Consider a pair of wires **ideal** wires)

Length $<< \lambda$

Length $>> \lambda$, say, infinitely long

Voltage along a cable can vary!

Voltage $v(z) \sim E(z)D$

We can actually get to this wave behavior by using circuit theory, w/o going into details of the EM fields!
There is capacitance between any two pieces of conductors. A pair of plates, wires, etc., or the core and shields of a co-ax cable.

\[ C = C' \Delta z \]

\( C' \): capacitance per length
There is capacitance between any two pieces of conductors. A pair of plates, wires, etc., or the core and shields of a co-ax cable.

\[ C = C' \Delta z \]

\[ C' \text{: capacitance per length} \]

A piece of wire is actually an inductor

When \( i \) changes with \( t \), so does \( B \):

\[ \frac{di}{dt} \rightarrow \frac{dB}{dt} \]

\[ v \times E \times \frac{\partial B}{\partial t} \Rightarrow v \times \frac{di}{dt} \]

Voltage \( v \) along the wire
There is capacitance between any two pieces of conductors. A pair of plates, wires, etc., or the core and shields of a co-ax cable. 

\[ C = C' A \]

\[ C' \text{: capacitance per length} \]

A piece of wire is actually an inductor

\[ B \times i \]

When \( i \) changes with \( t \), so does \( B \) \( \frac{di}{dt} \rightarrow \frac{dB}{dt} \)

\[ v \times E \times \frac{\partial B}{\partial t} \Rightarrow v \times \frac{di}{dt} \]

\[ v = L \frac{di}{dt} \]

Voltage \( v \) along the wire

Similarly, a pair of coupled wires

\[ \times \times \times \]

Voltage \( v \) around the loop!
There is capacitance between any two pieces of conductors. A pair of plates, wires, etc., or the core and shields of a co-ax cable.

\[ C = C' \Delta z \]

\( C' \): capacitance per length

A piece of wire is actually an inductor

\[ B \times i \]

\[ E \times i \]

When \( i \) changes with \( t \), so does \( B \)

\[ \frac{dv}{dt} \rightarrow \frac{di}{dt} \]

Voltage \( v \) along the wire

Similarly, a pair of coupled wires

\[ i \times i \times i \]

Capacitance also considered

\[ L' \Delta z \]

Inductance lumped to one side

\[ L = L' \Delta z \]

Voltage \( v \) around the loop!
To make things simple, we first consider a pair of *ideal* wires. No resistance, no shunt (leakage).

Pay close attention. We take a different approach than does the book.
To make things simple, we first consider a pair of ideal wires. No resistance, no shunt (leakage).

Pay close attention. We take a different approach than does the book. Now, zoom in on one segment:

We stopped here on Thu 9/1/2022.
Inductor Capacitor

Take derivatives with regard to $z, t$

$$
\Delta v = v(z+\Delta z, t) - v(z, t) = -L' \Delta z \frac{\partial i}{\partial t}
$$

$$
\frac{\partial v}{\partial z} = \lim_{\Delta z \to 0} \frac{\Delta v}{\Delta z} = -L' \frac{\partial i}{\partial t}
$$

$$
\frac{\partial^2 v}{\partial z^2} = -L' \frac{\partial^2 i}{\partial t \partial z}
$$

$$
\frac{\partial^2 v}{\partial z \partial t} = -L' \frac{\partial^2 i}{\partial t^2}
$$

$$
\Delta i = i(z+\Delta z, t) - i(z, t) = -C' \Delta z \frac{\partial v}{\partial t}
$$

$$
\frac{\partial i}{\partial z} = \lim_{\Delta z \to 0} \frac{\Delta i}{\Delta z} = -C' \frac{\partial v}{\partial t}
$$

$$
\frac{\partial^2 i}{\partial z^2} = -C' \frac{\partial^2 v}{\partial t \partial z}
$$

$$
\frac{\partial^2 i}{\partial z \partial t} = -C' \frac{\partial^2 v}{\partial t^2}
$$
Partial differential equations

Do these 2 equations look familiar to you? What are they?

Let \( v_p = \cfrac{1}{\sqrt{L'C}} \), we have

\[
\frac{\partial^2 v}{\partial z^2} = \cfrac{1}{v_p^2} \frac{\partial^2 v}{\partial x^2}
\]

\( v = f(v_p t - z) \) is the general solution to this equation.

