Quiz 2

Find the time-domain sinusoidal functions corresponding to the following phasors. (z is position. Y_0^+ and Y_0^- are real and positive.) Make sure you follow conventions adopted in this course.

$$\tilde{Y}_1(z) = Y_0^+ e^{-j\beta z}$$

$$\tilde{Y}_2(z) = Y_0^- e^{j\beta z}$$

$$\tilde{Y}(z) = -2jY_0^+ \sin(\beta z)$$

Find the phasor for the following function of position z and time t. (z is position. V_0^+ is real and positive)

$$v(z,t) = 2V_0^+ \cos(\beta z)\cos(\omega t)$$

Quiz 2

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$$\tilde{Y}_1(z) = Y_0^+ e^{-j\beta z} \quad \Rightarrow \quad y_1(z,t) = Y_0^+ \cos(\omega t - \beta z)$$

$$\tilde{Y}_2(z) = Y_0^- e^{j\beta z} \quad \Rightarrow \quad y_2(z,t) = Y_0^- \cos(\omega t + \beta z)$$

$$\tilde{Y}(z) = -2\enclose{circle}{j}Y_0^+ \sin(\beta z)$$

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$$\tilde{Y}(z) = -2\textcircled{j} Y_0^+ \sin(\beta z) \Rightarrow \quad y(z,t) = \mathrm{Re}[-2jY_0^+ \sin(\beta z) e^{j\omega t}] = 2Y_0^+ \sin(\beta z)\sin(\omega t)$$

This j means phase $\pi/2$

Find the phasor for the following function of position z and time t. (z is position. V_0^+ is real and positive)

$$v(z,t) = 2V_0^+ \cos(\beta z)\cos(\omega t)$$

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Find the phasor for the following function of position z and time t. (z is position. V_0^+ is real and positive)

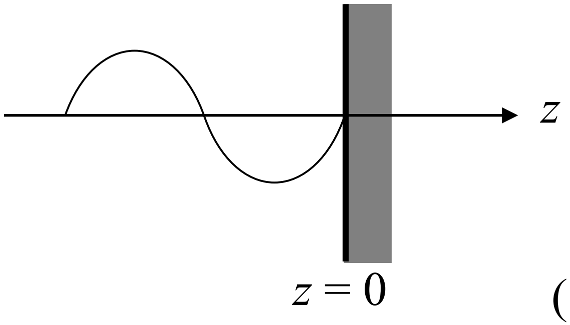
$$v(z,t) = 2V_0^+ \cos(\beta z)\cos(\omega t)$$

$$\Rightarrow \quad \tilde{V}(z) = 2V_0^+ \cos(\beta z)$$

Note: $v(z,t) = 2V_0^+ \cos(\beta z)\cos(\omega t) \neq \tilde{V}(z) = 2V_0^+ \cos(\beta z)$

Standing Wave

Interference between the incident & reflected waves → Standing wave



A string with one end fixed on a wall

Incident: $y_1(z,t)=Y_0^+\cos(\omega t-\beta z)$

$$\widetilde{Y}_1(z)=Y_0^+e^{-j\beta z}$$

(Set the incident wave's phase to be 0, i.e., Y_0^+ real & positive.)

Reflected: $y_2(z,t)=|Y_0^-|\cos(\omega t+\beta z+\phi)$

$$\widetilde{Y}_2(z)=Y_0^-e^{j\beta z}\text{, where } Y_0^-=|Y_0^-|e^{j\phi}=|Y_0^-|\angle\phi$$

The total displacement

$$\widetilde{Y}(z)=\widetilde{Y}_1(z)+\widetilde{Y}_2(z)=Y_0^+e^{-j\beta z}+Y_0^-e^{j\beta z}$$

We must have $\widetilde{Y}(0)=0 \quad \Rightarrow \quad Y_0^++Y_0^-=0$ i.e. $Y_0^-=-Y_0^+$

$$\boxed{\widetilde{Y}(z)=Y_0^+(e^{-j\beta z}-e^{j\beta z})}$$

$$\widetilde{Y}(z) = Y_0^+ (e^{-j\beta z} - e^{j\beta z})$$

Recall that $e^{j\theta} - e^{-j\theta} = 2j\sin\theta \quad \Rightarrow \quad \widetilde{Y}(z) = -2jY_0^+ \sin(\beta z)$

$$y(z,t) = \mathrm{Re}[-2jY_0^+ \sin(\beta z)] = 2Y_0^+ \sin(\beta z)\underline{\sin(\omega t)}$$

