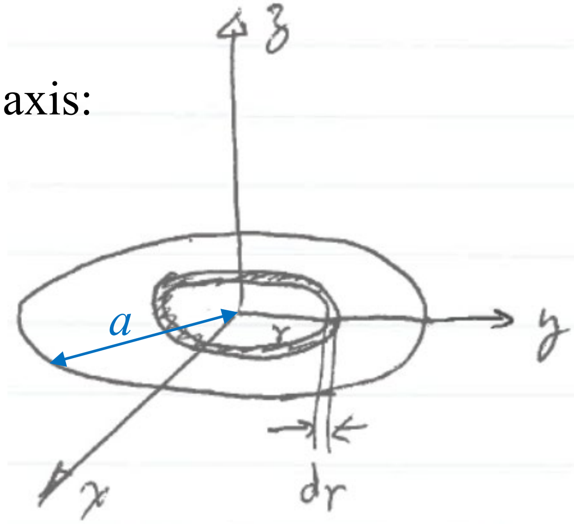
Actually, $z \to 0 \iff a \to \infty \quad \left(\frac{z}{a}\right) \to 0$

If the cylinder is elsewhere, the net flux is 0:

What goes in comes out; no charge inside the cylinder.

The field of a uniformly charged finite disk

Recall that we "did not pass the sanity test" for $E(z \rightarrow 0)$ along z axis:



$$E = \frac{1}{2\varepsilon_0} \frac{Q}{\pi a^2} \left(1 - \frac{z}{\sqrt{a^2 + z^2}} \right)$$

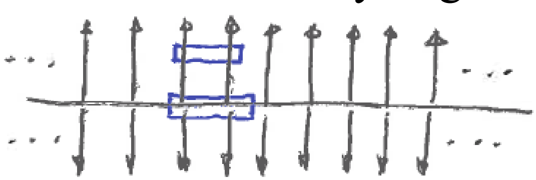
$$E(z \rightarrow 0) = \frac{1}{2\varepsilon_0} \cdot \frac{Q}{\pi a^2} \neq 0 \;!$$

Then we said what's important is the ratio z/a.
Everything is relative.

$$z \rightarrow 0 \quad \Leftrightarrow \quad a \rightarrow \infty \quad \left(\frac{z}{a}\right) \rightarrow 0$$

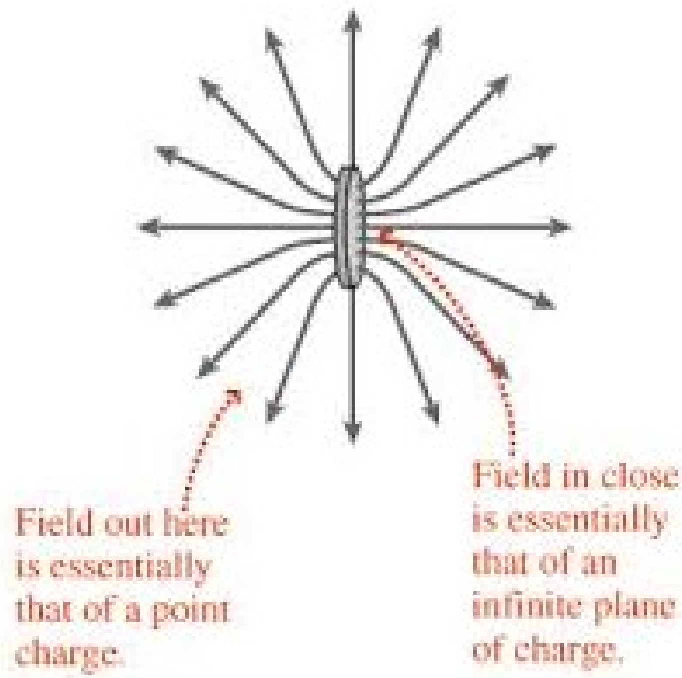
$$E\left(\frac{z}{a} \rightarrow 0\right) = \frac{1}{2\varepsilon_0} \cdot \frac{Q}{\pi a^2} = \frac{\rho_s}{2\varepsilon_0}$$

This is the field of an infinitely large sheet of charge.



From another point of view, this is the field at the center of a finite sheet. The donut is different from the pie no matter how small the hole is!

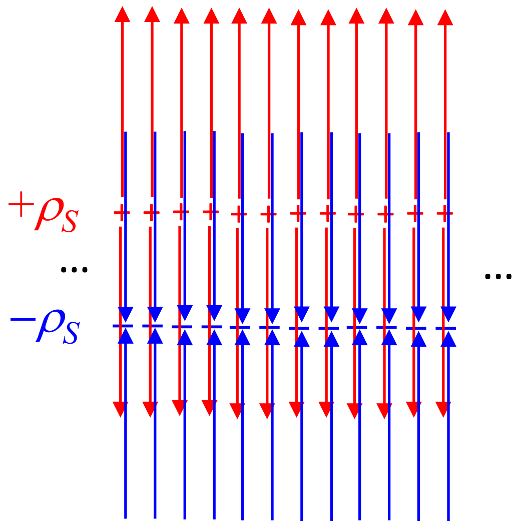
Visualize the field of the donut.



https://www.quora.com/Why-can-not-we-evaluate-electric-field-due-to-a-uniformly-charged-disc-at-any-point-on-its-axis-using-the-Gausss-law

Example 2: field of two infinitely large sheets with equal and opposite charge densities

What if there are two infinitely large sheets, one charged with a surface density $+\rho_S$, and the other $-\rho_S$. Assume ρ_S. is positive for convenience.



$$\leftarrow E = \frac{\rho_s}{2\varepsilon_0} - \frac{\rho_s}{2\varepsilon_0} = 0$$

$$\leftarrow E = -\frac{\rho_s}{2\varepsilon_0} + \frac{-\rho_s}{2\varepsilon_0} = -\frac{\rho_s}{\varepsilon_0}$$

$$\leftarrow E = -\frac{\rho_s}{2\varepsilon_0} + \frac{\rho_s}{2\varepsilon_0} = 0$$

Is this a familiar picture?

What circuit element is this picture a model of?

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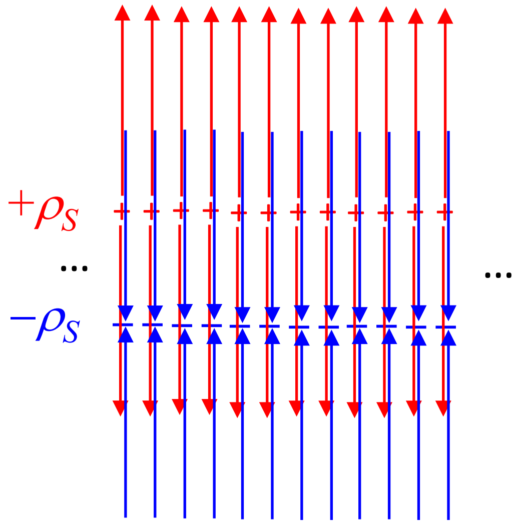

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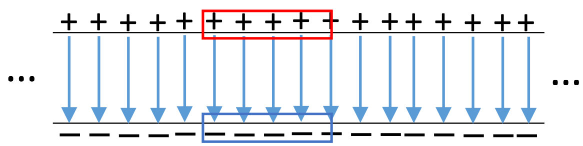
$$\leftarrow E = -\frac{\rho_s}{2\varepsilon_0} + \frac{\rho_s}{2\varepsilon_0} = 0$$

This a infinitely large parallel-plate capacitor.

It is the simplest model of the capacitor, ignoring the fringe effect.

by assuming infinite lateral size

Another look at the parallel-plate capacitor
(Two infinitely large sheets of opposite charges)



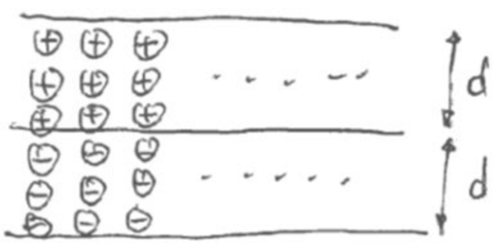
The electric field lines starts from a positive charge and ends at a negative charge.

Gauss's law leads to $E = \frac{\rho_s}{\varepsilon_0}$

You may use a negative sign to signify the "downward" direction. $E = -\frac{\rho_s}{\varepsilon_0}$

Sign conventions are kind of arbitrary. We just need to be self-consistent within the context.

An example for you to work out on your own:



Two charged slabs, one with a volume charge density $+\rho$, the other $-\rho$, where $\rho > 0$. Each slab has a thickness d and infinite area.

Find the electric field distribution.

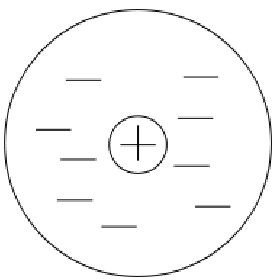
You may define the direction perpendicular to the slabs x, and set $x = 0$ for the interface between them.

So far we have limited our discussions to free space.
Now, let's talk about dielectrics (insulators).

