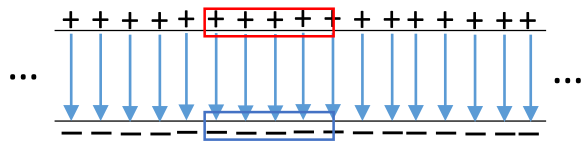
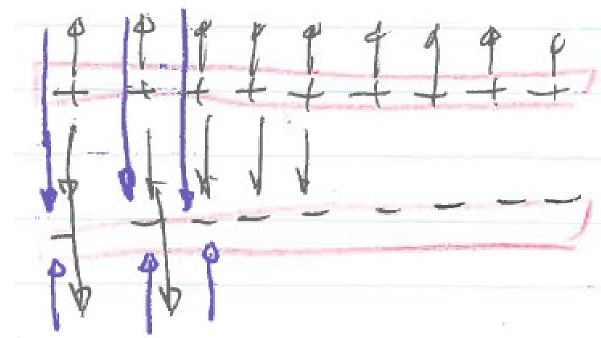
# Capacitors



The electric field lines starts from a positive charge and ends at a negative charge.

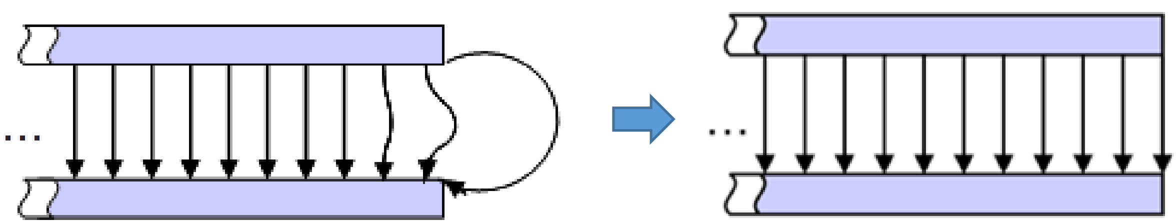
Gauss's law leads to $E = \frac{\rho_s}{\varepsilon_o}$ ← Surface charge density

If the two charge sheets are on two conductor plates, you have a parallel-plate capacitor.



Net charge can only exist on the surface of a(n) (ideal) conductor, never in the interior.

A real capacitor is finite in size.  Here we consider a parallel-plate capacitor infinitely large, just to ignore the fringe effect.



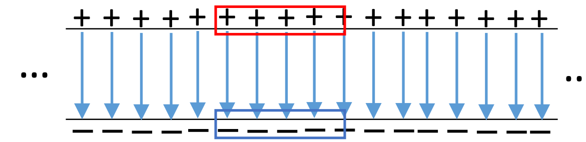
What physics do we violate when ignoring the fringe effect?

We actually mean that the plates' lateral dimensions are much, much larger than the distance, *d*, between the plates. This is actually true for many practical capacitors.

"Large" and "small" are relative.

Now, the field's direction is well defined, so we treat the field, *E*, as a scalar.