Why sin?

$$\widetilde{Y}(z) = Y_0^+(e^{-j\beta z} - e^{j\beta z})$$

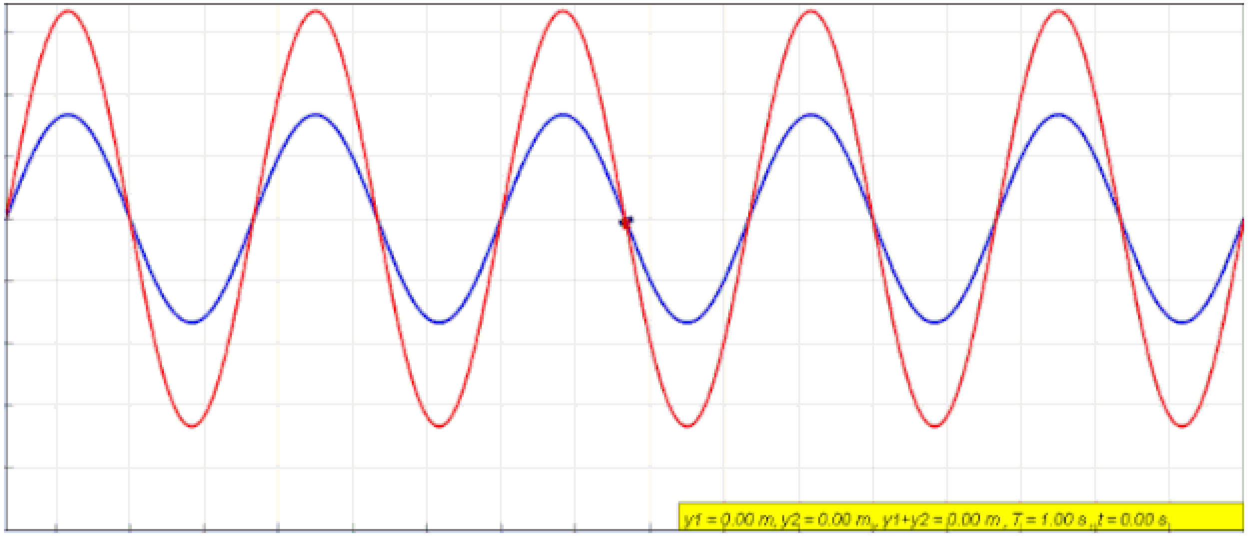
Recall that $e^{j\theta} - e^{-j\theta} = 2j\sin\theta \quad \Rightarrow \quad \widetilde{Y}(z) = -2jY_0^+\sin(\beta z)$

$$y(z,t) = \mathrm{Re}[-2jY_0^+\sin(\beta z)] = 2Y_0^+\sin(\beta z)\underline{\sin(\omega t)}$$

Why sin?

See Wikipedia Standing Wave animation to get visual picture:

Harmonic oscillation at each *z*, with amplitude following sin(βz)

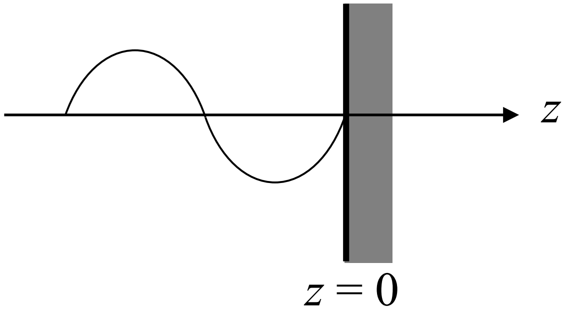


https://en.wikipedia.org/wiki/Standing_wave

This is Homework 1 Problem 4.

Here we just used the phasor tool to do it the easy way.