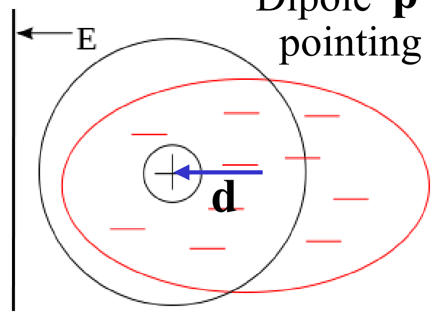
Electric Fields in Insulators (Dielectrics)

Polarization (defined to account for internal charges of media/materials)

1. Electronic polarization

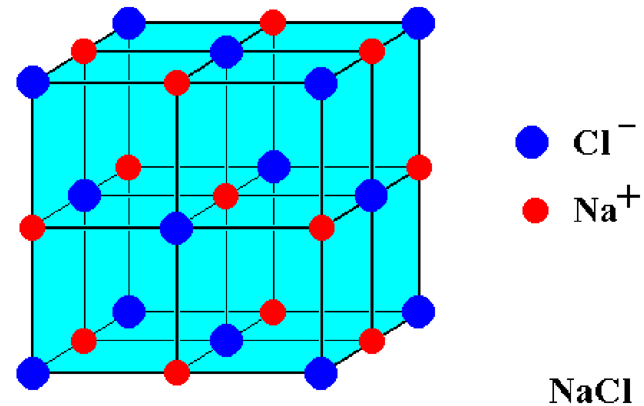


Dipole $\mathbf{p} = q\mathbf{d}$
pointing from – to +



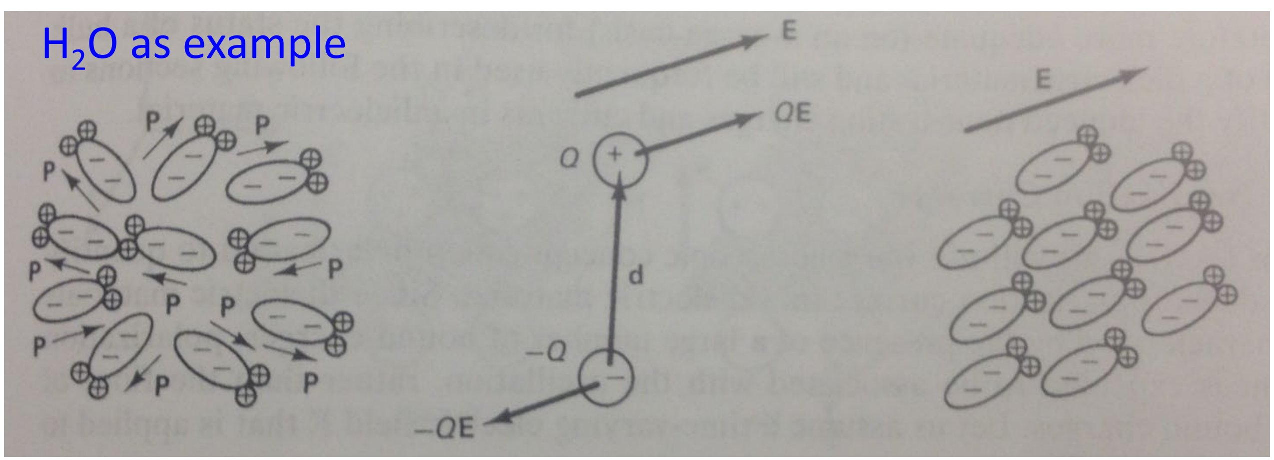
When $\mathbf{E} = 0$, $\mathbf{p} = 0$.

2. Ionic polarization



When $\mathbf{E} = 0$, net dipole is 0.

3. Orientational polarization



Again, no net dipole when $\mathbf{E} = 0$.

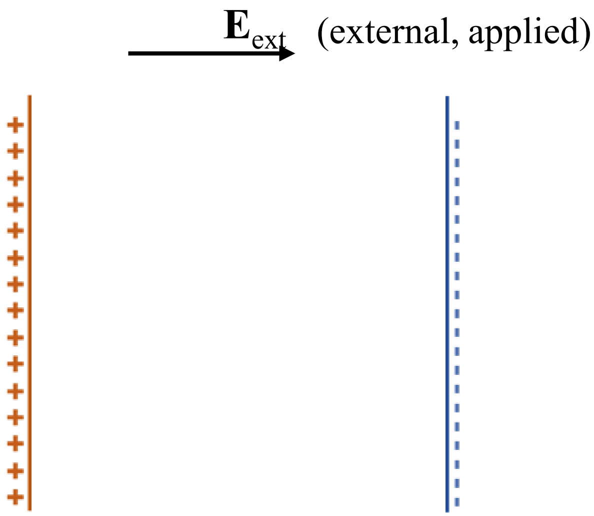
Define polarization

$$\mathbf{P} = \lim_{\Delta V \to 0} \frac{\sum_i \mathbf{p}_i}{\Delta V}$$

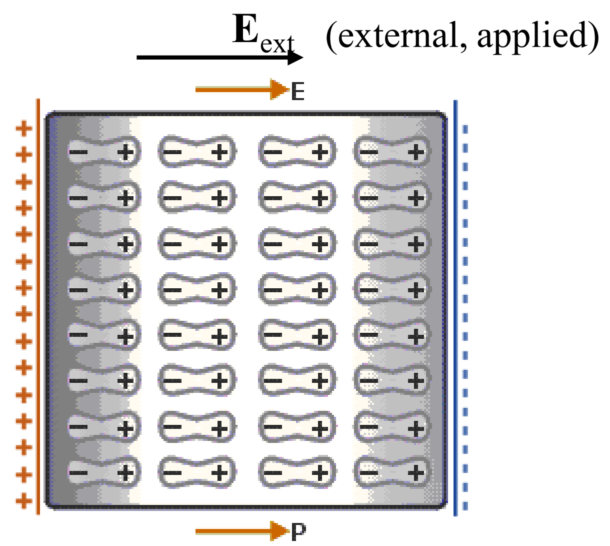
(**net** dipole per volume; notice **vector** summation)

For all three cases, when $\mathbf{E} = 0$, net dipole is 0, therefore $\mathbf{P} = 0$. A finite $\mathbf{E}$ will induce a net polarization $\mathbf{P}$.

Next, we use a simple model based on the parallel-plate capacitor to illustrate the behavior of a dielectric in the presence of an applied external electric field.
Big picture first, followed by details.



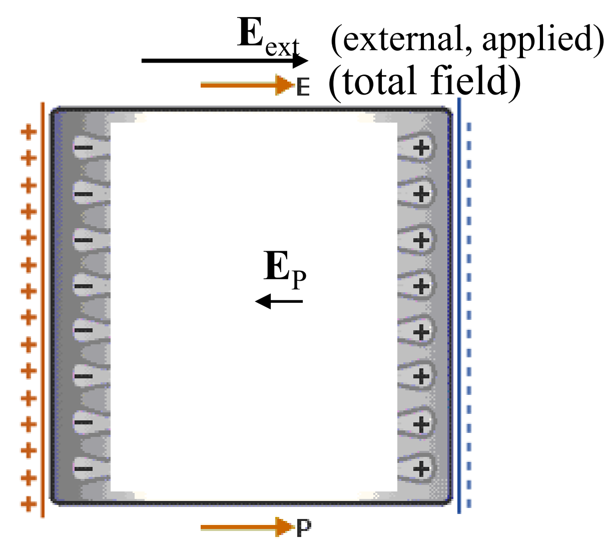
Capacitor with vacuum between plates



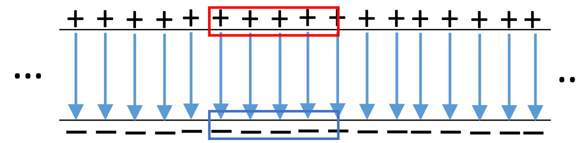
Capacitor with dielectric between plates

The big picture:

- External field induces polarization (net dipoles).
- Induced polarization is equivalent to a surface charge,
- which gives rise to an internal electric field.
- The internal field is against the external field.
- The net (total) field is what we care about.
- The net, external, and internal fields each follows Gauss's law with regard to the net, external, and internal charges.



Let's digress back to the parallel-plate capacitor with free space between plates, for the manifestation of Gauss's law at a surface.



Recall that the field of a parallel-plate capacitor is the consequence of Gauss's law applied to the surface charge densities:

$$\oint \mathbf{E}\cdot d\mathbf{s} = \int \frac{\rho}{\varepsilon_0} dV$$

or, in the differential form (the "small picture"),

$$\nabla \cdot \vec{E} = \frac{\rho}{\varepsilon_0}$$

Volume density

General forms of Gauss's law

$$\Longleftrightarrow \quad E = \frac{\rho_s}{\varepsilon_0} \quad \text{for the plate surfaces}$$

Surface density

Manifestation at a surface

Let's define the "electric displacement" $\mathbf{D} = \varepsilon_0 \mathbf{E}$. Then,

$$\oint_s \vec{D}\cdot d\vec{s} = \int_V \rho \, dV \qquad \nabla \cdot \mathbf{D} = \rho \qquad \Longleftrightarrow \qquad D = |\mathbf{D}| = \rho_s$$

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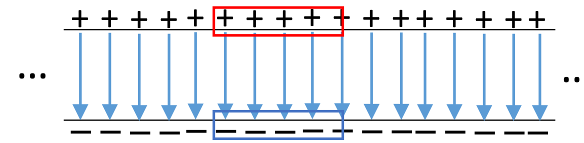

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Volume density

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Surface density

$$\oint_s \vec{D} \cdot d\vec{s} = \int_V \rho \, dV$$

$$\nabla \cdot \mathbf{D} = \rho$$

$$\longleftrightarrow \quad \mathrm{D} = |\mathbf{D}| = \rho_s$$

General forms of Gauss's law

Manifestation at a surface

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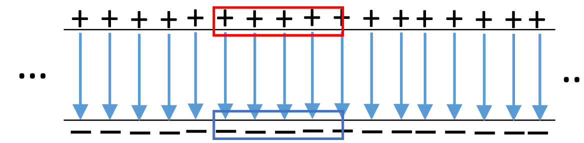

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or, in the differential form (the "small picture"),

$$\nabla \cdot \vec{E} = \frac{\rho}{\varepsilon_0}$$ ← Volume density

⟺ $E = \frac{\rho_s}{\varepsilon_0}$ (← Surface density) for the plate surfaces

$$\oint_s \vec{D} \cdot d\vec{s} = \int_v \rho \, dV$$

$$\nabla \cdot \mathbf{D} = \rho$$

⟺ $D = |\mathbf{D}| = \rho_s$

General forms of Gauss's law

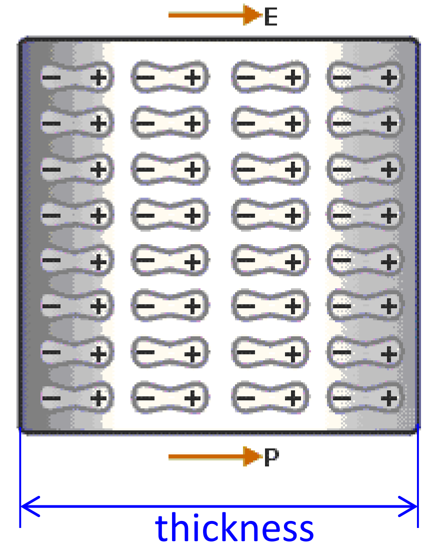
Manifestation at a surface

At a perfect conductor surface, we can write $D = |\mathbf{D}| = \rho_s$ as: $\rho_s = \mathbf{D} \cdot \hat{\mathbf{n}}$