Do it on your own: verify this.
Let \( \nu_p = \frac{1}{\sqrt{l'C'}} \), we have

\[
\frac{\partial^2 \nu}{\partial z^2} = \frac{1}{\nu_p^2} \frac{\partial^2 \nu}{\partial x^2}
\]

\( \nu = f(\nu_p t - z) \) is the general solution to this equation.
\[
\frac{\partial^2 v}{\partial z^2} = L' C' \frac{\partial^2 v}{\partial x^2}
\]

Let \( v_p = \frac{1}{\sqrt{L' C'}} \), we have

\[ v = f(v_p t - z) \]

is the general solution to this equation.

This is the wave equation in 1D!
More strictly, the **lossless, dispersionless, linear** wave equation.

Assume: no resistance, no leakage; \( v_p \) independent of frequency;
\( v_p \) independent of voltage \( v \)

The equation for \( i \) is in the same form – 
**formally** the same.
Therefore, formally same solution.

\[
\frac{\partial^2 i}{\partial z^2} = L' C' \frac{\partial^2 i}{\partial x^2}
\]

These two together are called the **telegrapher’s** equations.
Formally same wave equations for voltage & current

\[
\begin{align*}
\frac{\partial^2 u}{\partial z^2} &= L' C' \frac{\partial^2 U}{\partial x^2} \\
\frac{\partial^2 i}{\partial z^2} &= L' C' \frac{\partial^2 i}{\partial x^2}
\end{align*}
\]

Solution to both take general form \( f(\nu_p t - z) \).

Is this amazing?

We arrived at the wave equation from circuit theory, regardless of frequency. Why does this approach work?
Circuit theory is a simple part of EM (black boxes: lumped elements)

Inside the black boxes:

\[ Q \propto v \Rightarrow Q = C v \]

\( C \) is a proportional constant.

\[ C \frac{dv}{dt} = \frac{dQ}{dt} = i \]

Current is charge flow per time.

\[ v = L \frac{di}{dt} \]

Changing \( B \) field induces \( E \).

\[ B \propto i \]

\[ \frac{di}{dt} \propto \frac{dB}{dt} \propto E \propto v \]

A wire (or a pair of wires) is also an inductor!

These “lumped” element models are valid only when dimensions \(< < \) wavelength

A circuit element is a model of a physical phenomenon, not necessarily a circuit component.
Formally same wave equations for voltage & current

\[
\begin{align*}
\frac{\partial^2 v}{\partial x^2} &= L' C' \frac{\partial^2 v}{\partial t^2} \\
\frac{\partial^2 i}{\partial x^2} &= L' C' \frac{\partial^2 i}{\partial t^2}
\end{align*}
\]

Solution to both take general form \( f(v_p t - z) \).

Is this amazing?
We arrived at the wave equation from circuit theory, regardless of frequency. Why does this approach work?

One more agreement, for 2-wire cables:

\[
L' = \frac{\mu}{\pi} \ln \left[ \frac{D}{d} + \sqrt{(D/d)^2 - 1} \right]
\]

\[
C' = \frac{\pi \varepsilon}{\ln \left[ \frac{D}{d} + \sqrt{(D/d)^2 - 1} \right]}
\]

\[
\therefore v_p = \frac{1}{\sqrt{L' C'}} = \frac{1}{\sqrt{\mu \varepsilon}}
\]

See Table 2-1, pp. 52 in 8/E (pp. 45 in 7/E, pp.53 in 6/E)

Consistent with EM theory (to be discussed later)!
Check offline for other transmission lines.
(There will be homework problems for you to learn about other types of transmission lines, as well as non-ideal coax cables)
\[
\frac{\partial^2 v}{\partial z^2} = \frac{1}{v_p^2} \frac{\partial^2 v}{\partial t^2}
\]

(The equation for \(i\) is in the same form – formally the same.)

\(v = f(v_p t - z)\) is the general solution to this equation.

What are the single frequency, simple harmonic solutions?

Single frequency, simple harmonic solutions:

\[
v(z, t) = \left| V_0^+ \right| \cos(\omega t - \beta z + \phi_0^+)
\]

\[
i(z, t) = \left| I_0^+ \right| \cos(\omega t - \beta z + \phi_0^+)
\]

(formally the same equation, thus formally the same solution.)

Here, \(V_0^+\) and \(I_0^+\) are “complex amplitudes” that we will talk about later. For the waves, they are not the phasors of the waves. We will talk about the distinction.

\(|V_0^+|\) and \(|I_0^+|\) are the real amplitudes, or simply amplitudes.