Review Homework 1 Problem 4 (and also Quiz 2), relate the physical quantities to the phasors.



$$\tilde{Y}(z) = Y_0^+(e^{-j\beta z} - e^{j\beta z}) \quad \Rightarrow \quad \boxed{\tilde{Y}(z) = -2jY_0^+\sin(\beta z)}$$

$$y(z,t) = \mathrm{Re}[-2jY_0^+\sin(\beta z)] = 2Y_0^+\sin(\beta z)\sin(\omega t)$$

Similarly, a shorted transmission line:

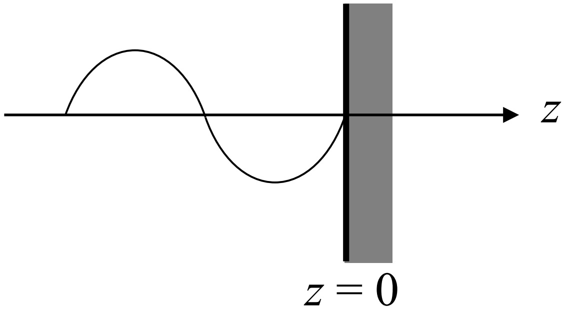
By definition of “short circuit”, $\tilde{V}(0) = 0 \Rightarrow V_0^+ + V_0^- = 0$

$$V_0^- = -V_0^+$$

$$\boxed{\tilde{V}(z) = -2jV_0^+\sin(\beta z)}$$

$$v(z,t) = \mathrm{Re}[-2jV_0^+\sin(\beta z)] = 2V_0^+\sin(\beta z)\sin(\omega t)$$

Like a mirror. What property of a mirror makes it a mirror?



$$\tilde{Y}(z) = Y_0^+(e^{-j\beta z} - e^{j\beta z}) \quad \Rightarrow \quad \boxed{\tilde{Y}(z) = -2jY_0^+ \sin(\beta z)}$$

$$y(z,t) = \mathrm{Re}[-2jY_0^+ \sin(\beta z)] = 2Y_0^+ \sin(\beta z)\sin(\omega t)$$

$$\Gamma = \frac{Z_L - Z_0}{Z_L + Z_0} = -1$$

Similarly, shorted transmission line:

By definition of "short circuit", $\tilde{V}(0) = 0 \;\Rightarrow\; V_0^+ + V_0^- = 0$

$$V_0^- = -V_0^+ \qquad\qquad \Gamma = -1$$

$$\boxed{\tilde{V}(z) = -2jV_0^+ \sin(\beta z)}$$

$$v(z,t) = \mathrm{Re}[-2jV_0^+ \sin(\beta z)] = 2V_0^+ \sin(\beta z)\sin(\omega t)$$

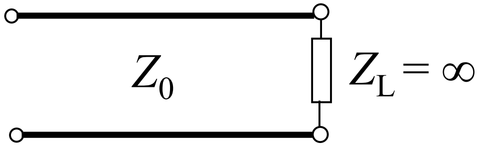
Like a mirror. What property of a mirror makes it a mirror?

But, $I_0^- = I_0^+ \qquad\qquad \dfrac{I_0^+}{I_0^-} = -\Gamma = 1$

Find $\tilde{I}(z)$ and $i(z, t)$ on your own.

What if the transmission line is terminated in open circuit?

Note: open ended ≠ open circuit for high frequencies!
(You will see how to make an open circuit later.)



$$V_0^- = V_0^+ \qquad \Gamma = \frac{Z_L - Z_0}{Z_L + Z_0} = 1$$

$$\tilde{V}(z) = V_0^+ (e^{-j\beta z} + e^{j\beta z}) = 2V_0^+ \cos(\beta z)$$

Find $v(z, t)$ on your own.

$I_0^- = -I_0^+$ (since total current is 0, by definition of open circuit)

Find $\tilde{I}(z)$ and $i(z, t)$ on your own.

In all the above examples, $\Gamma = \pm 1$. Completely reflected.

At very high frequencies, we often are only able to measure the amplitude or power (∝ amplitude squared), but not the instantaneous values or the waveform.