← Surface normal **pointing out (i.e. into the free space or vacuum)**; check signs for both plate surfaces

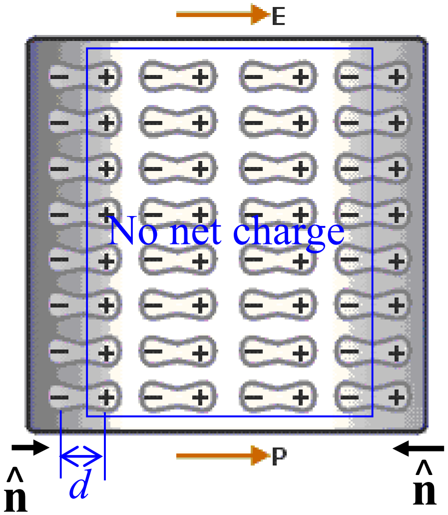
For the perfect conductor plate, $D = \rho_s$, as a result of $\nabla \cdot \mathbf{D} = \rho$

Consider a dielectric slab of infinite lateral size (cross section shown in figure).



Assume an electric field **E** (total field!) is present.

Regardless of the mechanism (electronic, ionic, orientational), **E** induces **P** by polarizing the dielectric.



No net charge in the interior.
Two sheet charges at surfaces by definition of polarization **P**:

$$(|\rho_{sP}|A)d = P(Ad) \quad \Rightarrow \quad |\rho_{sP}| = P$$

The polarization charge density, or "internal" charge density.

as opposed to external; not interior

We paused here on Thu 10/20/2022.

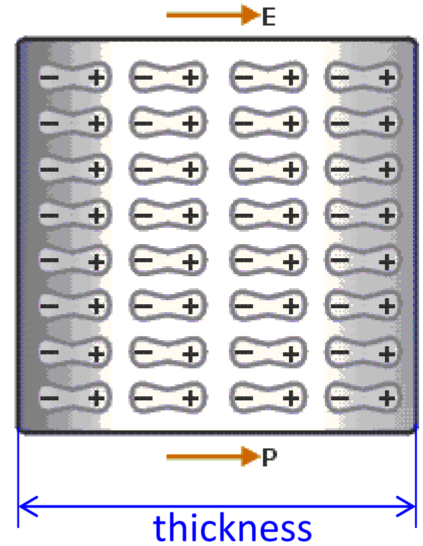
What's the unit of polarization charge density ρ_{sP}?

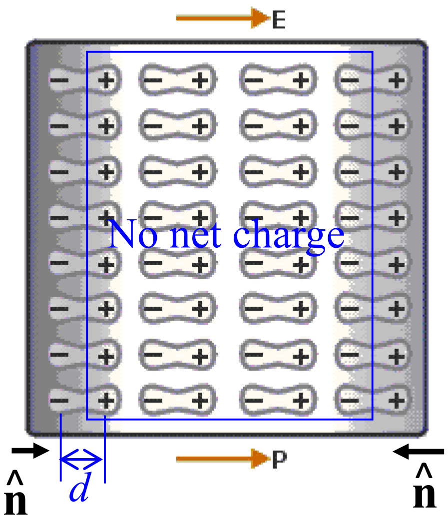
What's the unit of polarization P?

Consider a dielectric slab of infinite lateral size (cross section shown in figure).

Assume an electric field **E** (total field!) is present.

Regardless of the mechanism (electronic, ionic, orientational), **E** induces **P** by polarizing the dielectric.





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We paused here on Thu 10/20/2022.

What's the unit of polarization charge density ρ_{sP}?

What's the unit of polarization P?

As any **P** is from – to + (whereas any **D** is from + to –), we write:

$$\rho_{sP} = -P$$

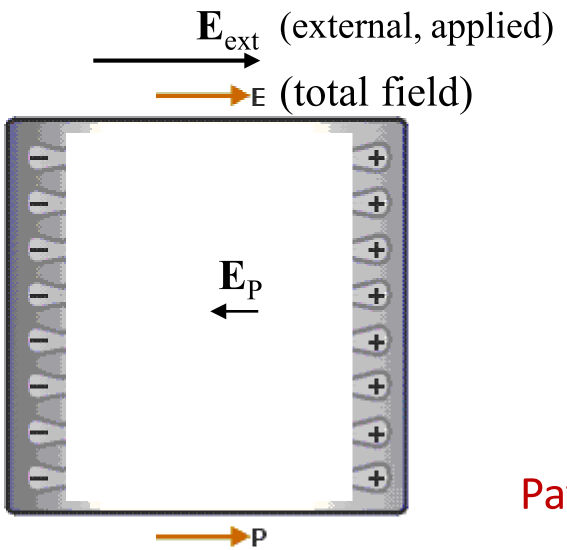
Or, more generally, $\rho_{sP} = -\mathbf{P}\cdot\hat{\mathbf{n}}$ ⟺ $\rho_P = -\nabla\cdot\mathbf{P}$

Manifestation at a surface

General form

Local surface normal **pointing into the dielectric**;
surface does not have to be planar

Now we apply Gauss's law to the polarization charge



By Gauss's law, this polarization (or "internal") charge leads to a polarization (or "internal") field

$$E_P = \rho_{sP}/\varepsilon_0 \;\Rightarrow\; \varepsilon_0 E_P = \rho_{sP} = -P \;\Rightarrow\; E_P = -P/\varepsilon_0$$

More generally, in the vector form: $\boxed{\varepsilon_0 \mathbf{E}_P = -\mathbf{P}}$

Pay attention to the sign.

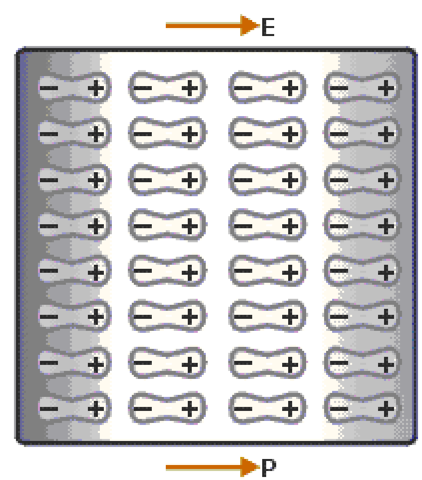
Pay attention to directions.

Now we relate the polarization to the total (net) field

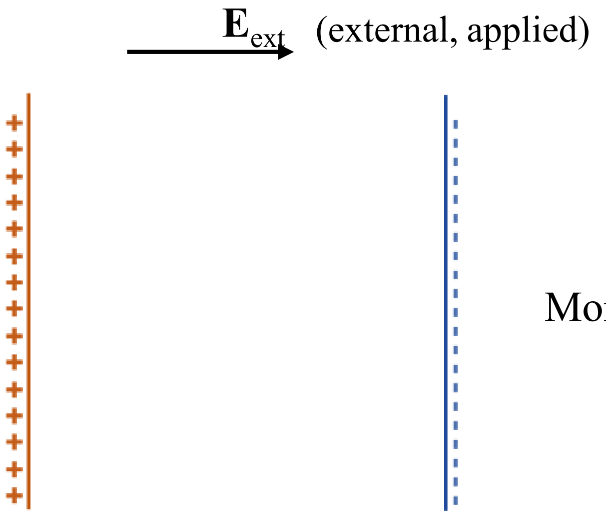
Regardless of the mechanism (electronic, ionic, orientational), **E** induces **P**.

No spontaneous polarization and not too strong **E**, $\mathbf{P} \propto \mathbf{E}$.

$\boxed{\mathbf{P} = \chi\varepsilon_0\mathbf{E}}$ (Will explain why later)



Now we apply Gauss's law to the external charge



Again, consider a parallel capacitor with vacuum/air between the two plates.

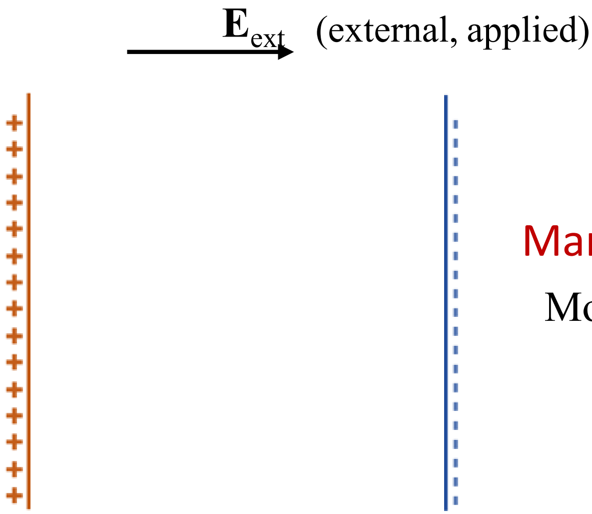
External surface charge density ρ_s induces external field

$$E_{ext} = \rho_s / \varepsilon_0.$$

More generally, $\rho_s = \varepsilon_0 \, \mathbf{E}_{ext} \cdot \hat{\mathbf{n}}$

Surface normal **pointing into the vacuum between plates**; check signs for both plate surfaces

Now we apply Gauss's law to the external charge



Again, consider a parallel capacitor with vacuum/air between the two plates.