We have not yet shown the voltage and current waves are in phase. But they are. You can take this as a conclusion for now. The proof is on next page.
Here we show that the voltage and current waves are in phase with each other:

\[ v(z,t) = \beta |V_0^+| \sin(\omega t - \beta z + \phi_{v0}^+) \]

\[ i(z,t) = |I_0^+| \cos(\omega t - \beta z + \phi_{i0}^+) \]

\[ \frac{\partial v}{\partial z} = \beta |V_0^+| \sin(\omega t - \beta z + \phi_{v0}^+) \]

\[ \frac{\partial i}{\partial t} = -\omega |I_0^+| \sin(\omega t - \beta z + \phi_{i0}^+) \]

Recall that

\[ \frac{\partial v}{\partial z} = \lim_{\Delta z \to 0} \frac{\Delta v}{\Delta z} = -L \frac{\partial i}{\partial t} \]

For this to hold for any arbitrary \( z \), we must have \( \phi_{v0}^+ = \phi_{i0}^+ \equiv \phi_0^+ \)

So, in phase!

We also get a by-product:

\[ v(z,t) = |V_0^+| \cos(\omega t - \nu_{p_{v0}^+}) \]

Unit: \( \frac{m}{s \cdot m} = \frac{H}{s} = \Omega \)

These conclusions are important!

Anywhere, any time \( \nu(z,t)/i(z,t) = \text{constant} \)
Define $\frac{|V_0^+|}{|I_0^+|} = \frac{\omega L'}{\beta} = v_p L' \equiv Z_0$

$(Z_0 = v(z,t)/i(z,t) \text{ is real, i.e., purely resistive, for lossless lines})$

Consider:

\[ \begin{array}{c}
Z_0 \quad v(z,t) \quad \ldots \ldots
\end{array} \]

\[ \begin{array}{c}
i(z,t) \quad +
\end{array} \]
Define $\frac{|V_0^+|}{|I_0^+|} = \frac{\omega L'}{\beta} = \nu_p L' \equiv Z_0$

$(Z_0 = \nu(z,t)/i(z,t)$ is real, i.e., purely resistive, for lossless lines)

Consider:

There is no way to tell the difference just by measuring $\nu$ and $i$:

Energy propagating away vs. energy dissipated

Analogy: laser beam going to infinity or hitting a totally black wall
Define \[ |V_0^+| = \frac{\omega L'}{\beta} = \nu_p L' \equiv Z_0 \]

\( (Z_0 = v(z,t)/i(z,t) \) is real, i.e., purely resistive, for lossless lines) 

Consider:

\[ Z_0 \rightarrow v(z,t) \rightarrow i(z,t) \]

Infinitely long

There is no way to tell the difference just by measuring \( v \) and \( i \):

Energy propagating away vs. energy dissipated

Analogy: laser beam going to infinity or hitting a totally black wall

THE most important new concept in the first half of this course
Define \( \left| \frac{V_0^+}{I_0^+} \right| = \frac{\omega L'}{\beta} = v_p L' \equiv Z_0 \)

\((Z_0 = v(z,t)/i(z,t) \) is real, i.e., purely resistive, for lossless lines\)

Consider:

There is no way to tell the difference just by measuring \( v \) and \( i \):

Energy propagating away vs. energy dissipated

Analogy: laser beam going to infinity or hitting a totally black wall

**THE most important new concept in the first half of this course**

You may also use

\[
\frac{\partial i}{\partial z} = \lim_{\Delta z \to 0} \frac{\Delta i}{\Delta z} = - C' \frac{\partial v}{\partial x}
\]

Doing the derivatives in a similar way as in last page, you will also see the voltage and current waves are in phase.

You will have a similar “by-product” about the \( v/i \) ratio.

It may look different, but you should be able to show they are equal.

Do it on your own. Hint: use \( v_p = \frac{1}{\sqrt{L'C'}} \)  

Use scratch paper!
Now you can work on HW2: P1, P2

The solution copied:

\[ v^+(z, t) = |V_0^+| \cos(\omega t - \beta z + \phi_0^+) \]

\[ i^+(z, t) = |I_0^+| \cos(\omega t - \beta z + \phi_0^+) \]

In what direction do these waves propagate?
Waves propagating the other way are also solutions to the same equations:

\[ v^-(z, t) = V_0^- \cos(\omega t + \beta z + \phi^-_0) \]

\[ i^-(z, t) = I_0^- \cos(\omega t + \beta z + \phi^-_0) \]

The superscript + or − signifies propagation direction: +z or −z.