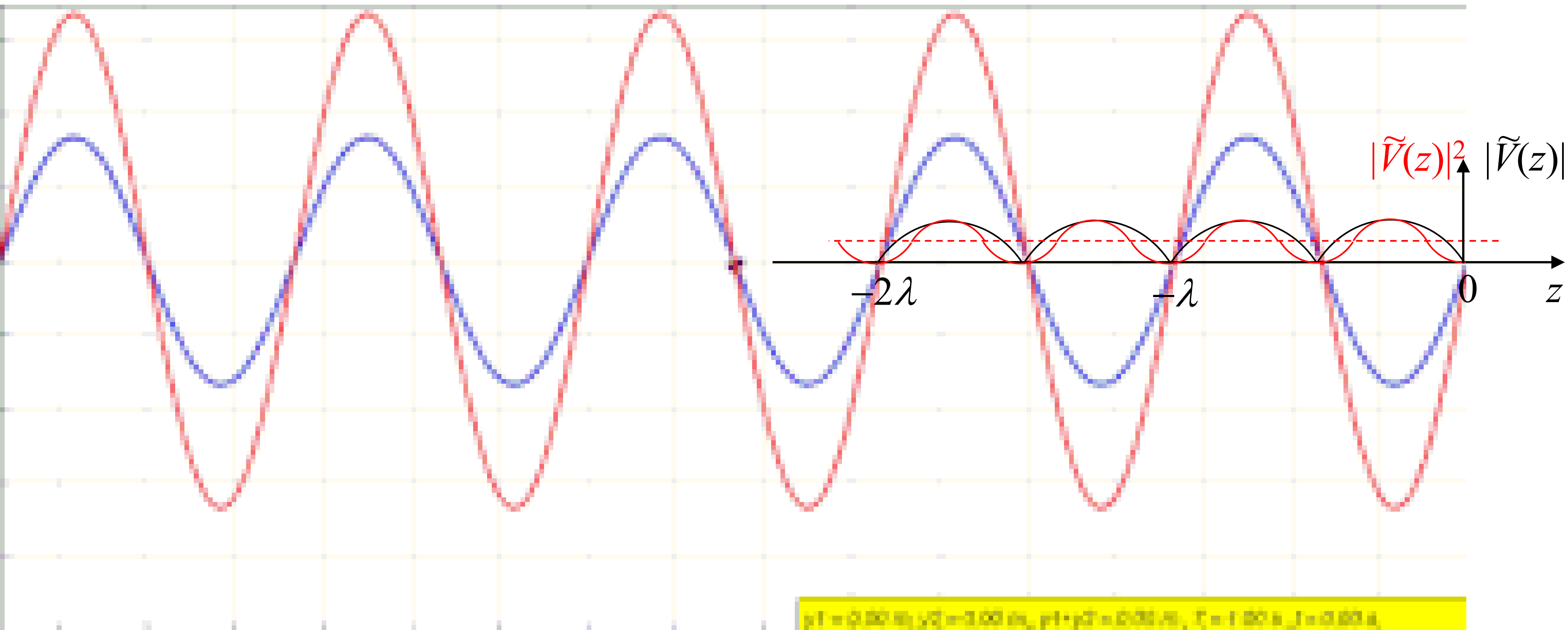
The following example is for the short circuit. The open circuit is similar (just with a shift of origin).

The amplitude of the voltage wave $v(z, t)$ at position z is

$$|\tilde{V}(z)| = \sqrt{\tilde{V}(z)\tilde{V}^*(z)}$$

"Local amplitude" – see the sanding wave animation again

The "complex amplitude" containing the phase



At very high frequencies, we often can only measure the amplitude or power ($\propto$ amplitude squared), but not the instantaneous values or the waveform.

The following example is for the short circuit. The open circuit is similar (just with a shift of origin).