External surface charge density ρ_s induces external field

$$E_{ext} = \rho_s / \varepsilon_0.$$

Manifestation at a surface

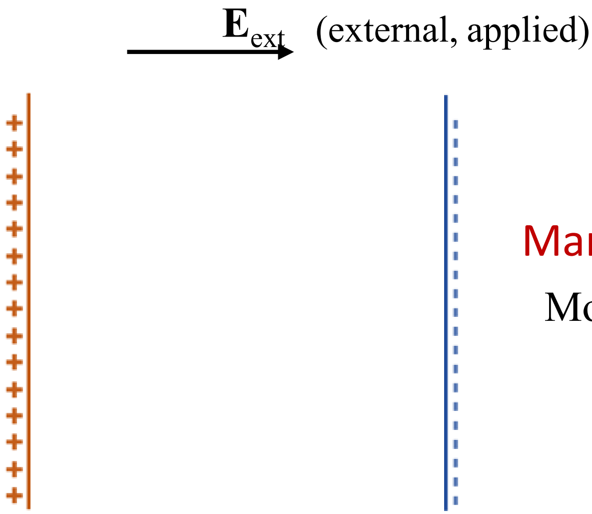
More generally, $\rho_s = \varepsilon_0 \mathbf{E}_{ext} \cdot \hat{\mathbf{n}}$

$$\nabla \cdot \vec{E} = \frac{\rho}{\varepsilon_0}$$

General form

Surface normal **pointing into the vacuum between plates**; check signs for both plate surfaces

Now we apply Gauss's law to the external charge



Again, consider a parallel capacitor with vacuum/air between the two plates.

External surface charge density ρ_s induces external field

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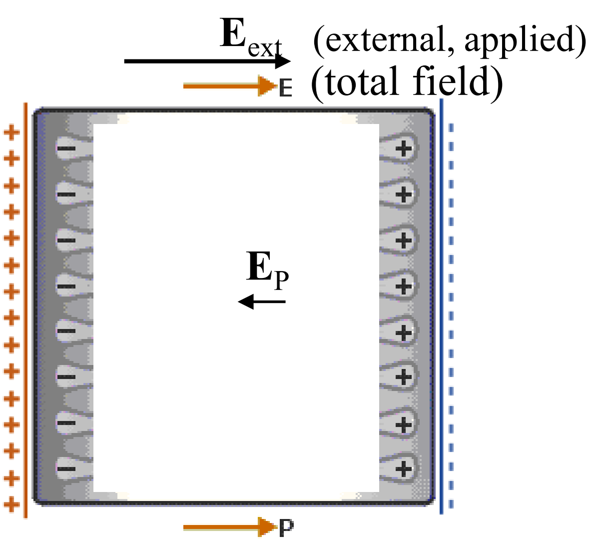
Manifestation at a surface

More generally, $\rho_s = \varepsilon_0 \mathbf{E}_{ext} \cdot \hat{\mathbf{n}}$ ⟷ $\nabla \cdot \vec{E}_{ext} = \dfrac{\rho}{\varepsilon_0}$

General form

Surface normal **pointing into the vacuum between plates**; check signs for both plate surfaces

Now, keep the capacitor isolated (so that ρ_s cannot change), and push a slab of dielectric into the space between the two plates.



Recall that the polarization (or internal) field is

$$E_P = \rho_{sP}/\varepsilon_0 = -P/\varepsilon_0.$$ (also by Gauss's law)

More generally, $\rho_{sP} = -\mathbf{P} \cdot \hat{\mathbf{n}}$ ⟷ $\rho_P = -\nabla \cdot \mathbf{P}$

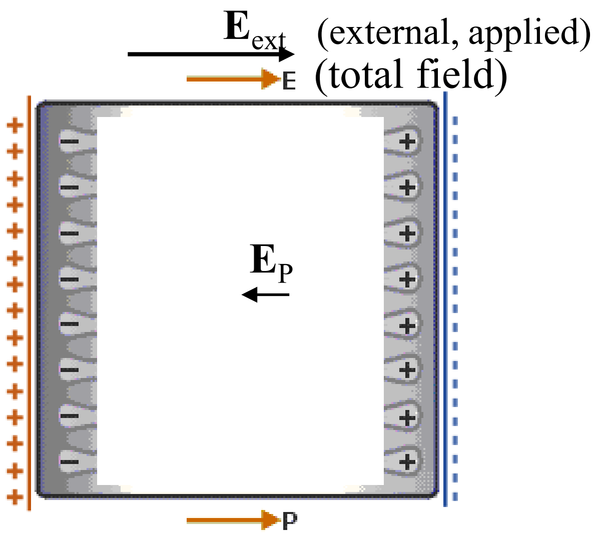
The total field, also following Gauss's law, is

$$E = E_{ext} + E_P = \rho_s/\varepsilon_0 - P/\varepsilon_0 = \rho_s/\varepsilon_0 + \rho_{sP}/\varepsilon_0$$

More generally, $\mathbf{E} \cdot \hat{\mathbf{n}} = \mathbf{E}_{ext} \cdot \hat{\mathbf{n}} + \mathbf{E}_P \cdot \hat{\mathbf{n}} = \rho_s/\varepsilon_0 - \mathbf{P} \cdot \hat{\mathbf{n}}/\varepsilon_0 = \rho_s/\varepsilon_0 + \rho_{sP}/\varepsilon_0$

(More conveniently seen for the left side, where $\rho_s > 0$ and $\rho_{sP} < 0$. But check this out for both sides/plates)

The total (net) field and the total charge follow Gauss's law



The total field $E = E_{ext} + E_P = \rho_s/\varepsilon_0 - P/\varepsilon_0 = \rho_s/\varepsilon_0 + \rho_{sP}/\varepsilon_0$ (1)

More generally, $\mathbf{E}\cdot\hat{\mathbf{n}} = \mathbf{E}_{ext}\cdot\hat{\mathbf{n}} + \mathbf{E}_P\cdot\hat{\mathbf{n}} = \rho_s/\varepsilon_0 - \mathbf{P}\cdot\hat{\mathbf{n}}/\varepsilon_0 = \rho_s/\varepsilon_0 + \rho_{sP}/\varepsilon_0$

(manifestation at a surface)

⟺ $\nabla\cdot\mathbf{E} = \frac{1}{\varepsilon_0}\rho_{total} = \frac{1}{\varepsilon_0}(\rho + \rho_P)$

(general form)

Write Eq. (1) again: $E = \rho_s/\varepsilon_0 - P/\varepsilon_0$

Recall that $\mathbf{P} = \chi\varepsilon_0\mathbf{E}$

Notice this is the total field

⇒ $E = \rho_s/\varepsilon_0 - P/\varepsilon_0 = \rho_s/\varepsilon_0 - \chi E$

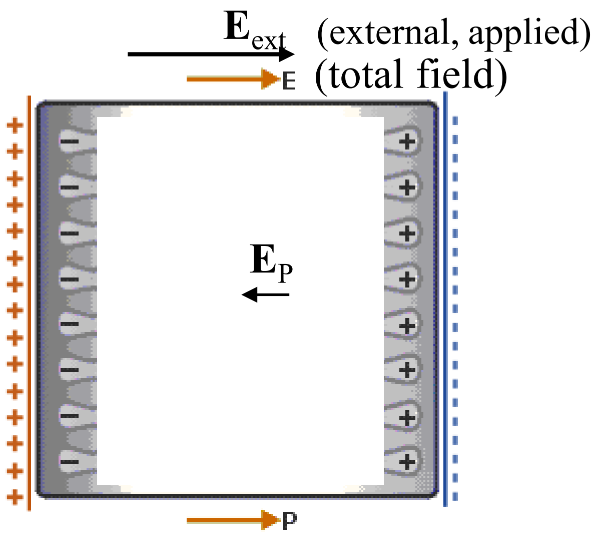
⇒ $(1+\chi)E = \rho_s/\varepsilon_0$

Define $\varepsilon_r = 1+\chi$, then $\varepsilon_r E = \rho_s/\varepsilon_0$

⇒ $\varepsilon_r\varepsilon_0 E = \rho_s$

Define $\varepsilon = \varepsilon_0\varepsilon_r = \varepsilon_0(1+\chi)$, then $\varepsilon E = \rho_s$

The total (net) field and the total charge follow Gauss's law



The total field $E = E_{\text{ext}} + E_P = \rho_s/\varepsilon_0 - P/\varepsilon_0 = \rho_s/\varepsilon_0 + \rho_{sP}/\varepsilon_0$ (1)

More generally, $\mathbf{E}\cdot\hat{\mathbf{n}} = \mathbf{E}_{\text{ext}}\cdot\hat{\mathbf{n}} + \mathbf{E}_\text{P}\cdot\hat{\mathbf{n}} = \rho_s/\varepsilon_0 - \mathbf{P}\cdot\hat{\mathbf{n}}/\varepsilon_0 = \rho_s/\varepsilon_0 + \rho_{sP}/\varepsilon_0$

(manifestation at a surface)

$$\Leftrightarrow \quad \nabla\cdot\mathbf{E} = \frac{1}{\varepsilon_0}\rho_{\text{total}} = \frac{1}{\varepsilon_0}(\rho + \rho_P)$$

(general form)

Write Eq. (1) again: $E = \rho_s/\varepsilon_0 - P/\varepsilon_0$

Recall that $\mathbf{P} = \chi\varepsilon_0\mathbf{E}$

Notice this is the total field

More generally,

$\Rightarrow$ $E = \rho_s/\varepsilon_0 - P/\varepsilon_0 = \rho_s/\varepsilon_0 - \chi E$ $\rightarrow$ $\mathbf{E}\cdot\hat{\mathbf{n}} = \rho_s/\varepsilon_0 - \chi\mathbf{E}\cdot\hat{\mathbf{n}}$ $\Leftrightarrow$ $\nabla\cdot\mathbf{E} = \frac{\rho}{\varepsilon_0} - \nabla\cdot\chi\mathbf{E}$

$\Rightarrow$ $(1+\chi)E = \rho_s/\varepsilon_0$ $\rightarrow$ $(1+\chi)\mathbf{E}\cdot\hat{\mathbf{n}} = \rho_s/\varepsilon_0$ $\Leftrightarrow$ $\nabla\cdot(1+\chi)\mathbf{E} = \frac{\rho}{\varepsilon_0}$

