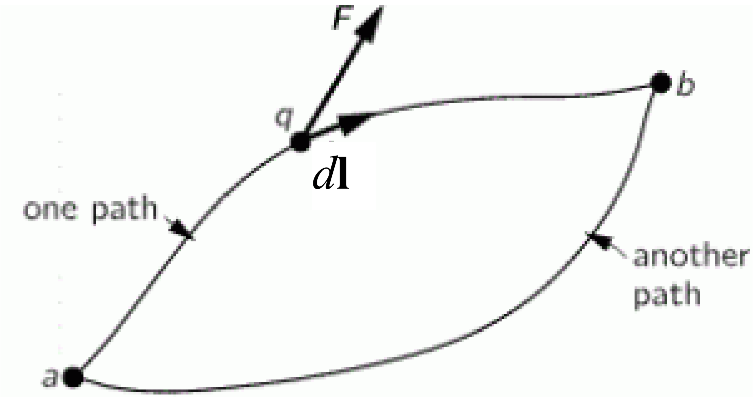
We continue to compare the electrostatic and magnetostatic fields.

The electrostatic field is conservative: $\oint \vec{E}\cdot d\vec{l} = 0$ $\nabla \times \vec{E} = 0$



This allows us to define the potential *V*:

$$\nabla V = -\vec{E}$$

because

$$V_a - V_b = -\int \vec{E}\cdot d\vec{l}$$

is independent of the path.

If a vector field has no curl (i.e., is conservative), it must be something's gradient.

V is called the “scalar potential”.

Gravity is conservative. Therefore you do not see water flowing in such a loop without a pump in the physical world.

Height can be regarded the potential in the gravitational field near earth surface.

Vector potential of magnetic field (read offline)

For the magnetic field, $\nabla\cdot\vec{B}=0$ $\oint\vec{B}\cdot d\vec{S}=0$

If a vector field has no divergence (i.e., is solenoidal),
it must be something's curl.

$$\nabla\cdot(\nabla\times\vec{A})=0$$

In other words, the curl of a vector field has zero divergence.

Let's use another physical context to help you understand this math:

$$\vec{J}=\nabla\times\vec{H}$$

Ampère's law

$$\nabla\cdot\vec{J}=0 \Leftrightarrow \oint_S \mathbf{J}\cdot d\mathbf{s}=0$$

Kirchhoff's current law (KCL)

Since $\nabla\cdot\vec{B}=0$, we can define a vector field **A** such that $\vec{B}=\nabla\times\vec{A}$

A is the vector potential.

Notice that for a given **B**, **A** is not unique. For example, if $\vec{B}=\nabla\times\vec{A}$

then $\vec{B}=\nabla\times(\vec{A}+\vec{A}_0)$, because $\nabla\times\vec{A}_0=0$

Similarly, for the electrostatic field, the scalar potential *V* is not unique:

If $\nabla V=-\vec{E}$ then $\nabla(V+V_0)=-\vec{E}$

You have the freedom to choose the reference V_0

$$\left.\begin{array}{l}\nabla \times \vec{H} = \vec{J} \quad \text{(Ampère's law)} \\ \vec{B} = \mu \vec{H}\end{array}\right\} \Rightarrow \nabla \times \vec{B} = \mu \vec{J} \Rightarrow$$

$$\nabla \times (\nabla \times \vec{A}) = \mu \vec{J}$$

Going through the math, you will get $\nabla^2 \vec{A} - \nabla(\nabla \cdot \vec{A}) = -\mu \vec{J}$

Here is what $\nabla^2 \vec{A}$ means: $\nabla^2 \vec{A} = \hat{x} \nabla^2 A_x + \hat{y} \nabla^2 A_y + \hat{z} \nabla^2 A_z$

Just notation. Notice that $\nabla^2 \vec{A}$ is a vector.

Still remember what ∇^2 means for a scalar field?