The amplitude of the voltage wave $v(z, t)$ at position z is

$$|\boxed{\tilde{V}(z)}| = \sqrt{\tilde{V}(z)\tilde{V}^*(z)}$$

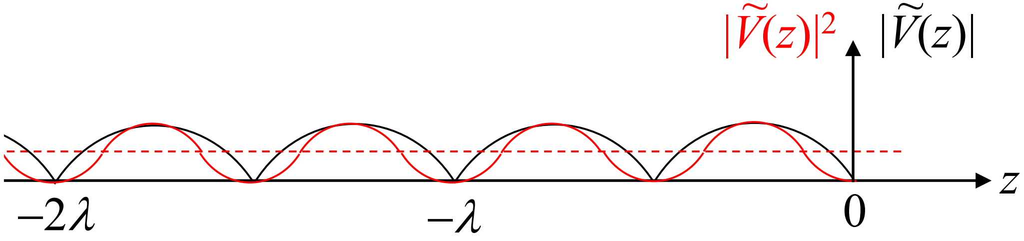
"Local amplitude" – see the sanding wave animation again

The "complex amplitude" containing the phase

$$\tilde{V}(z) = -2jV_0^+ \sin(\beta z)$$

$$\Rightarrow \quad |\tilde{V}(z)| = |-2\boxed{j}V_0^+ \sin(\beta z)|$$

$$= 2|V_0^+||\sin(\beta z)|$$



$$|\tilde{V}(z)|^2 = 4|V_0^+|^2 \sin^2(\beta z) = 2|V_0^+|^2 [1 - 2\cos(2\beta z)]$$

Question: What's the spatial period of the standing wave?

At very high frequencies, we often can only measure the amplitude or power (∝ amplitude squared), but not the instantaneous values or the waveform.

The following example is for the short circuit. The open circuit is similar (just with a shift of origin).

The amplitude of the voltage wave $v(z, t)$ at position z is

$$|\tilde{V}(z)| = \sqrt{\tilde{V}(z)\tilde{V}^*(z)}$$

"Local amplitude" – see the sanding wave animation again

The "complex amplitude" containing the phase

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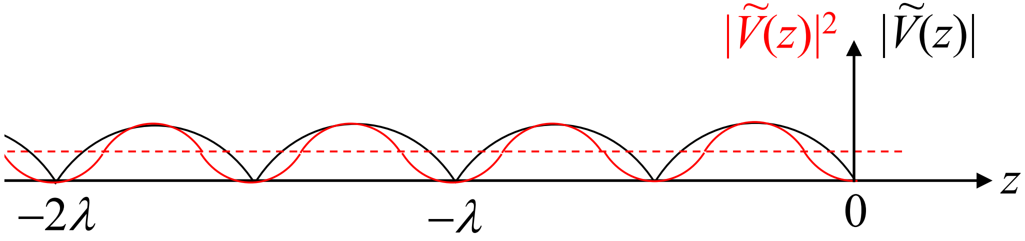
$$\Rightarrow \quad |\tilde{V}(z)| = |-2jV_0^+ \sin(\beta z)|$$

$$= 2|V_0^+||\sin(\beta z)|$$



$$|\tilde{V}(z)|^2 = 4|V_0^+|^2 \sin^2(\beta z) = 2|V_0^+|^2 [1-2\cos(2\beta z)]$$

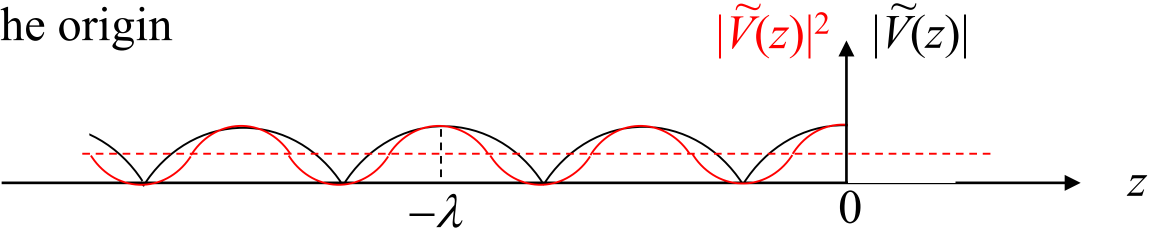
Question: What's the spatial period of the standing wave?