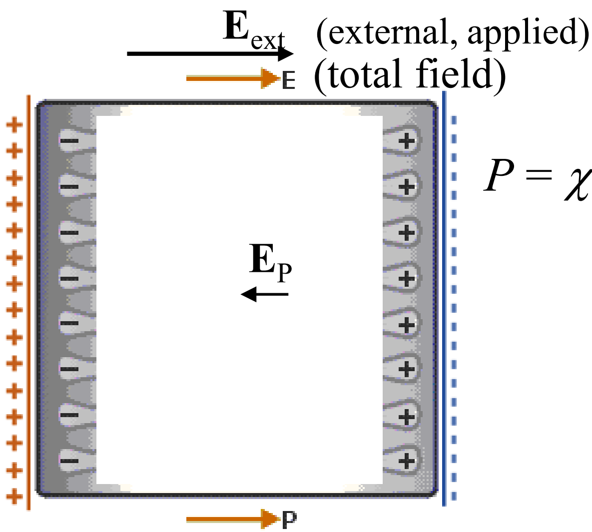
Define $\varepsilon_r = 1+\chi$, then $\varepsilon_r E = \rho_s/\varepsilon_0$

$\Rightarrow$ $\varepsilon_r\varepsilon_0 E = \rho_s$ $\rightarrow$ $\varepsilon_r\varepsilon_0\mathbf{E}\cdot\hat{\mathbf{n}} = \rho_s$ $\Leftrightarrow$ $\nabla\cdot(\varepsilon_r\varepsilon_0\mathbf{E}) = \rho$

Define $\varepsilon = \varepsilon_0\varepsilon_r = \varepsilon_0(1+\chi)$, then $\varepsilon E = \rho_s$ $\rightarrow$ $\varepsilon\mathbf{E}\cdot\hat{\mathbf{n}} = \rho_s$ $\Leftrightarrow$ $\nabla\cdot(\varepsilon\mathbf{E}) = \rho$

manifestation at a surface

general form



A Quick Summary

For this simple geometry, we have shown:

$P = \chi\varepsilon_0 E \Rightarrow (1+\chi)E = \rho_s/\varepsilon_0$

Define $\varepsilon_r = 1+\chi$, and we have:

$\varepsilon_r E = \rho_s/\varepsilon_0 \Rightarrow$ $\boxed{\varepsilon_r\varepsilon_0 E = \rho_s}$

Define $\varepsilon = \varepsilon_0\varepsilon_r = \varepsilon_0(1+\chi)$, and we have: $\boxed{\varepsilon E = \rho_s}$

More generally, $\varepsilon\mathbf{E}\cdot\hat{\mathbf{n}} = \rho_s$

Surface normal **pointing into the dielectric**; check signs for both plate surfaces

Notice that $\mathbf{E}$ is the total field.

> We lump the polarization effect of a dielectric into a parameter ε and replace ε_0 (for free space) with ε (for the dielectric) in equations, which otherwise remain the same.

We often write χ as χ_e, thus $\boxed{\varepsilon \equiv \varepsilon_0(1+\chi_e) \equiv \varepsilon_0\varepsilon_r}$

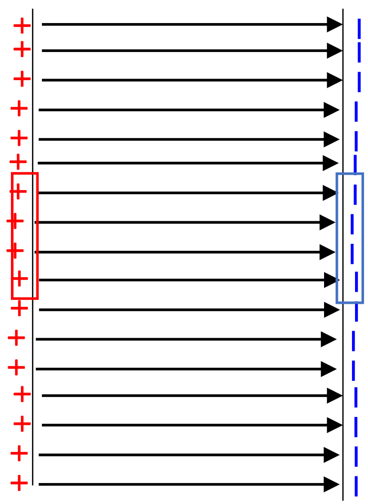
The field distribution of an infinitely large parallel-plate capacitor with a vacuum gap is the manifestation of Gauss's law. $\varepsilon_0\mathbf{E}\cdot\hat{\mathbf{n}} = \rho_s$

The above relations for the capacitor filled with a dielectric result from Gauss's law and the properties of the dielectric. $\varepsilon\mathbf{E}\cdot\hat{\mathbf{n}} = \rho_s$

Generalization of these relations leads to Gauss's law in dielectrics.

Gauss's law in free space

Recall that the field of a parallel-plate capacitor is the consequence of Gauss's law applied to the surface charge densities:



$$\nabla \cdot \vec{E} = \frac{\rho}{\varepsilon_0} \quad \Longleftrightarrow \quad \varepsilon_0 \mathbf{E} \cdot \hat{\mathbf{n}} = \rho_s$$

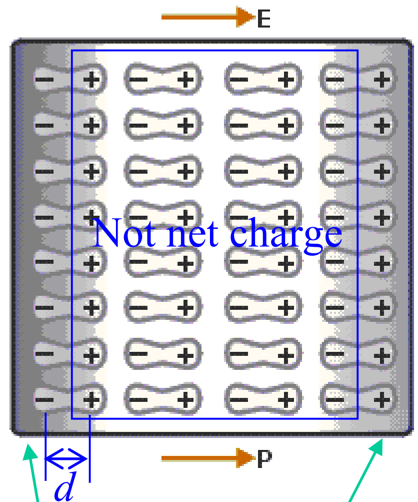
$$\nabla \cdot \mathbf{D} = \rho \quad \Longleftrightarrow \quad \mathbf{D} \cdot \hat{\mathbf{n}} = \rho_s$$

Free (external) surface charge density

Free (external) **volume** charge density

Gauss's law in dielectrics

On both surfaces of a dielectric slab



$$\rho_{sP} = -\mathbf{P} \cdot \hat{\mathbf{n}}$$

Polarization (internal) surface charge density

$$\boxed{\rho_P = -\nabla \cdot \mathbf{P}} \quad \Longleftrightarrow \quad \rho_{sP} = -\mathbf{P} \cdot \hat{\mathbf{n}}$$

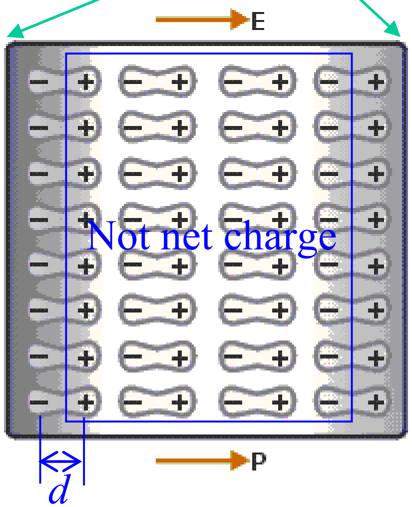
Notice the sign

Polarization (internal) **volume** charge density

Gauss's law in dielectrics

On both surfaces of a dielectric slab

$\rho_{sP} = -\mathbf{P}\cdot\hat{\mathbf{n}}$



$$\rho_P = -\nabla\cdot\mathbf{P} \iff \rho_{sP} = -\mathbf{P}\cdot\hat{\mathbf{n}}$$

Polarization (internal) surface charge density

Notice the sign

Polarization (internal) **volume** charge density

Free (external) **volume** charge density

Generalize to other geometries

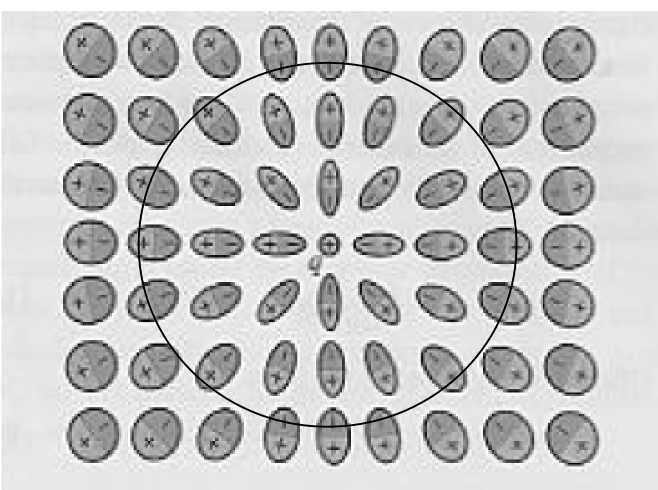


Figure 1-6: Polarization of the atoms of a dielectric material by a positive charge q.

$$\nabla\cdot\mathbf{E} = \frac{1}{\varepsilon_0}\rho_{\text{total}} = \frac{1}{\varepsilon_0}(\rho + \rho_P)$$

(treat external and polarization charges equally)

$$\rho_P = -\nabla\cdot\mathbf{P}$$

$$\Rightarrow \nabla\cdot\mathbf{E} = \frac{\rho}{\varepsilon_0} - \frac{1}{\varepsilon_0}\nabla\cdot\mathbf{P}$$

$$\Rightarrow \varepsilon_0\nabla\cdot\mathbf{E} + \nabla\cdot\mathbf{P} = \rho$$

$$\mathbf{P} = \chi_e\varepsilon_0\mathbf{E} \quad \Rightarrow$$

$$\nabla\cdot(\varepsilon_0 + \chi_e\varepsilon_0)\mathbf{E} \equiv \nabla\cdot\mathbf{D} = \rho,$$

where $\varepsilon \equiv \varepsilon_0(1+\chi_e) \equiv \varepsilon_0\varepsilon_r$,

$$\mathbf{D} \equiv \varepsilon_0\varepsilon_r\mathbf{E} \equiv \varepsilon\mathbf{E}$$

(lump polarization effect into ε, consider external charge only)

Highlights

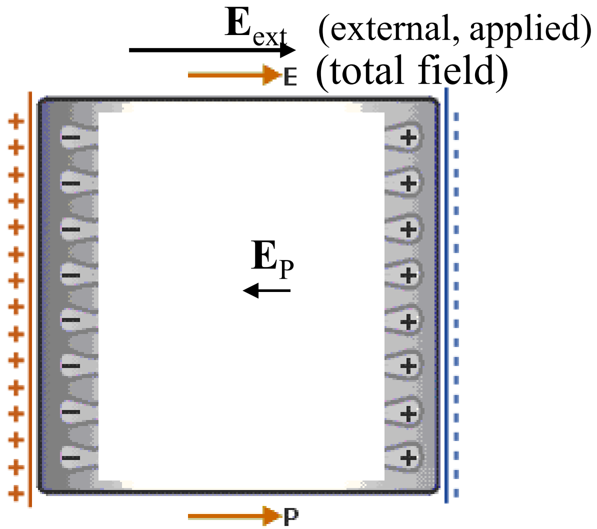
$$\nabla\cdot(\varepsilon_0+\chi_e\varepsilon_0)\mathbf{E}\equiv\nabla\cdot\mathbf{D}=\rho,$$
where $\varepsilon\equiv\varepsilon_0(1+\chi_e)\equiv\varepsilon_0\varepsilon_r$,
$$\mathbf{D}\equiv\varepsilon_0\varepsilon_r\mathbf{E}\equiv\varepsilon\mathbf{E}$$

Take-home message:

We lump the polarization effect of a dielectric material into a parameter ε, and substitute ε_0 (for free space) with ε (for the dielectric) in equations.

