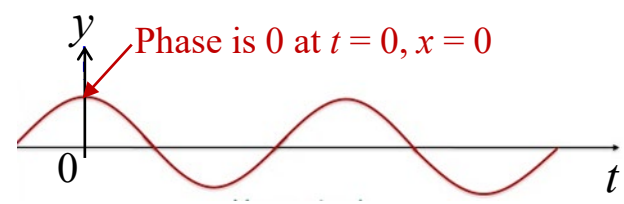
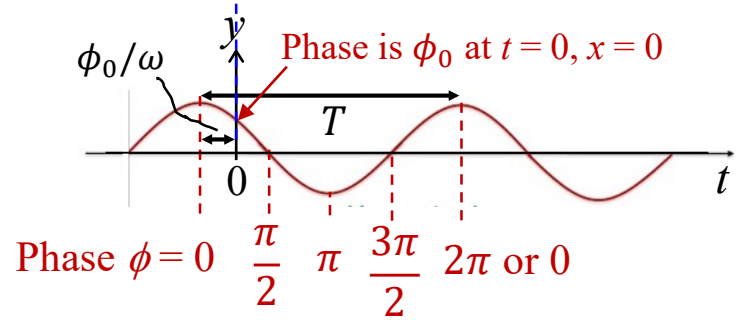
We now review "phasors" and consider how to use them to describe waves

Consider a function of time, or "signal", $y(t) = A\cos(\omega t)$



Shift by a phase ϕ_0, then it becomes
$y(t) = A\cos(\omega t + \phi_0)$

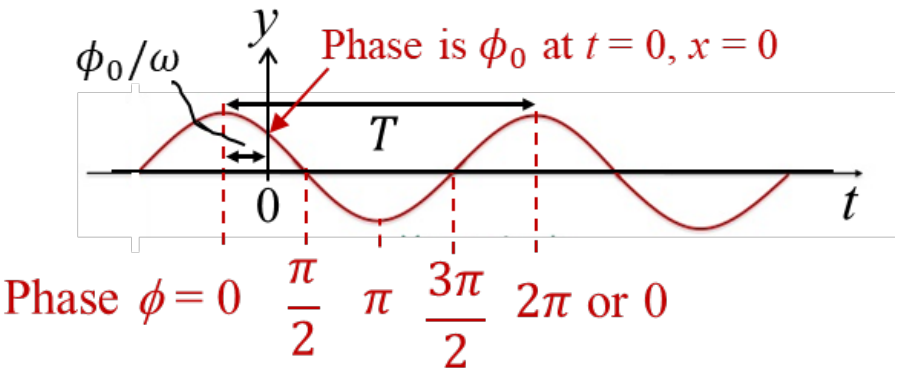


Using $\cos(\alpha + \beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$, we write

$$y(t) = A\cos(\omega t + \phi_0) = A\cos\phi_0\cos(\omega t) - A\sin\phi_0\sin(\omega t)$$

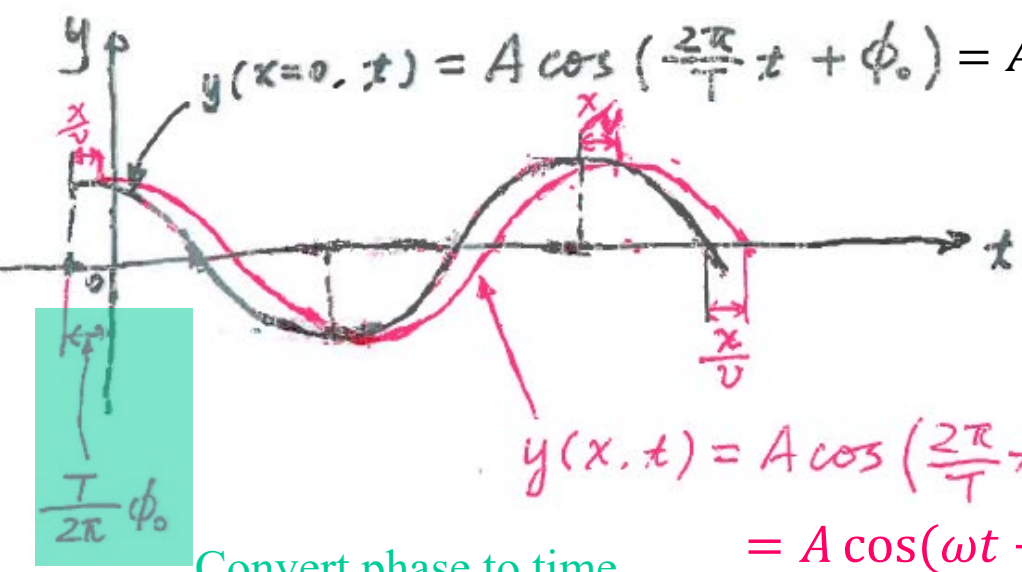
A linear combination of $\cos(\omega t)$ and $\sin(\omega t)$.