$$E = \frac{\rho_s}{\varepsilon}$$

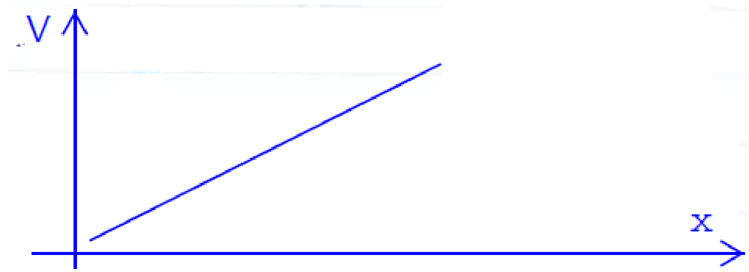


*E* is the (negative; ignore sign here) slope of the potential. Therefore,

$$V = Ed = \frac{\rho_s d}{\varepsilon} = \frac{Q}{A}\frac{d}{\varepsilon}$$

$$\Rightarrow \quad Q = \varepsilon \frac{A}{d} V$$

$$\Rightarrow \quad C = \varepsilon \frac{A}{d}$$

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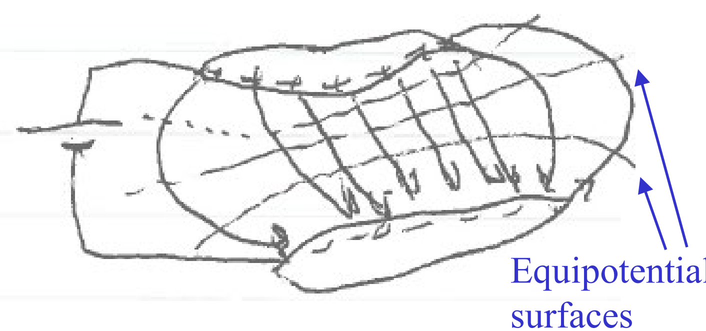
Any pair of conductors makes a capacitor.  Just more complicated calculation of the capacitance.

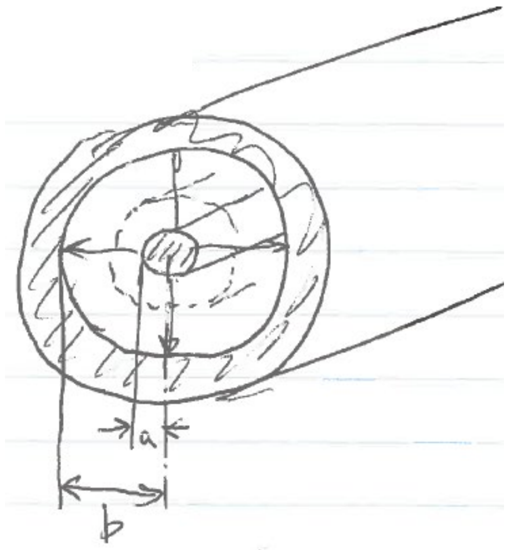
Next, we show some special cases that are "analysis friendly."




What are the equipotential surfaces for the parallel-plate capacitor?

**Example**: Capacitance (per length) of co-ax cable

(Read offline on your own with Example 4-12 in textbook, and compare result to Table 2-1)




By symmetry, we know the electric field lines are in the radial direction.

(Therefore, equipotential surfaces are cylindrical surfaces coaxial with the core and the shield)

Capacitance scales with length. So we normalize the capacitance against length.

For a $\Delta l$ long segment of the infinitely long cable, imagine a cylinder of radius $r$ as the Gaussian surface. By Gauss's law,

linear charge density: charge per length

Why don't we define surface charge densities?

$$\varepsilon E\,(2\pi r)\,\Delta l = \rho_l\,\Delta l$$

$$\therefore\ E = \frac{\rho_l}{\varepsilon}\,\frac{1}{2\pi r} \Rightarrow$$

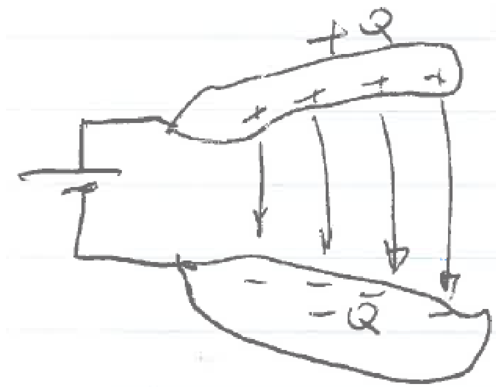
$$V = \int_a^b E\,dr = \frac{\rho_l}{\varepsilon}\,\frac{1}{2\pi}\int_a^b \frac{1}{r}\,dr = \frac{\rho_l}{2\pi\varepsilon}\ln r\Big|_a^b = \frac{\rho_l}{2\pi\varepsilon}\ln\frac{b}{a}$$

$$\Delta Q = \rho_l\,\Delta l = \Delta C\,V \Rightarrow$$

$$\Delta C = \frac{\Delta Q}{V} = \frac{\rho_l\,\Delta l}{V} = \frac{\rho_l\,\Delta l}{\frac{\rho_l}{2\pi\varepsilon}\ln\frac{b}{a}} = \frac{2\pi\varepsilon\,\Delta l}{\ln\frac{b}{a}} \Rightarrow \boxed{C' = \frac{\Delta C}{\Delta l} = \frac{2\pi\varepsilon}{\ln\frac{b}{a}}}$$

Now we show that a capacitor stores energy.