The polarization charge ρ_{sP} (or ρ_P in general) works against the external charge ρ_s (or ρ in general).
The polarization field $\mathbf{E_P}$ is always against the externa field $\mathbf{E}_{ext}$. Therefore the name dielectric.
$\varepsilon_r = 1+\chi > 1$, meaning larger D, therefore more charge, i.e. larger ρ needed to get to the same total filed E.

$$\varepsilon > \varepsilon_0$$



Limitations of our discussion:

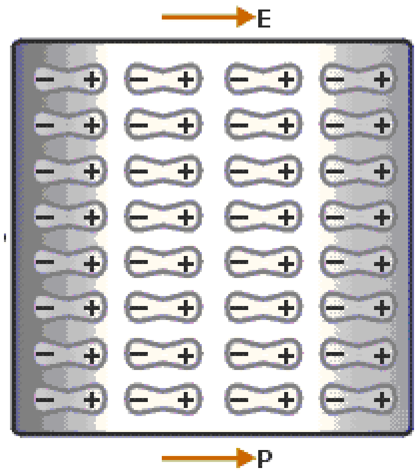
- No spontaneous polarization or piezoelectric polarization: whenever $\mathbf{E} = 0$, $\mathbf{P} = 0$
- Linearity: $\mathbf{P} = \chi\varepsilon_0\mathbf{E}$, $\mathbf{D} = \varepsilon\mathbf{E}$
- Isotropy: The proportional constants are the same in all directions, thus $\mathbf{P}//\mathbf{E}$, $\mathbf{D}//\mathbf{E}$

We consider a dielectric slab of infinite lateral size.

Regardless of the mechanism (electronic, ionic, orientational), **E** induces **P**.

For materials without spontaneous polarization and for not too strong **E**, $\mathbf{P} \propto \mathbf{E}$.

$$\mathbf{P} = \chi \varepsilon_0 \mathbf{E}$$



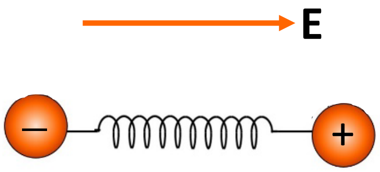
Side notes:

Spontaneous polarization: Some materials exhibit finite **P** even when **E** = 0, due to low symmetry of their structures. Although not covered in this course, this phenomenon (pyroelectricity) is important. GaN and related semiconductors (AlGaN, InGaN) are such materials. If the spontaneous polarization can be switched by an external electric field, such a material is ferroelectric.

Related to this, we can mechanically strain some material to break/lower its symmetry thus induce finite **P** at **E** = 0. This is called piezoelectricity.

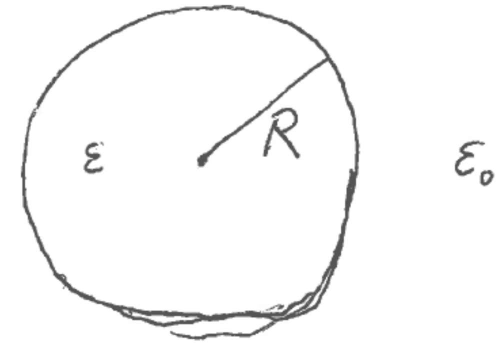
Spontaneous and piezoelectric polarizations are exploited in GaN-based power electronics devices (to obtain carriers without doping the semiconductors).

$\mathbf{P} \propto \mathbf{E}$: For a dielectric without spontaneous polarization, each dipole can be modeled as the positive and negative charges connected by a Hookean spring, near their equilibrium positions. Electric force $\mathbf{F} \propto \mathbf{E}$ results in displacement **d** from equilibrium for each dipole, thus $\mathbf{p} \propto \mathbf{d}$ and the total dipole moment per volume $\mathbf{P} \propto \mathbf{d}$. At steady state, the Hookean force $-K\mathbf{d}$ is balanced by the **F**, thus $\mathbf{F} = K\mathbf{d}$. Since $\mathbf{F} \propto \mathbf{E}$, $\mathbf{F} \propto \mathbf{d}$, and $\mathbf{P} \propto \mathbf{d}$, we have $\mathbf{P} \propto \mathbf{E}$.



Example 3: **E** and **D** of a uniformly charged sphere

For a charged dielectric sphere with charge density ρ, dielectric constant ε_r (thus $\varepsilon = \varepsilon_0 \varepsilon_r$), and radius R, find $\mathbf{E}(\mathbf{r})$ and $\mathbf{D}(\mathbf{r})$ for all $\mathbf{r}$.

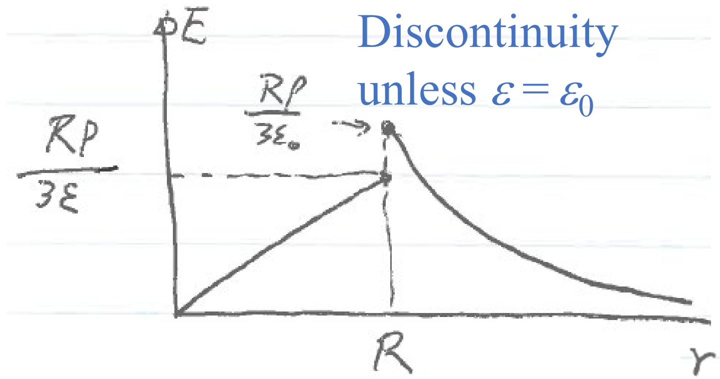


The system is spherically symmetric, therefore $\mathbf{E}(\mathbf{r}) = E(r)\hat{\mathbf{r}}$ and $\mathbf{D}(\mathbf{r}) = D(r)\hat{\mathbf{r}}$.

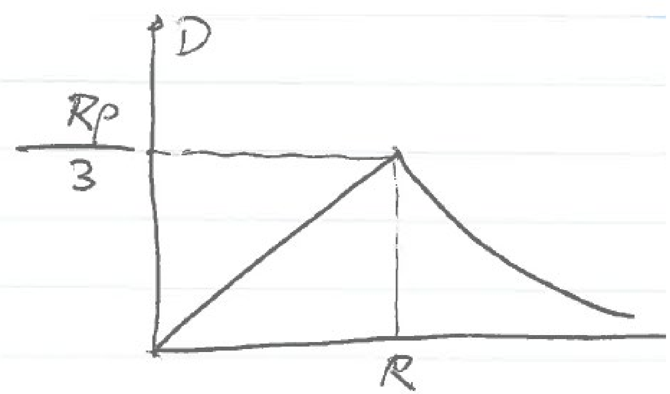
For $r \leq R$, $\quad 4\pi r^2 \varepsilon E = \frac{4}{3}\pi r^3 \rho$

$$\therefore E = \frac{1}{3\varepsilon} r\rho \qquad \Rightarrow \qquad E(R) = \frac{R\rho}{3\varepsilon}$$

$$D = \frac{1}{3} r\rho \qquad\qquad D(R) = \frac{R\rho}{3}$$



Discontinuity unless $\varepsilon = \varepsilon_0$



For $r > R$, $\quad 4\pi r^2 \varepsilon_0 E = \frac{4}{3}\pi R^3 \rho = Q$

$$\therefore \boxed{E = \frac{Q}{4\pi r^2 \varepsilon_0}} = \frac{R^3 \rho}{3\varepsilon_0 r^2}$$

Same as point charge

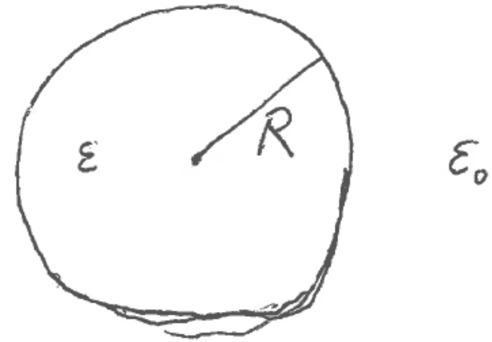
Why is E discontinuous and D continuous?

Read textbook Section 4-7 overview & Subsection 4-7.1.