From a previous lecture: $\nabla^2 V = \dfrac{\partial^2 V}{\partial x^2} + \dfrac{\partial^2 V}{\partial y^2} + \dfrac{\partial^2 V}{\partial z^2}$

Recall that the choice for **A** is not unique. It turns out that we can always choose **A** such that

$$\nabla \cdot \vec{A} = 0$$

$$\left.\begin{array}{l} \nabla \times \vec{H} = \vec{J} \quad \text{(Ampère's law)} \\ \vec{B} = \mu \vec{H} \end{array}\right\} \Rightarrow \nabla \times \vec{B} = \mu \vec{J} \Rightarrow$$

$$\left.\begin{array}{r} \nabla \times (\nabla \times \vec{A}) = \mu \vec{J} \quad \Rightarrow \quad \nabla^2 \vec{A} - \nabla(\nabla \cdot \vec{A}) = -\mu \vec{J} \\ \text{The choice for } \mathbf{A} \text{ is not unique. We choose } \mathbf{A} \text{ such that } \nabla \cdot \vec{A} = 0 \end{array}\right\} \Rightarrow$$

$$\nabla^2 \vec{A} = -\mu \vec{J}$$

Here is what $\nabla^2 \vec{A}$ means: $\nabla^2 \vec{A} = \hat{x}\nabla^2 A_x + \hat{y}\nabla^2 A_y + \hat{z}\nabla^2 A_z$

Notice that $\nabla^2 \vec{A}$ is a vector. Thus this is actually three equations:

$$\begin{cases} \nabla^2 A_x = -\mu J_x \\ \nabla^2 A_y = -\mu J_y \\ \nabla^2 A_z = -\mu J_z \end{cases}$$

Recall the definition of ∇^2 for a scalar field from a previous lecture:

$$\nabla^2 V = \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2}$$

Poisson's equation for the magnetic field $\nabla^2 \vec{A} = -\mu \vec{J}$

is actually three equations:

$$\begin{cases} \nabla^2 A_x = -\mu J_x \\ \nabla^2 A_y = -\mu J_y \\ \nabla^2 A_z = -\mu J_z \end{cases}$$

Compare Poisson's equation for the magnetic field with that for the electrostatic field:

$$\nabla^2 V = -\frac{\rho}{\varepsilon}$$

$$\nabla^2 V = \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = -\frac{\rho}{\varepsilon}$$

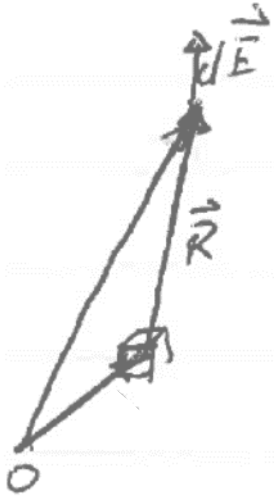
Given **J**, you can solve **A**, from which you get **B** by $\vec{B} = \nabla \times \vec{A}$

Given ρ, you can solve V, from which you get **E** by $\vec{E} = -\nabla V$

Exams (Test 2 & Final) problems will not involve the vector potential.
This is an important topic, however, if you have further interest in microwave engineering, antennas, etc. Moreover, the concepts we just discussed help you to better understand the magnetic field. For your interest, therefore, review these notes & Section 5-4 of textbook.