Of course, any linear combinations of waves in opposite directions are also solutions:

\[ v(z, t) = V_0^+ \cos(\omega t - \beta z + \phi^+_0) + V_0^- \cos(\omega t + \beta z + \phi^-_0) \]

\[ i(z, t) = I_0^+ \cos(\omega t - \beta z + \phi^+_0) + I_0^- \cos(\omega t + \beta z + \phi^-_0) \]

They may represent combinations of incident and reflected waves.

Recall that we have a mathematical tool to
1. Avoid the pain of dealing trigonometric functions, and
2. Turn partial differential equations to ordinary differential equations by putting aside the known time variation
Express waves with phasors

\[ v(z, t) = |V_0^+| \cos(\omega t - \beta z + \phi_0^+) + |V_0^-| \cos(\omega t + \beta z + \phi_0^-) \]

\[ \tilde{V} = \tilde{V}^+ + \tilde{V}^- = V_0^+ e^{-j\beta z} + V_0^- e^{j\beta z} \]

Where do the phases go?
Express waves with phasors

\[ \mathbf{v}(z,t) = |V_0^+| \cos(\omega t - \beta z + \phi_0^+) + |V_0^-| \cos(\omega t + \beta z + \phi_0^-) \]

Where do the phases go?

Recall the details how we convert a time-domain function to a phasor:

By adding an “imaginary partner”

\[ \mathbf{v}(z,t) \rightarrow |V_0^+| e^{j(\omega t - \beta z + \phi_0^+)} + |V_0^-| e^{j(\omega t + \beta z + \phi_0^-)} = (|V_0^+| e^{j\phi_0^+}) e^{-j \beta z} + (|V_0^-| e^{j\phi_0^-}) e^{j \beta z} e^{j \omega t} \]

This is not the phasor yet.

Throw away the known time variation \( e^{j \omega t} \)

and define the complex amplitudes \( V_0^+ = |V_0^+| e^{j\phi_0^+} \) and \( V_0^- = |V_0^-| e^{j\phi_0^-} \)

We get the phasor:

\[ \widetilde{\mathbf{V}} = \widetilde{\mathbf{V}}^+ + \widetilde{\mathbf{V}}^- = V_0^+ e^{-j \beta z} + V_0^- e^{j \beta z} \]

Notice sign and direction
Express waves with phasors

The current wave is similar

\[ i(z, t) = \left| I_0^+ \right| \cos(\omega t - \beta z + \phi_0^+) + \left| I_0^- \right| \cos(\omega t + \beta z + \phi^-) \]

phasor

\[ \tilde{I} = \tilde{I}^+ + \tilde{I}^- = I_0^+ e^{-j\beta z} + I_0^- e^{j\beta z} \]

This tool makes our life much easier when we deal with a more complicated situation. Here’s the more complicated situation:

No wires are ideal.
Any wire has some resistance.

\[ R = R' \Delta z \]

Resistance per length

There is always some shunt conductance between two wires

\[ G = G' \Delta z \]

Shunt conductance per length

Notice that \( R' \) and \( G' \) describes two different things. \( R' \neq 1/G' \)
Analyze the circuit in the phasor way

\[
\frac{d\tilde{V}}{dz} = \lim_{\Delta z \to 0} \frac{\Delta \tilde{V}}{\Delta z} = -(R' + j\omega L') \tilde{I}
\]

\[
\frac{d\tilde{I}}{dz} = \lim_{\Delta z \to 0} \frac{\Delta \tilde{I}}{\Delta z} = -(G' + j\omega C') \tilde{V}
\]

Take derivatives on one equation and insert it into the other, you get

\[
\frac{d^2 \tilde{V}}{dz^2} - (R' + j\omega L') (G' + j\omega C') \tilde{V}(x) = 0
\]

and a formally same equation for the current.

This is an ordinary differential equation. Because we used phasors.