We just learned: A wave propagating by a distance x is just a phase shift of $-\beta x$.



"Waveform" of $y(x,t) = A\cos(\omega t - \beta x + \phi_0)$ at $x = 0$:

$$y(0,t) = A\cos(\omega t + \phi_0)$$



$= A\cos(\omega t + \phi_0)$

Measure waveforms at different locations

$$y(x,t) = A\cos\left(\frac{2\pi}{T}t + \phi_0 - \frac{2\pi}{\lambda}x\right) = A\cos\left[\frac{2\pi}{T}t + \left(\phi_0 - \frac{2\pi}{\lambda}x\right)\right]$$

$$= A\cos(\omega t + \phi_0 - \beta x)$$

Convert phase to time

Using $\cos(\alpha + \beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$, we write

$$y(x,t) = A\cos(\omega t + \phi_0 - \beta x) = \boxed{A\cos(\phi_0 - \beta x)}\cos(\omega t) - \boxed{A\sin(\phi_0 - \beta x)}\sin(\omega t)$$

Also a linear combination of $\cos(\omega t)$ and $\sin(\omega t)$, just with weights dependent on x.

Phasors

Handling the trigonometric functions can be tedious:

$$\cos(\alpha + \beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$$

This is just a simple phase shift, say, $\alpha \leftarrow \omega t$, $\beta \leftarrow \phi_0$.

(pronounced like oyler)

We have Euler's identity: $e^{ix} = \cos x + i\sin x$ *i* versus *j*. Will talk about this later.

that relates trigonometric functions to complex exponentials,

the math of which is a bit simpler:

$$e^{j(\alpha+\beta)} = e^{j\alpha}e^{j\beta}$$

$$e^{j(\omega t+\phi)} = e^{j\phi}e^{j\omega t}$$

Take α as ωt and β as ϕ.

At least, phase shifting is much easier!

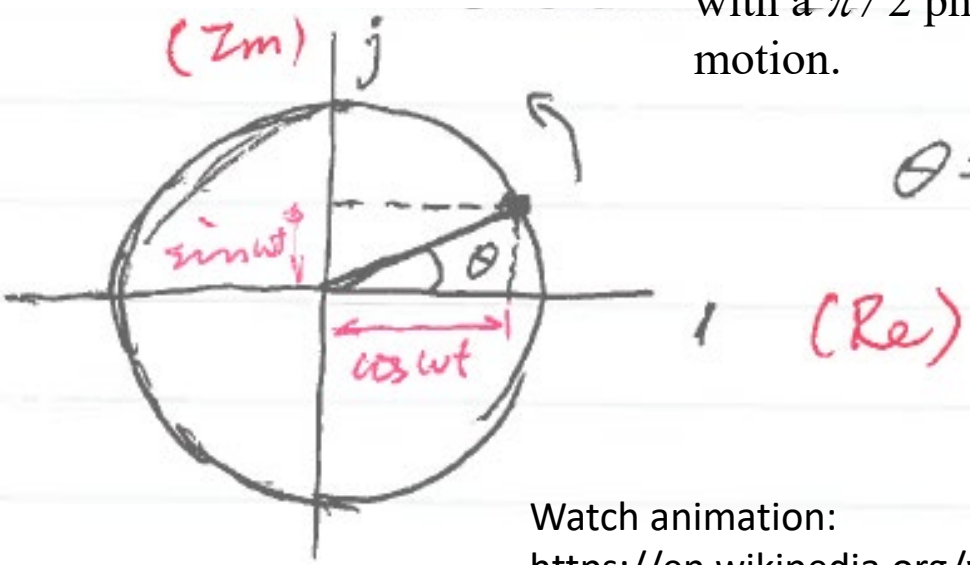
So, we use $e^{j\alpha}$ to represent $\cos\alpha$ for mathematical convenience.