The open circuit: Just a shift of the origin

$$|\tilde{V}(z)| = 2|V_0^+||\cos(\beta z)|$$

$$|\tilde{V}(z)|^2 = 4|V_0^+|^2 \cos^2(\beta z)$$

$$= 2|V_0^+|^2 [1+2\cos(2\beta z)]$$



For both the short circuit (SC) and open circuit (OC),

$$\boxed{|\tilde{V}(z)|_{\max} = 2|V_0^+|}$$

Constructive

$$\boxed{|\tilde{V}(z)|_{\min} = 0}$$

Destructive

$$\boxed{|\Gamma| = 1}$$

Complete reflection. Completely a standing wave.

There are cases where $|\Gamma| = 1$ but $\Gamma \neq \pm 1$.
Also complete reflection. We'll talk about those cases later.

What if $|\Gamma| \neq 1$?

Partially standing, partially traveling.

Now, let's look at the maxima and minima of this combination of a standing wave and a traveling wave.
Recall that, in general, Γ is a complex number:

$$\Gamma = |\Gamma| e^{j\theta_r}$$

$$\widetilde{V}(z)=\underbrace{V_0^+e^{-j\beta z}}_{\text{Incident}}+\underbrace{V_0^-e^{j\beta z}}_{\text{Reflected}}=V_0^+e^{-j\beta z}+\Gamma V_0^+e^{j\beta z}=V_0^+e^{-j\beta z}+|\,\Gamma\,|\,e^{j\theta_r}V_0^+e^{j\beta z}$$

$$\Gamma=\left|\Gamma\right|e^{j\theta_r}$$

$$\tilde{V}(z) = \underbrace{V_0^+ e^{-j\beta z}}_{\text{Incident}} + \underbrace{V_0^- e^{j\beta z}}_{\text{Reflected}} = V_0^+ e^{-j\beta z} + \Gamma V_0^+ e^{j\beta z} = V_0^+ e^{-j\beta z} + |\Gamma| e^{j\theta_r} V_0^+ e^{j\beta z}$$

$$\Gamma = |\Gamma| e^{j\theta_r}$$

$$|\tilde{V}(z)| = \sqrt{\tilde{V}(z) \; \tilde{V}^*(z)}$$

$$= \sqrt{V_0^+ (e^{-j\beta z} + |\Gamma| e^{j\theta_r} e^{j\beta z}) \; (V_0^+)^* (e^{j\beta z} + |\Gamma| e^{-j\theta_r} e^{-j\beta z})}$$

Notice that $V_0^+ (V_0^+)^* = |V_0^+|^2$

V_0^+ is a complex amplitude

$$\therefore |\tilde{V}(z)| = |V_0^+| \sqrt{(\cdots)(\cdots)}$$

$$\tilde{V}(z) = \underset{\text{Incident}}{\boxed{V_0^+ e^{-j\beta z}}} + \underset{\text{Reflected}}{\boxed{V_0^- e^{j\beta z}}} = V_0^+ e^{-j\beta z} + \Gamma V_0^+ e^{j\beta z} = V_0^+ e^{-j\beta z} + |\Gamma| e^{j\theta_r} V_0^+ e^{j\beta z}$$

$$\Gamma = |\Gamma| e^{j\theta_r}$$

$$|\tilde{V}(z)| = \sqrt{\tilde{V}(z) \quad \tilde{V}^*(z)}$$

$$= \sqrt{V_0^+ (e^{-j\beta z} + |\Gamma| e^{j\theta_r} e^{j\beta z}) \quad (V_0^+)^* (e^{j\beta z} + |\Gamma| e^{-j\theta_r} e^{-j\beta z})}$$

Notice that $V_0^+ (V_0^+)^* = |V_0^+|^2$

V_0^+ is a complex amplitude

$$\therefore |\tilde{V}(z)| = |V_0^+| \sqrt{(\cdots)(\cdots)}$$

$$= |V_0^+| \sqrt{1 + |\Gamma|^2 + \underbrace{2|\Gamma| \cos(2\beta z + \theta_r)}_{\text{Interference term}}}$$

$$\tilde{V}(z) = \underbrace{V_0^+ e^{-j\beta z}}_{\text{Incident}} + \underbrace{V_0^- e^{j\beta z}}_{\text{Reflected}} = V_0^+ e^{-j\beta z} + \Gamma V_0^+ e^{j\beta z} = V_0^+ e^{-j\beta z} + |\Gamma| e^{j\theta_r} V_0^+ e^{j\beta z}$$

$$\Gamma = |\Gamma| e^{j\theta_r}$$

$$|\tilde{V}(z)| = \sqrt{\tilde{V}(z)\ \tilde{V}^*(z)}$$

$$= \sqrt{V_0^+\left(e^{-j\beta z} + |\Gamma| e^{j\theta_r} e^{j\beta z}\right)\ (V_0^+)^*\left(e^{j\beta z} + |\Gamma| e^{-j\theta_r} e^{-j\beta z}\right)}$$