We have arrived at this equation just by circuit analysis using phasors. You could also first do the circuit analysis in the time domain, arriving at partial differential equations, and then convert quantities to phasors and arrive at the same ordinary differential equations, as done in the book (Sections 2-3 & 2-4).
The partial differential equations for the general, more complicated situation are, well, too complicated. We don’t even bother to tackle them. Let’s look at the simpler ordinary differential equation:

\[
\frac{d^2 \tilde{V}}{dz^2} - (R' + j\omega L') (G' + j\omega C') \tilde{V}(z) = 0
\]

Before solving this equation, let’s first have a digression back to the ideal case

\[
R' = 0 \quad G' = 0
\]

\[
\frac{d^2 \tilde{V}}{dz^2} + \omega^2 L' C' \tilde{V} = 0 \quad \Rightarrow \quad \tilde{V}(z) = V_0 \pm e^{\pm j\omega \sqrt{L'/C'}z}
\]

Recall that \( \frac{\omega}{\beta} = v_p = \frac{1}{\sqrt{L'/C'}} \quad \Rightarrow \quad \beta = \omega \sqrt{L'/C'} \)

we have \( \tilde{V} = V_0 \pm e^{\pm j\beta z} \)

Notice that these are actually two solutions. What are the difference between the two solutions?
Waves propagating in two opposite directions: \( \tilde{V} = \tilde{V}^+ + \tilde{V}^- = V_0^+ e^{-j\beta z} + V_0^- e^{j\beta z} \)

No surprise. The solutions we got earlier for the ideal case.

Now, back to the more complicated, general case

\[
\frac{d^2 \tilde{V}}{dz^2} - (R' + j\omega L') (G' + j\omega C') \tilde{V}(z) = 0
\]

Let \( \gamma^2 = (R' + j\omega L') (G' + j\omega C') \)

we can write

\[
\frac{d^2 \tilde{V}}{dz^2} - \gamma^2 \tilde{V}(z) = 0
\]

Compare this to the ideal case

\[
\beta = \omega \sqrt{L'C'}
\]

\[
\beta^2 \rightarrow -\gamma^2 \quad \text{thus} \quad -j\beta \rightarrow -\gamma \quad \text{and} \quad j\beta \rightarrow \gamma
\]

These two equations are “formally” the same, except \(-\gamma^2\) is complex.

So, the solutions are \( \tilde{V} = V_0^+ e^{-\gamma z} + V_0^- e^{\gamma z} \)

What kind of waves are they?
\[ \tilde{V} = V_0^+ e^{-\gamma z} + V_0^- e^{\gamma z} \]

What kind of waves are they?

Let \( \gamma = \alpha + j\beta \) (we are doing nothing. Any complex number can be written as this), we have the first solution

\[ \tilde{V}^+ = V_0^+ e^{-\gamma z} = V_0^+ e^{-(\alpha+j\beta)z} = V_0^+ e^{-\alpha z} e^{-j\beta z} \]

What kind of wave is this?

Similarly,
\[ \tilde{I}^+ = I_0^+ e^{-\gamma z} = I_0^+ e^{-(\alpha+j\beta)z} = I_0^+ e^{-\alpha z} e^{-j\beta z} \]

Why do the waves attenuate when there is resistance or shunt leakage? (Why is there no attenuation when the wire is made of a perfect conductor and the medium between them is a perfect insulator)?

Recall that, in circuit theory, reactive versus resistive..., ...

How to express the decaying wave propagating in the \(-z\) direction?
Again, there can be waves going the other way.

\[ \tilde{V} = V_0^+ e^{-\alpha x} e^{-j\beta z} + V_0^- e^{\alpha x} e^{j\beta z} \]

\[ \tilde{I} = I_0^+ e^{-\alpha x} e^{-j\beta z} + I_0^- e^{\alpha x} e^{j\beta z} \]

We were here at the end of class on Tue 9/6/2022.
Again, there can be waves going the other way.

\[
\tilde{V} = V_0^+ e^{-\alpha z} e^{-j\beta z} + V_0^- e^{\alpha z} e^{j\beta z} \quad \tilde{I} = I_0^+ e^{-\alpha z} e^{-j\beta z} + I_0^- e^{\alpha z} e^{j\beta z}
\]

Again, we discuss the most important concept of the first half of the semester:

\[
\tilde{V} = V_0^+ e^{-\gamma z} + V_0^- e^{\gamma z} \quad \frac{d\tilde{V}}{dz} = -\gamma V_0^+ e^{-\gamma z} + \gamma V_0^- e^{\gamma z}
\]
Again, there can be waves going the other way.

\[
\tilde{V} = V_0^+ e^{-\alpha z} e^{-j\beta z} + V_0^- e^{\alpha z} e^{j\beta z} \quad \text{sign} \\
\tilde{I} = I_0^+ e^{-\alpha z} e^{-j\beta z} + I_0^- e^{\alpha z} e^{j\beta z} \quad \text{sign}
\]