First, the general case of a pair of conductors of any shape.




Charging a capacitor is separating positive and negative charges.
They attract each other therefore work needs to be done.

The work done is energy spent on separating them.
This energy is stored in the capacitor.

$$Q = CV$$

To add a bit more charge $dQ$, we need to do work $dW = V\,dQ$

To charge a capacitor from $V' = 0$ to $V' = V$, we need to do work

$$W = \int_{V'=0}^{V} V'\,dQ' = \int_0^V V'\,d(CV') = C\int_0^V V'\,dV' = \frac{1}{2}CV'^2\Big|_0^V = \frac{1}{2}CV^2$$

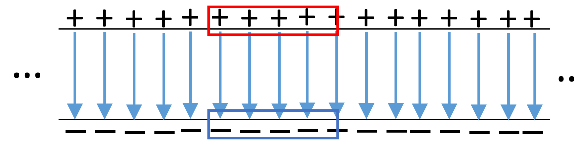
Harder and harder to add the same bit of charge

We need to take the integral since the voltage is increasing as we charge.

Notice that the factor ½ is due to the integral.
It is there for a dc voltage.

Now let's calculate the energy stored in a parallel-plate capacitor with a different method.

Ignore fringe effect, $E = \dfrac{\rho_s}{\varepsilon} = \dfrac{Q}{\varepsilon A}$



To pull the two plates a distance *d* apart, we need to do work

$$\left.\begin{array}{r} W = Fd = QEd = Q\dfrac{Q}{\varepsilon A}d = Q^2\dfrac{d}{\varepsilon A} \\ C = \dfrac{\varepsilon A}{d} \end{array}\right\} \Rightarrow$$

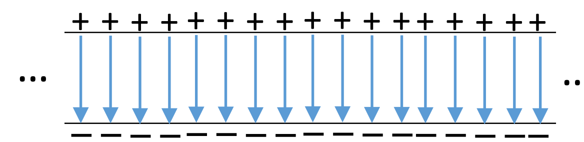
$$\left.\begin{array}{c} W = \dfrac{Q^2}{C} \\ Q = CV \end{array}\right\} \Rightarrow W = CV^2$$

Compare with previous result, $W = \dfrac{1}{2}CV^2$

Did we do anything wrong?

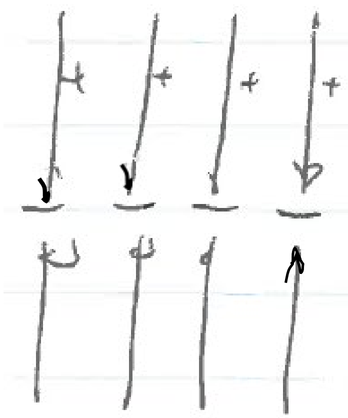
What we did wrong was double counting, a sort of “creative accounting”.

$$E = \frac{\rho_s}{\varepsilon} = \frac{Q}{\varepsilon A}$$ is the total field due to both plates.



The force is exerted on one plate by the field due to the other. Therefore,

$$E = \frac{\rho_s}{2\varepsilon} = \frac{Q}{2\varepsilon A}$$



The positive charge sheet in the field of the negative charge sheet

$$\left.\begin{aligned} W &= Fd = QEd = Q \cdot \frac{Q}{2\varepsilon A} d = \frac{1}{2} Q^2 \frac{d}{\varepsilon A} \\ C &= \frac{\varepsilon A}{d} \end{aligned}\right\}$$

$$\Rightarrow \left.\begin{aligned} W &= \frac{1}{2}\frac{Q^2}{C} \\ Q &= CV \end{aligned}\right\} \Rightarrow W = \frac{1}{2} C V^2$$

This another way to appreciate the factor ½.

Energy storage is important, since renewable energy sources are not synchronized with our demands. Let's see how good the capacitor is at energy storage.