But, why can we do this? What's behind this?

$$e^{j\omega t} = \cos\omega t + j\sin\omega t$$

The projections of constant-speed circular motion are harmonic oscillations.

Conversely, the superposition of two harmonic oscillations with a $\pi/2$ phase difference is a constant-speed circular motion.



$$\theta = \omega t$$

$$\mathrm{Re}\, e^{j\omega t} = \cos\omega t$$
$$\mathrm{Im}\, e^{j\omega t} = \sin\omega t$$

Watch animation:
https://en.wikipedia.org/wiki/Phasor#/media/File:Unfasor.gif

For every oscillation $A\cos(\omega t + \phi)$

we add an "imaginary partner" $jA\sin(\omega t + \phi)$

$$Ae^{j(\omega t+\phi)}$$
$$= A\cos(\omega t+\phi) + jA\sin(\omega t+\phi)$$

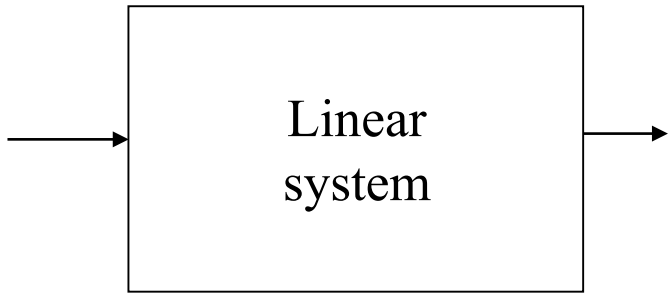
Instead of considering $A\cos(\omega t + \phi)$, which is mathematically more difficult to handle, we consider $Ae^{j(\omega t+\phi)}$, which is easier.

After we solve the problem, we throw away the "imaginary partner," keeping only the "real" part.

But why are we allowed to do this? What's the justification?

The answer is "linearity."

$$Ae^{j(\omega t+\phi)}$$
$$= A\cos(\omega t+\phi) + jA\sin(\omega t+\phi)$$



Response to a sum is the sum of responses.

Simple example of nonlinearity: $f(x) = x^2$

$(e^{j\omega t})^2 = ?$

$(\cos\omega t + j\sin\omega t)^2 = ?$

$$(\cos\omega t + j\sin\omega t)^2 \neq \cos^2\omega t + j\sin^2\omega t$$

Based on Euler's identity, we have a mathematical tool to more easily handle harmonic oscillations for linear systems or media.

It can be a pain to handle cos and sin functions.
So we add an "imaginary partner" to the cosine function:

Say, for a voltage

$$v(t) = A\cos(\omega t + \phi_0) \quad \rightarrow \quad Ae^{j(\omega t+\phi_0)}$$

$$= Ae^{j\phi_0}e^{j\omega t}$$

This "rotation" part is always there – a background.
So leave it out.

The phasor, or "complex amplitude"

$$\tilde{V} = Ae^{j\phi_0} = A\angle\phi_0$$

carries two pieces of information:

- The real amplitude A
- The reference phase ϕ_0

Convert a phasor to a time domain function:

$$\tilde{V} = Ae^{j\phi_0} = A\angle\phi_0$$

$$v(t) = \mathrm{Re}(\tilde{V}e^{j\omega t})$$

1. Put the rotation part, or background, back

2. Take the real part (i.e., throw away the "imaginary partner")

Note: here we take a "cosine reference." A different convention is the "sine reference," where you add and throw away a "real partner."

Use a phasor to represent a traveling wave

$$v(x, t) = A\cos(\omega t - \beta x + \phi_0) \quad \rightarrow \quad Ae^{j(\omega t - \beta x + \phi_0)}$$

$$= Ae^{j(-\beta x + \phi_0)} e^{j\omega t}$$

Leave out

The phasor, or complex amplitude at each position x

$$\tilde{V}(x) = Ae^{j(-\beta x + \phi_0)} = A\angle(-\beta x + \phi_0)$$

Notice x dependence

$$= (Ae^{j\phi_0}) e^{-j\beta x}$$

Propagation factor

"complex amplitude" of wave

For both the time domain function $v(t) = A\cos(\omega t + \phi_0)$ and the wave $v(x, t) = A\cos(\omega t - \beta x + \phi_0)$, the time dependence is left out, since we are dealing with a single frequency.