Again, we discuss the most important concept of the first half of the semester:

\[
\tilde{V} = V_0^+ e^{-\gamma z} + V_0^- e^{\gamma z} \quad \text{Take derivatives} \quad \frac{d\tilde{V}}{dz} = -\gamma V_0^+ e^{-\gamma z} + \gamma V_0^- e^{\gamma z}
\]

From circuit analysis

\[
\frac{d\tilde{V}}{dz} = \lim_{\Delta z \to 0} \frac{\Delta \tilde{V}}{\Delta z} = -(R' + j\omega L') \tilde{I} = -(R' + j\omega L')I_0^+ e^{-\alpha z} e^{-j\beta z} - (R' + j\omega L')I_0^- e^{\alpha z} e^{j\beta z}
\]
Again, there can be waves going the other way.

\[
\tilde{V} = V_0^+ e^{-\alpha z} e^{-j\beta z} + V_0^- e^{\alpha z} e^{j\beta z} \quad \tilde{I} = I_0^+ e^{-\alpha z} e^{-j\beta z} + I_0^- e^{\alpha z} e^{j\beta z}
\]

Again, we discuss **the most important concept** of the first half of the semester:

\[
\tilde{V} = V_0^+ e^{-\gamma z} + V_0^- e^{\gamma z}
\]

**Take derivatives**

\[
\frac{d\tilde{V}}{dz} = -\gamma V_0^+ e^{-\gamma z} + \gamma V_0^- e^{\gamma z}
\]

From circuit analysis

\[
\lim_{\Delta z \to 0} \frac{\Delta \tilde{V}}{\Delta z} = -(R' + j\omega L') \tilde{I} = -(R' + j\omega L') I_0^+ e^{-\alpha z} e^{-j\beta z} - (R' + j\omega L') I_0^- e^{\alpha z} e^{j\beta z}
\]
Again, there can be waves going the other way.

\[
\widetilde{V} = V_0^+ e^{-\alpha z} e^{-j\beta z} + V_0^- e^{\alpha z} e^{j\beta z} \quad \widetilde{I} = I_0^+ e^{-\alpha z} e^{-j\beta z} + I_0^- e^{\alpha z} e^{j\beta z}
\]

Again, we discuss **the most important concept** of the first half of the semester:

\[
\widetilde{V} = V_0^+ e^{-\gamma z} + V_0^- e^{\gamma z}
\]

Take derivatives

\[
\frac{d\widetilde{V}}{dz} = -\gamma V_0^+ e^{-\gamma z} + \gamma V_0^- e^{\gamma z}
\]

From circuit analysis

\[
\left(\frac{d\widetilde{V}}{dz}\right) = \lim_{\Delta z \to 0} \frac{\Delta \widetilde{V}}{\Delta z} = -(R' + j\omega L') \widetilde{I} = -(R' + j\omega L') I_0^+ e^{-\alpha z} e^{-j\beta z} - (R' + j\omega L') I_0^- e^{\alpha z} e^{j\beta z}
\]

For the identity to hold for all \(z\), we must have the following:

For the \(e^{-\gamma z}\) term,

\[
\gamma V_0^+ = (R' + j\omega L') I_0^+
\]

\[
\implies \frac{V_0^+}{I_0^+} = \frac{R' + j\omega L'}{\gamma} = \frac{R' + j\omega L'}{\sqrt{(R' + j\omega L')(G' + j\omega C')}} = \sqrt{\frac{R' + j\omega L'}{G' + j\omega C'}}
\]
For the $e^{\gamma z}$ term, 

$$\gamma V_0^- = -(R' + j\omega L')I_0^-$$

$$\Rightarrow \frac{V_0^-}{I_0^-} = -\frac{R' + j\omega L'}{\gamma} = -\frac{R' + j\omega L'}{\gamma} \sqrt{(R' + j\omega L')(G' + j\omega C')} = -\frac{R' + j\omega L'}{\sqrt{G' + j\omega C'}}$$

Notice negative signs. Just because of sign convention (see circuit diagram)

Define 

$$Z_0 \equiv \frac{R' + j\omega L'}{\sqrt{G' + j\omega C'}}$$

the characteristic impedance

Complex and explicitly dependent on frequency in the general (lossy) case.
For the wave traveling towards $+z$, 
\[ \tilde{V}^+ = V_0^+ e^{-\gamma z} \]
\[ \tilde{I}^+ = I_0^+ e^{-\gamma z} \]
At any $z$, \[ \frac{\tilde{V}^+(z)}{\tilde{I}^+(z)} = \frac{V_0^+}{I_0^+} = Z_0 \]

For the wave traveling towards $-z$, 
\[ \tilde{V}^- = V^- e^{\gamma z} \]
\[ \tilde{I}^- = I_0^- e^{\gamma z} \]
At any $z$, \[ \frac{\tilde{V}^-(z)}{\tilde{I}^-(z)} = \frac{V_0^-}{I_0^-} = -Z_0 \]
Again, notice this negative sign.