Using the parallel-plate capacitor model,

$$W = \frac{1}{2} C V^2 = \frac{1}{2} \varepsilon \frac{A}{d} (Ed)^2 = \frac{1}{2} \varepsilon \underbrace{(Ad)}_{\text{volume}} E^2$$

So, the energy density that is store is $\frac{1}{2} \varepsilon E^2$

This conclusion can be extended to capacitors of any geometry.

We can charge a capacitor to higher and higher voltage,
so that unlimited energy can be stored. Right?

The electrostatic energy density stored in a dielectric is $\frac{1}{2}\varepsilon E^2$

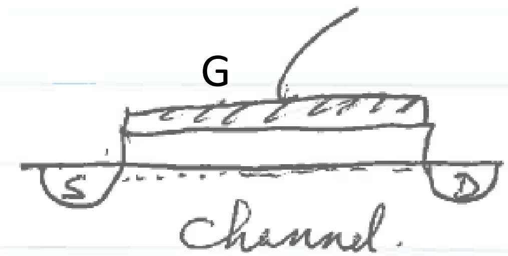
The limit to the highest possible energy density is the breakdown voltage (V/m), which is actually breakdown field.

$$\frac{1}{2}\varepsilon E_{br}^2$$

Energy is one important thing we EEs deal with. Information is another.
The field-effect transistor is the workhorse of modern microelectronics.

The areal charge density of the channel is

$$\rho_s = \frac{Q}{A} = \frac{C}{A}V = \frac{\varepsilon}{d}V = \varepsilon E$$




The highest possible induced charge density of the FET channel is limited by the breakdown field.

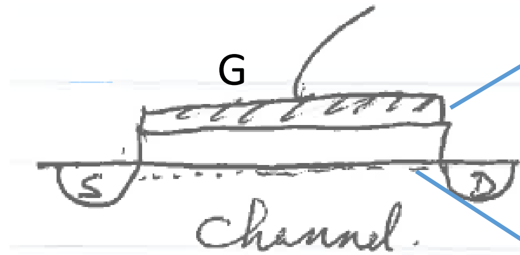
For example, $E_{br} = 10^9$ V/m = 1 V/nm for $SiO_2$. Same order of magnitude for other dielectrics.

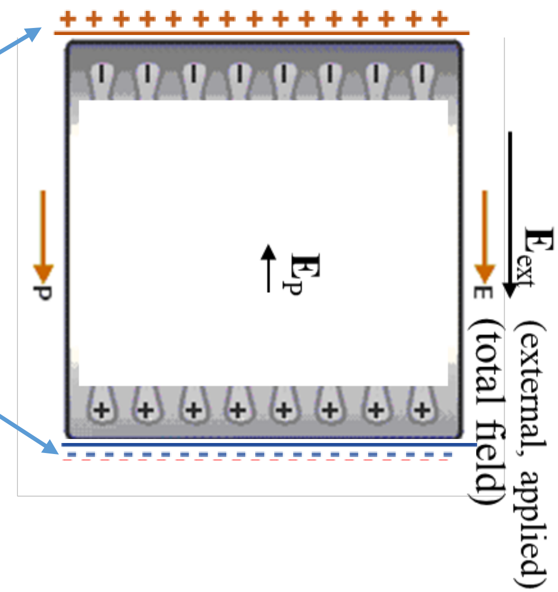
In order to increase the channel charge density, we can increase $\varepsilon$.

$\varepsilon = \varepsilon_0 \varepsilon_r$

$\varepsilon_r$ is a material property. We may choose materials with a high $\varepsilon_r$ as the gate insulator. The semiconductor industry uses $k$ for $\varepsilon_r$. Therefore, the term "high-k dielectrics".

$\rho_s = D = \varepsilon E = \varepsilon_0 \varepsilon_r E$




High $\varepsilon_r$ (high-k) gate dielectric

$\Downarrow$

More dipoles induced by same $E$ (therefore voltage)

$\Downarrow$

Lager $P$, therefore larger $\rho_{sP}$

$\Downarrow$

Need larger $\rho_s$ to cancel this $\rho_{sP}$ (recall polarization always against external field)

Review textbook Sections 4-9, 4-10, 4-7.2. Do Homework 9 up to Problems 6.

We finished this slide set on Thu 11/3/2022.