Exercise

Find the phasor for decaying wave $y(x, t) = Ae^{-\alpha x}\cos(\omega t - \beta x + \phi_0)$.

Exercise

Find the phasor for decaying wave $y(x,t) = Ae^{-\alpha x}\cos(\omega t - \beta x + \phi_0)$.

Answer

The tilde

Attenuation Propagation

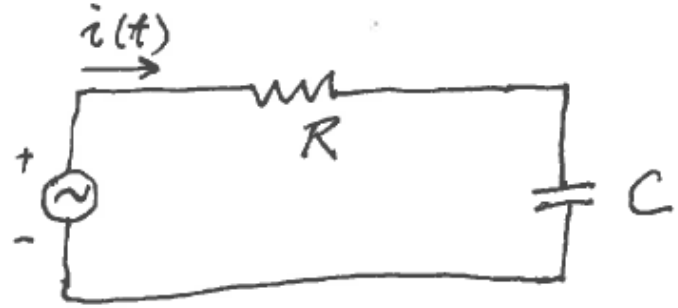
$$\tilde{Y}(x) = Ae^{-\alpha x - j\beta x + j\phi_0} = Ae^{j\phi_0}e^{-\alpha x}e^{-j\beta x}$$

$$= (A\angle\phi_0)\, e^{-\alpha x}e^{-j\beta x}$$

$$= Ae^{-\alpha x}\angle(\phi_0 - \beta x)$$

Advantages — an example

$v(t) = V_0 \cos \omega t$



Find $i(t)$.

(This is a special case of the example of Fig. 1-20 in the textbook with $\phi_0 = \frac{\pi}{2}$)

↳ the ϕ_0 defined there, in a sine reference.

$$\boxed{\tilde{V} = V_0}$$

$$v(t) = \mathrm{Re}\left(\tilde{V} e^{j\omega t}\right) = \mathrm{Re}\left(V_0 e^{j\omega t}\right) = V_0 \cos \omega t$$

Try to solve $i(t)$ in the time domain:

$$\frac{1}{C}\int i(t)\,dt + R\,i(t) = V_0 \cos \omega t$$

$$\frac{1}{C}\, i(t) + R\frac{di}{dt} = -\omega V_0 \sin \omega t$$

Need to solve an "ordinary" differential equation.

A good thing about complex exponentials:

$$\frac{d}{dt} e^{j\omega t} = j\omega e^{j\omega t}, \quad \int e^{j\omega t}\, dt = \frac{1}{j\omega} e^{j\omega t}$$

Differential equations are turned into algebra.

$$\frac{1}{C}\int i(t)\, dt + R i(t) = V_0 \cos\omega t$$

Adding "imaginary partners" to both $i(t)$ & $v(t)$:

$$\frac{1}{C}\int \tilde{I} e^{j\omega t}\, dt + R\tilde{I} e^{j\omega t} = V_0 e^{j\omega t}$$

$$\frac{1}{j\omega C}\tilde{I}\cancel{e^{j\omega t}} + R\tilde{I}\cancel{e^{j\omega t}} = V_0 \cancel{e^{j\omega t}} = \tilde{V}\cancel{e^{j\omega t}}$$

$$\frac{1}{j\omega C}\tilde{I}\cancel{e^{j\omega t}} + R\tilde{I}\cancel{e^{j\omega t}} = V_0\cancel{e^{j\omega t}} = \tilde{V}\cancel{e^{j\omega t}}$$

$$\boxed{\tilde{I}\left(\tilde{R} + \frac{1}{j\omega C}\right) = V_0 = \tilde{V}} \Rightarrow \tilde{I} = \frac{V_0}{R + \frac{1}{j\omega C}}$$

Notice that $\tilde{I}$ is a "constant" — no t dependence

(complex amplitude)