These relations are **separately** held by the two waves in opposite directions.

In general, \( Z_0 \equiv \sqrt{\frac{R' + j\omega L'}{G' + j\omega C'}} \) is complex and **explicitly** dependent on frequency.

Are \( V_0^+ \) and \( I_0^+ \) in phase in the general case?

For the **lossless** transmission line, \( R' = 0 \) and \( G' = 0 \),

\[ Z_0 = \sqrt{\frac{L'}{C'}} \]
Real. No **explicit** frequency dependence.
Take-home messages

• Voltage $v$ and current $i$ follow the same differential equation
• Therefore solutions in same form
• Therefore there is a constant ratio between their amplitudes and there is a constant shift between their phases for harmonic waves going in one direction
• In the phasor form, the complex amplitude ratio is $\pm Z_0$
• Being a voltage/current ratio, $Z_0$ has the dimension of impedance
• In general (lossy case), $Z_0$ is complex and explicitly depends on $\omega$
• In the lossless case, $Z_0$ is real w/o explicit frequency dependence
• $Z_0$ being real means the voltage and the current are in phase. It also means the equivalent circuit for a (semi-)infinitely long transmission line is simply a resistor with a resistance value $Z_0$.

At this point, review textbook up to Section 2-4.
Finish P1 through P6 of HW2.
The cables

Coaxial cable

Ribbon twin lead:
(used to be) used to connect a television receiving antenna to a home television set.

The model

The inductors (and resistors in lossy lines) are on only one side.
Which side is which wire???
The model

A pair of coupled wires

Voltage $\nu$ around the loop!

The inductors (and resistors in lossy lines) are on only one side. Which side is which wire???

$L' = L' \Delta z$

$L'$ is inductance per length.
The cables

Coaxial cable

Ribbon twin lead: (used to be) used to connect a television receiving antenna to a home television set.

\[ Z_0 = 75 \, \Omega \]


[https://www.moonraker.eu/antennas/antenna-hardware/ribbon-feeder/75-m-75-ohm-twin-feeder](https://www.moonraker.eu/antennas/antenna-hardware/ribbon-feeder/75-m-75-ohm-twin-feeder)

**Question:**

How to vary the \( Z_0 \) of the transmission line of a given construction?

Hint: Consider the simple lossless case

\[ Z_0 = \sqrt{\frac{L'}{C'}} \]

and use Table 2-1 in Textbook.

\[ Z_0 = 300 \, \Omega \]

Coaxial cable

Carries f’s up to 1GHz

Power transmission line

11 kV, 18 kV, 23 kV, 23.5 kV output (CLP power)
12.5 kV to 22 kV output (HEC)

132 kV
11 kV

underground cables and overhead lines

power station step up transformer
step down transformer
step down transformer
residential building

https://en.wikipedia.org/wiki/Coaxial_cable
There is no way to tell the difference just by measuring $v$ and $i$.
Energy propagating away vs. energy dissipated
Analogy: laser beam going to infinity or hitting a totally black wall

**Impedance match**

By the way, transmission line (thick line) versus “wire wires” (thin lines)

We want impedance match! (Reasons?)
Now, let’s look at a transmission line with a source and a load.

If $Z_L = Z_0$, impedance matched. All energy delivered to load. Good!

(we can view this from the vantage point of equivalent circuits)

Now, we just focused on the load. Will talk about the line and generator later.
Now, let’s look at a transmission line with a source and a load.

If $Z_L = Z_0$, impedance matched. All energy delivered to load. Good!

What if $Z_L \neq Z_0$?

The load says $\frac{\tilde{V}(0)}{\tilde{I}(0)} = \frac{\tilde{V}_L}{\tilde{I}_L} = Z_L$

If there were only the incident wave, $\frac{\tilde{V}^+(0)}{\tilde{I}^+(0)} = \frac{V_0^+}{I_0^+} = Z_0$

Something has to happen to resolve this “conflict.” That something is reflection.