Notice that $V_0^+ (V_0^+)^* = |V_0^+|^2$

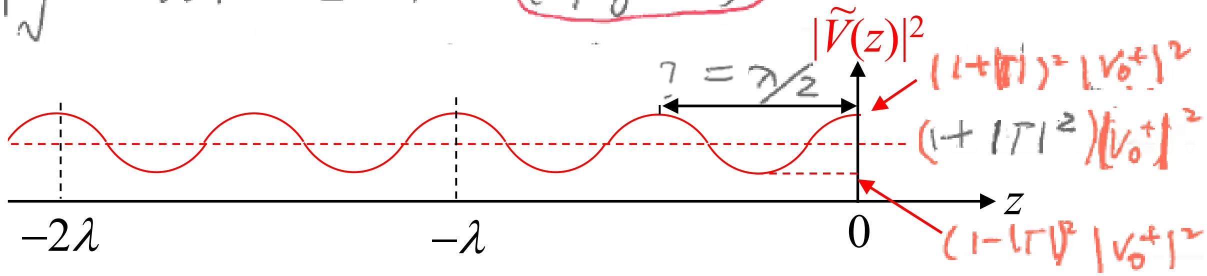
V_0^+ is a complex amplitude

$$\therefore |\tilde{V}(z)| = |V_0^+| \sqrt{(\cdots)(\cdots)}$$

$$= |V_0^+| \sqrt{1 + |\Gamma|^2 + \underbrace{2|\Gamma| \cos(2\beta z + \theta_r)}_{\text{Interference term}}}$$

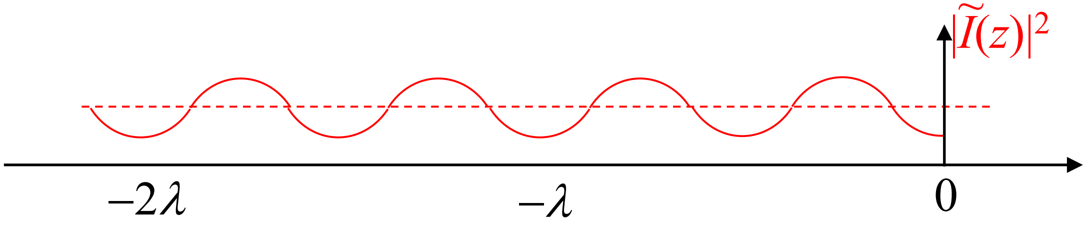
Similarly, $|\tilde{I}(z)| = |I_0^+| \sqrt{1 + |\Gamma|^2 - 2|\Gamma| \cos(2\beta z + \theta_r)}$

It's more convenient to plot $|\tilde{V}(z)|^2$ and $|\tilde{I}(z)|^2$ than the amplitudes.