Now, relate the phasor back to the time domain:

$$\frac{1}{R + \frac{1}{j\omega C}} = \frac{j\omega C}{1 + j\omega RC} = \frac{\omega C e^{j\frac{\pi}{2}}}{\sqrt{1+\omega^2R^2C^2}\, e^{j\phi_1}}$$

$$= \frac{\omega C}{\sqrt{1+\omega^2R^2C^2}} e^{j\left(\frac{\pi}{2} - \phi_1\right)},$$

where $\tan\phi_1 = \omega RC$, i.e. $\phi_1 = \tan^{-1}(\omega RC) = \arctan(\omega RC)$

$$\therefore \quad \tilde{I} = \frac{\omega C V_0}{\sqrt{1+\omega^2 R^2 C^2}} e^{j(\frac{\pi}{2} - \phi_1)}$$

$$i(t) = \mathrm{Re}\left(\tilde{I} e^{j\omega t}\right)$$

Put the $e^{j\omega t}$ back & take the real part

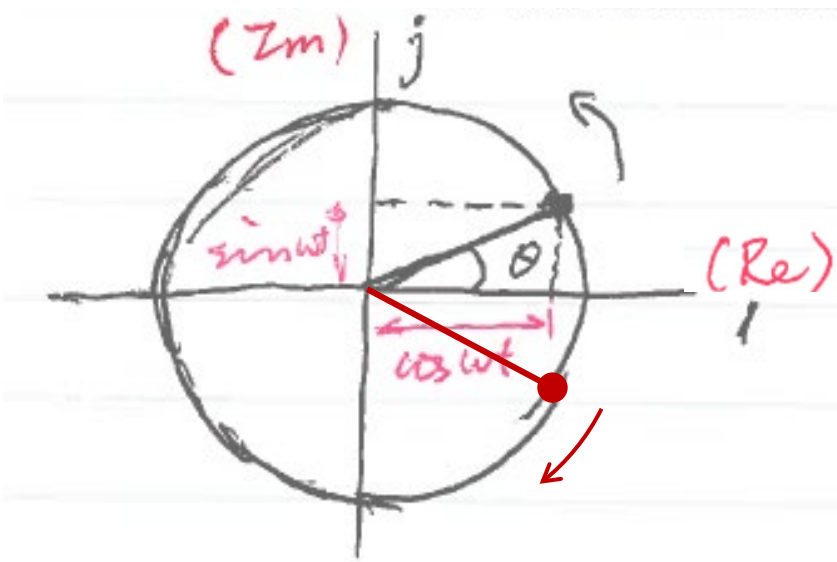
$$= \mathrm{Re}\left[\frac{\omega C V_0}{\sqrt{1+\omega^2 R^2 C^2}} e^{j(\omega t + \frac{\pi}{2} - \phi_1)}\right]$$

$$= \underbrace{\frac{\omega C V_0}{\sqrt{1+\omega^2 R^2 C^2}}}_{\text{Amplitude}} \cos\left(\omega t + \underbrace{\frac{\pi}{2} - \phi_1}_{\text{phase}}\right)$$

Instead of solving an ordinary differential equation, we just did some algebra.

Finish HW1 before next class meeting starts.

An alternative way to think about phasors



$$\cos\omega t = \frac{e^{j\omega t} + e^{-j\omega t}}{2}$$
$$= \frac{e^{j\omega t}}{2} + \left(\frac{e^{j\omega t}}{2}\right)^*$$

Signal $v(t) = A\cos(\omega t + \phi_0)$

$$= \frac{Ae^{j(\omega t+\phi_0)} + Ae^{-j(\omega t+\phi_0)}}{2}$$

corresponding to phasor

$$\tilde{V} = Ae^{j\phi_0}$$

Therefore,

$$v(t) = \frac{\tilde{V}e^{j\omega t} + \tilde{V}^* e^{-j\omega t}}{2}$$
$$= \frac{\tilde{V}}{2}e^{j\omega t} + \left(\frac{\tilde{V}}{2}e^{j\omega t}\right)^*$$
$$= 2\mathrm{Re}\left(\frac{\tilde{V}}{2}e^{j\omega t}\right) = \mathrm{Re}\left(\tilde{V}e^{j\omega t}\right)$$

For fun: i versus j