$\tilde{V}_L = \tilde{V}(z=0) = V_0^+ + V_0^-$

$\tilde{I}_L = \tilde{I}(z=0) = I_0^+ + I_0^- = \frac{V_0^+}{Z_0} - \frac{V_0^-}{Z_0}$

Sign due to convention

Both waves separately follow $V_0^\pm = \pm I_0^\pm Z_0$
Now, let’s look at a transmission line with a source and a load.

If $Z_L = Z_0$, impedance matched.
All energy delivered to load. Good!
(we can view this from the vantage point of equivalent circuits)

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Something has to happen to resolve this “conflict.” That something is reflection.

Sign due to convention

Both waves separately follow $V_0^\pm = \pm I_0^\pm Z_0$

By definition, $Z_L = \frac{\tilde{V}_L}{\tilde{I}_L} = \left(\frac{V_0^+ + V_0^-}{V_0^+ - V_0^-}\right) Z_0$

Solve it and we have $V_0^- = \frac{Z_L - Z_0}{Z_L + Z_0} V_0^+$
If $Z_L \neq Z_0$, there has to be a reflected wave.

$$V_0^- = \frac{Z_L - Z_0}{Z_L + Z_0} V_0^+$$

The load does not get all energy carried by the incident wave.

Where does the rest of the energy go?
If $Z_L \neq Z_0$, there has to be a reflected wave.

$$V_0^- = \frac{Z_L - Z_0}{Z_L + Z_0} V_0^+$$

The load does not get all energy carried by the incident wave.

**Where does the rest of the energy go?**

Consider analogy: laser beam hitting wall not totally black/dark.

Define the **voltage reflection coefficient**

$$\Gamma \equiv \frac{V_0^-}{V_0^+} = \frac{Z_L - Z_0}{Z_L + Z_0} = \frac{Z_L - 1}{Z_L + 1}$$

$$\frac{Z_L}{Z_0} = \frac{1 + \Gamma}{1 - \Gamma}$$
If $Z_L \neq Z_0$, there has to be a reflected wave.

$$V_0^- = \frac{Z_L - Z_0}{Z_L + Z_0} V_0^+$$

The load does not get all energy carried by the incident wave.

**Where does the rest of the energy go?**

Consider analogy: laser beam hitting wall not totally black/dark.

Define the **voltage reflection coefficient** $\Gamma \equiv \frac{V_0^-}{V_0^+} = \frac{Z_L - Z_0}{Z_L + Z_0} = \frac{Z_L}{Z_0} \frac{-1}{1 - \frac{Z_L}{Z_0}}$

- One-to-one mapping between $\Gamma$ and $\frac{Z_L}{Z_0}$
- The ratio $\frac{Z_L}{Z_0}$ more important than $Z_L$ itself
Therefore we define the **normalized load impedance** \( Z_L = \frac{Z_L}{Z_0} \)

Thus,

\[
\Gamma \equiv \frac{V_0^-}{V_0^+} = \frac{Z_L - 1}{Z_L + 1} \quad \text{and} \quad Z_L = \frac{1 + \Gamma}{1 - \Gamma}
\]

Notice the one-to-one mapping. This is very important!

For the current

\[
\frac{I_0^-}{I_0^+} = -\frac{V_0^-}{V_0^+} = -\Gamma
\]

Where does the negative sign come from?

\[
Z_0 = \sqrt{\frac{L'}{C'}} \text{ is real for a lossless line, but} \n\]

\[
Z_0 \equiv \sqrt{\frac{R' + j\omega L'}{G' + j\omega C'}} \text{ is complex in general.}
\]

\( Z_L \) is complex in general. Thus, \( \Gamma \) is complex in general.

Read the textbook: Section 2-6 overview, 2-6.1

We finished this lecture on Thu 9/8/2022
Project

Circuit simulations to transition you from lumped element-based circuit theory

Part 1

Generator: 1 V step, rise time = 0.1 ns. Internal impedance 50 Ω.

Plot the two voltages $V_1$ and $V_2$ for the above two cases.

Hint: You may make mistakes. Do a sanity check by a “back of an envelope” analysis. At the very least, find out the steady state. Does the simulation give you more or less what you expect?

Ongoing project. Stay tuned for next steps.
ADS Tutorial

Tue 9/13, in class. Bring your lap top.

Follow instructions on Canvas to get access to ADS before the class.

If you need more info or miss the class, see tutorial document on course website. In the document, there is a link to a YouTube video.

Contact TAs for help if needed: GTA Tsotne Kvelashvili tkvelash@utk.edu and UGTA Graham Travis tgraham9@vols.utk.edu