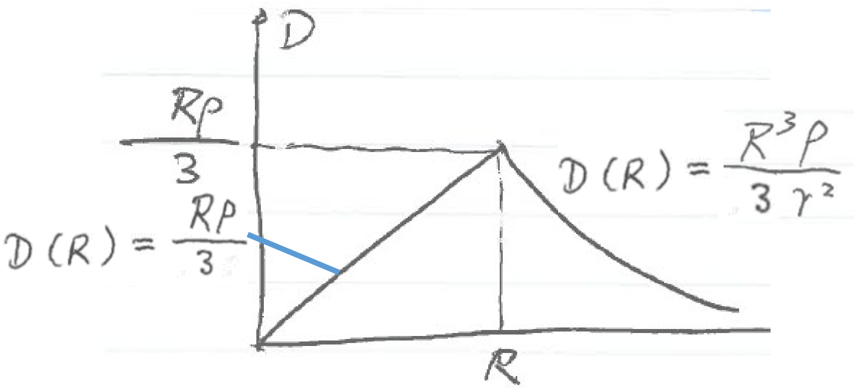
E and **D** of a uniformly charged sphere

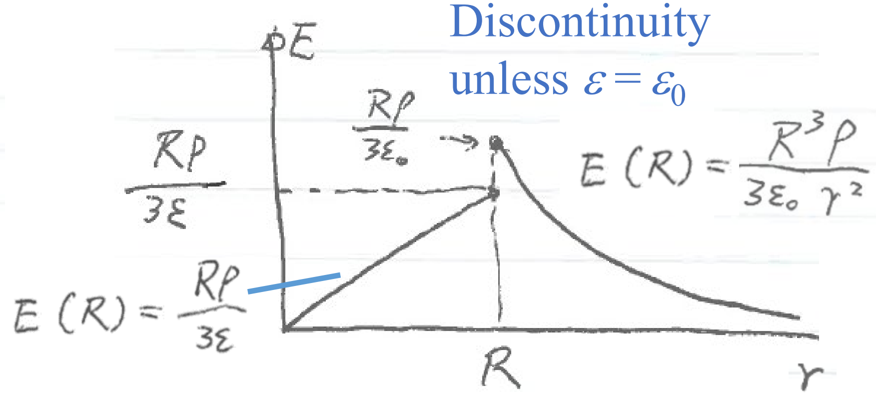
A charged dielectric sphere with charge density ρ, dielectric constant ε_r (thus $\varepsilon = \varepsilon_0 \varepsilon_r$), and radius R.

"external"



Why is E discontinuous and D continuous?





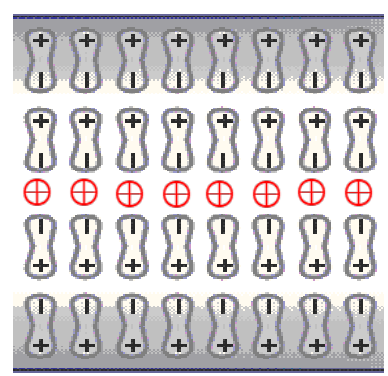
Discontinuity unless $\varepsilon = \varepsilon_0$

The quick answer:

There is polarization charge on the sphere surface, accounting for the extra field. ☞

Consider an easier-to-visualize planar case. ☞

Can you plot the E field distribution?

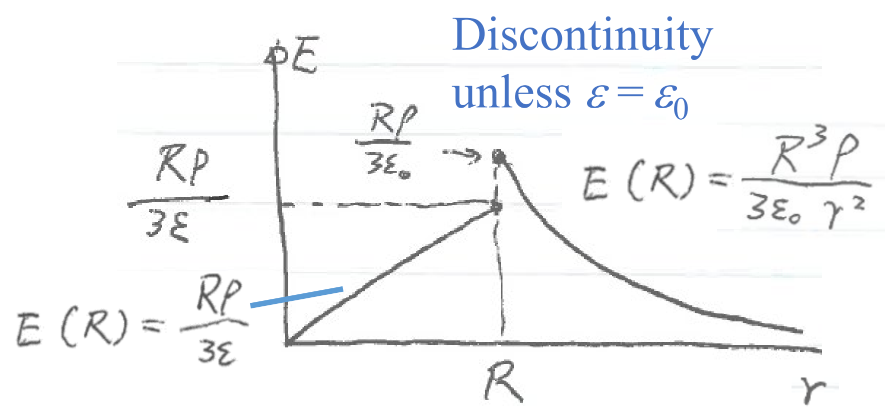


You are strongly encouraged to go through the Explanation and Summary offline.

The quick answer:

There is polarization charge on the sphere surface, ☞ 
accounting for the extra field,

$$\Delta E = \frac{R\rho}{3\varepsilon_0} - \frac{R\rho}{3\varepsilon} = \frac{R\rho}{3\varepsilon_0}\left(1 - \frac{1}{\varepsilon_r}\right) = \frac{\varepsilon_r - 1}{\varepsilon_r}\frac{R\rho}{3\varepsilon_0}$$



Explanation:

$$\mathbf{P} = \chi\varepsilon_0\mathbf{E}$$

$\Rightarrow$ $\mathbf{P}(R\hat{\mathbf{r}}) = \chi\varepsilon_0\frac{R\rho}{3\varepsilon}\hat{\mathbf{r}} = \frac{\varepsilon_r - 1}{\varepsilon_r}\frac{R\rho}{3}\hat{\mathbf{r}}$ on the inner side of the sphere surface,

or simply $P(R) = \frac{\varepsilon_r - 1}{\varepsilon_r}\frac{R\rho}{3}$. The surface density of polarization charge is

$$\rho_{sP} = -\mathbf{P}\cdot\hat{\mathbf{n}} = \mathbf{P}\cdot\hat{\mathbf{r}} = \frac{\varepsilon_r - 1}{\varepsilon_r}\frac{R\rho}{3}.$$

Obviously, $\boxed{\Delta E = \frac{\rho_{sP}}{\varepsilon_0}}$

The field at the surface due to this surface density of polarization charge is

$$\mathbf{E}_{\mathrm{P}} = \frac{\rho_{sP}}{\varepsilon_0}\hat{\mathbf{r}} = \frac{\varepsilon_r - 1}{\varepsilon_r}\frac{R\rho}{3\varepsilon_0}\hat{\mathbf{r}} \quad \text{or simply} \quad E_{\mathrm{P}} = \frac{\varepsilon_r - 1}{\varepsilon_r}\frac{R\rho}{3\varepsilon_0}.$$

(By applying Gauss's law to a patch of the sphere surface)

Comparing this to the discontinuity $\Delta E = \frac{\varepsilon_r - 1}{\varepsilon_r}\frac{R\rho}{3\varepsilon_0}$,

you see this field E_{P} due to the polarization charge exactly accounts for the discontinuity.

Summary & important comments:

$$\oint_S \vec{D} \cdot d\vec{S} = \int_V \rho \, dV \qquad \nabla \cdot \mathbf{D} = \rho$$

D is only about the external charge; ε or ε_0 does not enter the equations.

E is due to both the external charge and the internal (polarization) charge.

Additional details

In our simple parallel-plate capacitor model, external charges are located on the plates, not in the interior of the dielectric. As a result, the polarization charges are only at the two surfaces of the dielectric.

More generally, there is a relation between the external charge and the polarization charge.

$$\rho_P = -\nabla\cdot\mathbf{P} \quad \Rightarrow \quad \rho_P = -\chi\varepsilon_0\nabla\cdot\mathbf{E} \quad \Rightarrow \quad \boxed{\rho_P = -\chi\frac{\rho}{\varepsilon_r} = -\frac{(\varepsilon_r-1)}{\varepsilon_r}\rho}$$

↑ insert $\mathbf{P} = \chi\varepsilon_0\mathbf{E}$

↑ insert $\nabla\cdot\mathbf{E} = \rho/\varepsilon$

Notice the negative sign:
Polarization charge **against** external charge

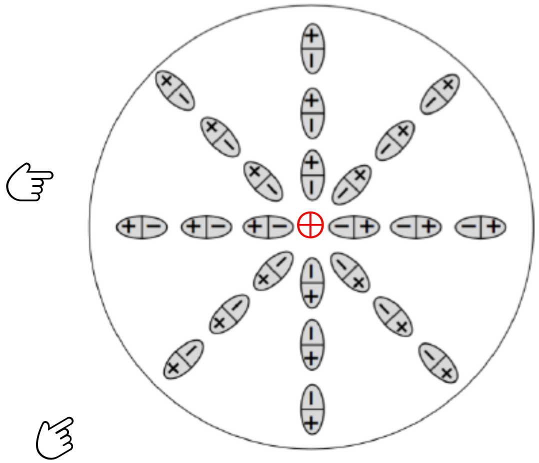
The uniformly charge sphere follows this relation.

For the parallel-plate capacitor, $\rho = 0 \Rightarrow \rho_P = 0$.

Similarly, a dielectric sphere with all "external" charge concentrated at the center,

$$\rho = 0 \Rightarrow \rho_P = 0$$

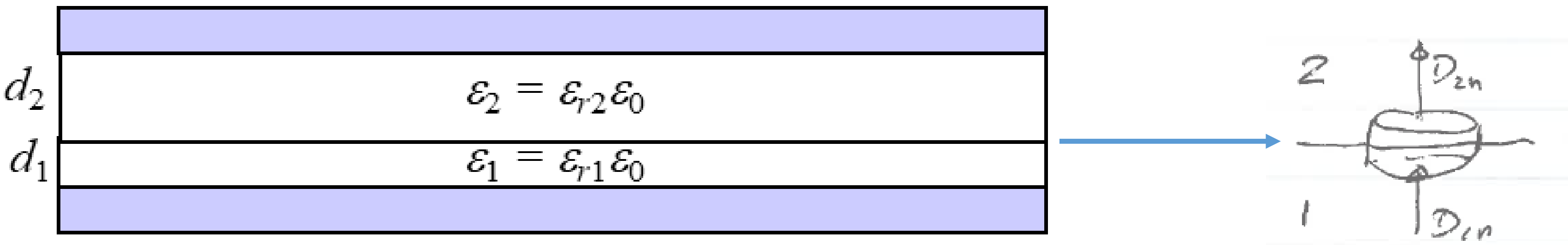
for $0 < r < R$.



Exercise:

Find $\mathbf{E}$ and $\mathbf{D}$ at arbitrary positions $\mathbf{r}$ ($0 < r < \infty$) with regard to the center of a dielectric sphere where a point charge q is located. The dielectric constant is ε_r and the radius is R.

For simple geometry, consider a parallel-plate capacitor like this:



Imagine a tiny pie, with a zero thickness and an area *ΔS*, and with the bottom and top on opposite sides of the boundary.