Summary of methods to find magnetostatic and electrostatic fields

Electrostatics



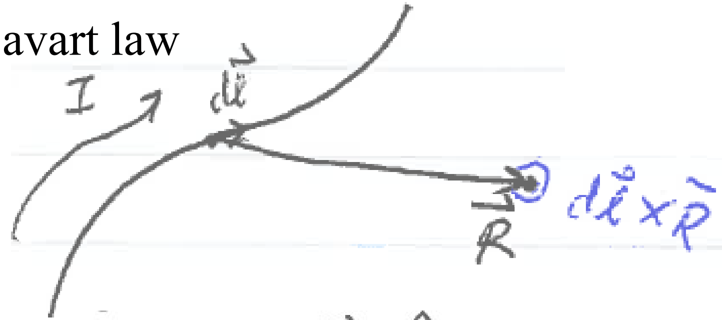
Coulomb's law

$$d\vec{E} = \frac{1}{4\pi\varepsilon}\frac{\rho dV}{R^2}\hat{R}$$

$$d\vec{D} = \frac{1}{4\pi}\frac{\rho dV}{R^2}\hat{R}$$

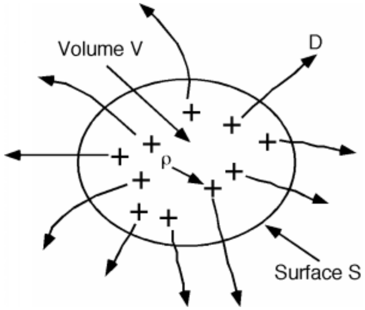
Magnetostatics

Biot-Savart law



$$d\vec{H} = \frac{1}{4\pi}\frac{I d\vec{\ell}\times\hat{R}}{R^2}$$

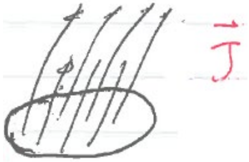
Take integral to get total field.



Gauss's law

$$\oint \vec{D}\cdot d\vec{S} = \int \rho dV = Q$$

Ampère's law



$$\oint \vec{H}\cdot d\vec{\ell} = \int \vec{J}\cdot d\vec{S} = I$$

Often used to take advantage of symmetry. Simple math.

$$\nabla^2 V = -\frac{\rho}{\varepsilon}$$

$$\vec{E} = -\nabla V$$

$$\nabla^2 \vec{A} = -\mu\vec{J}$$

$$\vec{B} = \nabla\times\vec{A}$$

Poisson's equations. Solve partial differential equations to get potentials, and then fields.

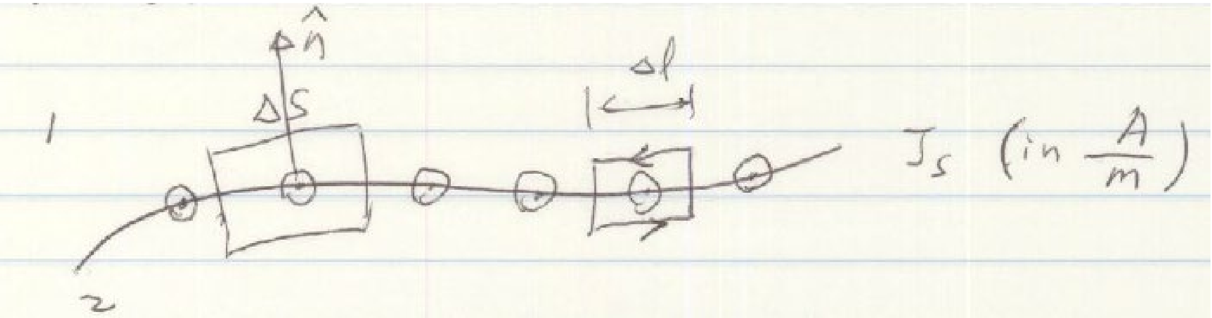
The last topic in magnetostatics: Magnetic boundary conditions

Consider a cylinder with zero thickness, with infinitesimal top/bottom surface area *ΔS*

$\nabla \cdot \vec{B} = 0$ $\Rightarrow$

$\vec{B}_1 \cdot \hat{n} \, \Delta S = \vec{B}_2 \cdot \hat{n} \, \Delta S$

$\Rightarrow$ $B_{1n} = B_{2n}$



Consider a rectangle with zero width, with infinitesimal length *Δl*

$\oint \vec{H} \cdot d\vec{l} = I$ $\Rightarrow$ $-H_{1t}\Delta l + H_{2t}\Delta l = J_s \Delta l$ $\Rightarrow$ $H_{2t} - H_{1t} = J_s$

> Notice that J_s is the sheet current density. The current is confined to flow along the interface, which has no thickness. Thus the dimension of J_s is current / length, and thus its unit A/m. The "cross section" of the current sheet is a curve.

Notice that the normal boundary condition is expressed in field **B**, while the tangential one in field **H**. For now we just relate the two fields by $\mathbf{B} = \mu\mathbf{H}$, where μ is proportional constant determined by materials properties and $\mu = \mu_0$ for free space. The magnetism of materials will be discussed later.

Review textbook Section 5-6. Do Homework 11 Problem 1 (**Problem 5.33** in textbook).

We finished this slide set on Thu 11/15/2022.