Recall Gauss's law: $\oint_s \vec{D} \cdot d\vec{S} = Q \Rightarrow (D_{2n} - D_{1n})\Delta S = \rho_s \Delta S$

Subscript n means normal, for general case.
Not needed in this case.

$$\Rightarrow \quad D_{2n} - D_{1n} = \rho_s$$
$$\varepsilon_2 E_{2n} - \varepsilon_1 E_{1n} = \rho_s$$

Does not include polarization charge

At interface between two dielectrics with $\rho_s = 0$

$$D_{2n} = D_{1n}, \quad \Leftrightarrow \quad \varepsilon_2 E_{2n} = \varepsilon_1 E_{1n}$$

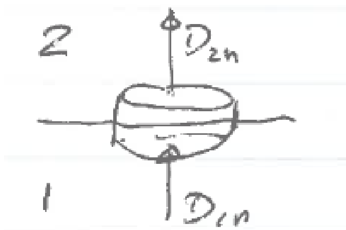
When external interface charge density $\rho_s = 0$,
D_n is continuous.
E_n is discontinuous due to polarization charge at interface.

External charge

At interface between two dielectrics with $\rho_s = 0$

$$D_{2n} = D_{1n} \quad \Longleftrightarrow \quad \varepsilon_2 E_{2n} = \varepsilon_1 E_{1n}$$

Subscript n means normal, for general case.



The interface need not be planar.
For non-planar interface, let the zero-thickness pie's area $\Delta S \rightarrow 0$, and you'll get the same conclusion.

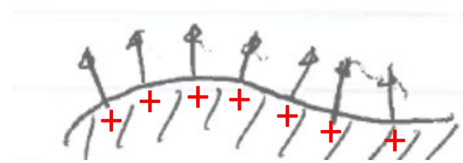
Similarly, at interface between a perfect conductor and a dielectric or vacuum

(Medium 1 is the conductor)

$$E_1 = 0 \Rightarrow E_{1n} = 0$$

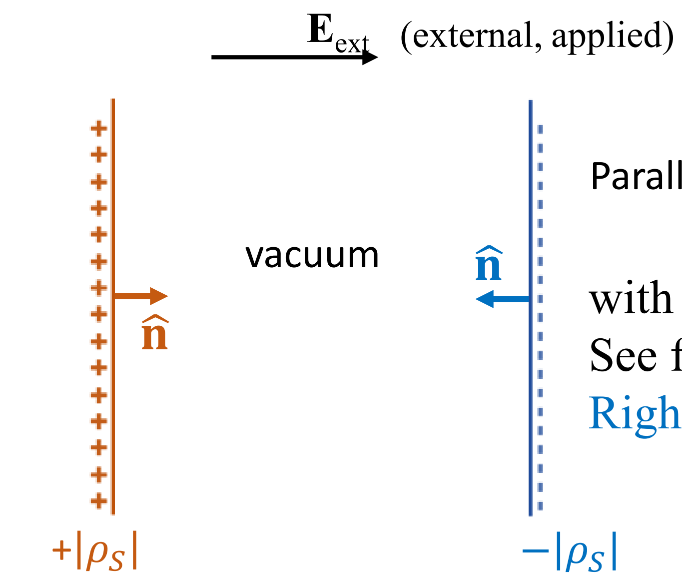
$$\varepsilon_2 E_{2n} = \rho_s$$

External charge



These are boundary conditions of the electrostatic field.

Summary on Dielectrics



- Gauss's law in free space

Parallel-plate capacitor model

$$\varepsilon_0 \mathbf{E}_{ext} \cdot \hat{\mathbf{n}} = \rho_S$$ external

with $\hat{\mathbf{n}}$ defined as pointing to interior.
See figure. Left: $+|\rho_S|$ thus $\mathbf{E}_{ext}$ along $\hat{\mathbf{n}}$.
Right: $+|\rho_S|$ thus $\mathbf{E}_{ext}$ along $\hat{\mathbf{n}}$.

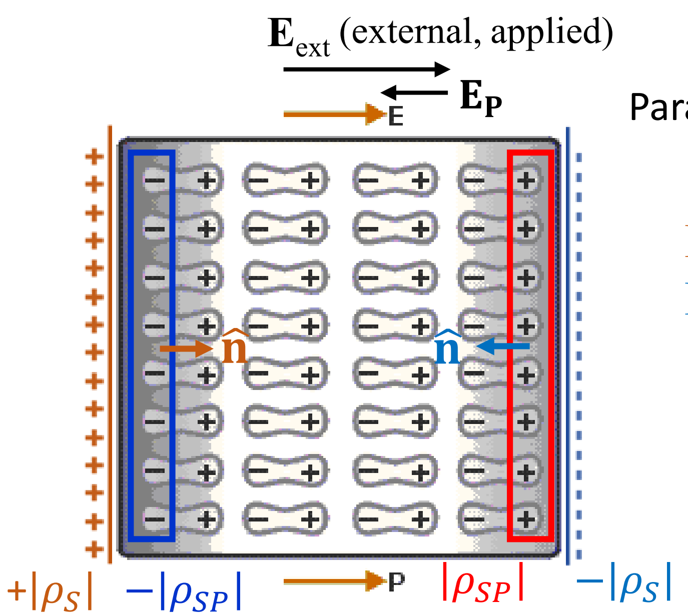
General

$$\varepsilon_0 \nabla \cdot \mathbf{E}_{ext} = \rho$$ external

- Dielectric in presence of external field

$$\mathbf{P} = \lim_{\Delta V \to 0} \frac{\sum_i \mathbf{p}_i}{\Delta V}$$

Vector sum, net!



Parallel-plate capacitor model

$$\rho_{SP} = -\mathbf{P} \cdot \hat{\mathbf{n}}$$

Left: $\rho_{SP} = -\mathbf{P} \cdot \hat{\mathbf{n}} = -|\rho_{SP}|$.
Right: $\rho_{SP} = -\mathbf{P} \cdot \hat{\mathbf{n}} = |\rho_{SP}|$

General

$$\nabla \cdot \mathbf{P} = -\rho_P$$

- Polarization charges give rise to internal field $\mathbf{E_P}$

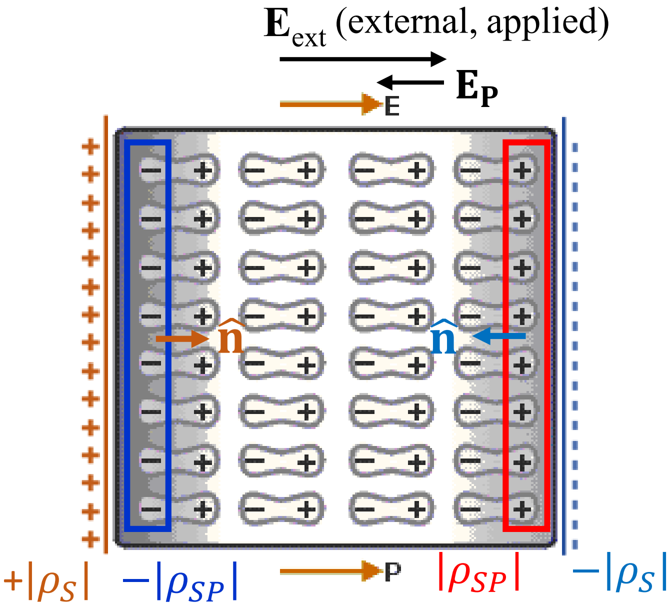
$$\mathbf{E_P} = \sigma_P \hat{\mathbf{n}}/\varepsilon_0$$
$$= -(\mathbf{P} \cdot \hat{\mathbf{n}})\hat{\mathbf{n}}/\varepsilon_0 = -\mathbf{P}/\varepsilon_0$$

for both left and right plates.

$$\left.\begin{array}{r} \varepsilon_0 \nabla \cdot \mathbf{E_P} = \rho_P \\ \nabla \cdot \mathbf{P} = -\rho_P \end{array}\right\}$$

$$\Rightarrow \ \mathbf{E_P} = -\mathbf{P}/\varepsilon_0$$

○ Polarization $\mathbf{P}$ is a linear response to the total field $\mathbf{E} = \mathbf{E}_{\text{ext}} + \mathbf{E}_{\mathbf{P}}$, rather than just $\mathbf{E}_{\text{ext}}$.



$$\mathbf{D} = \rho_S \hat{\mathbf{n}}$$

or

$$\mathbf{D} \cdot \hat{\mathbf{n}} = \rho_S$$

Define the proportional constant as $\chi_e \varepsilon_0$, i.e., $\mathbf{P} = \chi_e \varepsilon_0 \mathbf{E}$.

Parallel-plate capacitor model

$$\mathbf{E} = \mathbf{E}_{\text{ext}} + \mathbf{E}_{\mathbf{P}}$$
$$= \rho_S \hat{\mathbf{n}}/\varepsilon_0 - \mathbf{P}/\varepsilon_0$$

↑

$$\mathbf{P} = \chi_e \varepsilon_0 \mathbf{E}$$

⇓

$$\varepsilon_0 (1 + \chi_e) \mathbf{E} = \rho_S \hat{\mathbf{n}}$$

(This holds for both left and right plates)

General

$$\varepsilon_0 \nabla \cdot \mathbf{E}$$
$$= \varepsilon_0 \nabla \cdot \mathbf{E}_{\text{ext}} + \varepsilon_0 \nabla \cdot \mathbf{E}_{\mathbf{P}}$$
$$= \rho + \rho_P = \rho - \nabla \cdot \mathbf{P}$$

↑

$$\mathbf{P} = \chi_e \varepsilon_0 \mathbf{E}$$

⇓

$$\nabla \cdot [(1 + \chi_e)\varepsilon_0] \mathbf{E} = \rho$$

Define $1 + \chi_e = \varepsilon_r$ and $\varepsilon = \varepsilon_r \varepsilon_0$, then we have Gauss's law considering external charge only:

$$\varepsilon \mathbf{E} = \rho_S \hat{\mathbf{n}} \qquad\qquad \nabla \cdot (\varepsilon \mathbf{E}) = \rho$$

Define $\mathbf{D} = \varepsilon \mathbf{E} = \varepsilon_0 \mathbf{E} + \chi_e \varepsilon_0 \mathbf{E} = \varepsilon_0 \mathbf{E} + \mathbf{P}$, then

$$\mathbf{D} = \rho_S \hat{\mathbf{n}} \qquad\qquad \nabla \cdot \mathbf{D} = \rho$$

(This holds for both left and right plates)

We finished this slide set on Tue 11/1/2022.