$$2\beta ? = 2\pi$$

$$\therefore \boxed{? = \frac{2\pi}{2\beta} = \frac{1}{2}\lambda}$$

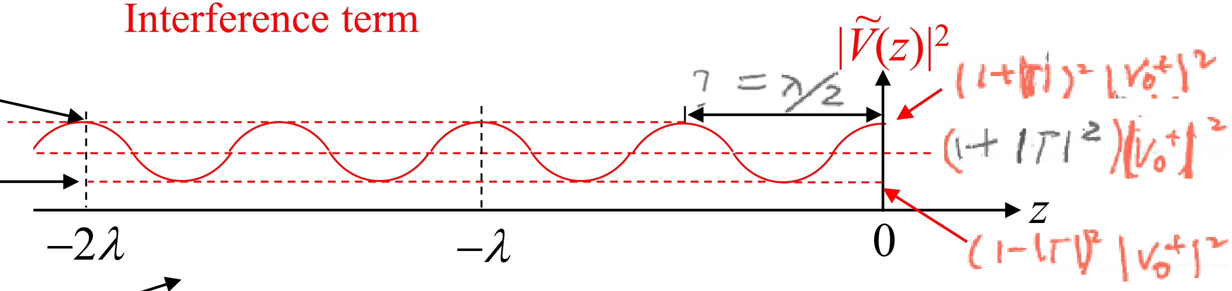


$$|\tilde{V}(z)| = |V_0^+|\sqrt{(\cdots)(\cdots)}$$

$$= |V_0^+|\sqrt{1+|\Gamma|^2 + \underbrace{2|\Gamma|\cos(2\beta z+\theta_r)}_{\text{Interference term}}}$$

Constructive interference $|\tilde{V}|^2_{max}$

Destructive interference $|\tilde{V}|^2_{min}$

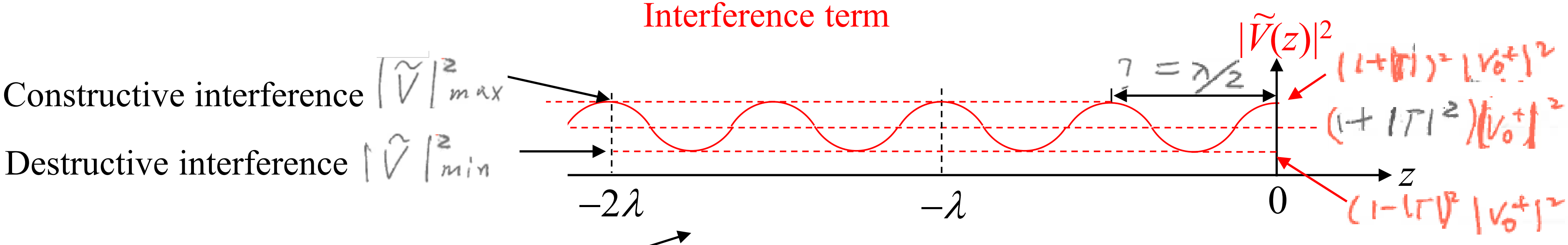


Pay attention to the max, min, and average values

In this plot, we have assumed a special case $\theta_r = 0$.
Can you think of a kind of load that leads to $\theta_r = 0$?
Question: In general, what's the condition for $\theta_r = 0$?

$$|\tilde{V}(z)| = |V_0^+| \sqrt{(\cdots)(\cdots)}$$

$$= |V_0^+| \sqrt{1 + |\Gamma|^2 + \underbrace{2|\Gamma| \cos(2\beta z + \theta_r)}_{\text{Interference term}}}$$



Pay attention to the max, min, and average values

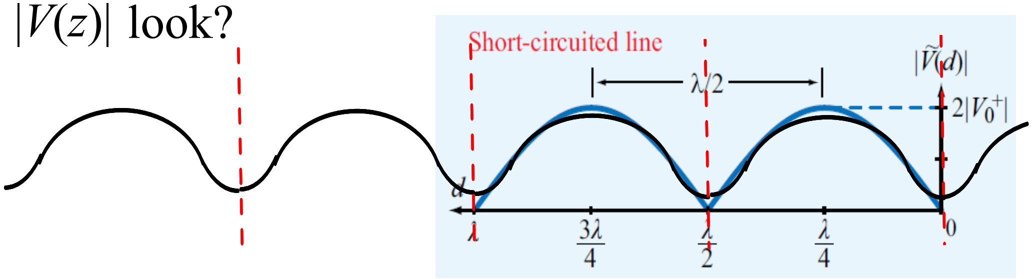
In this plot, we have assumed a special case $\theta_r = 0$.
Can you think of a kind of load that leads to $\theta_r = 0$?
Question: In general, what's the condition for $\theta_r = 0$?

$$\Gamma \equiv \frac{V_0^-}{V_0^+} = \frac{Z_L - Z_0}{Z_L + Z_0}$$

We stated that it's more convenient to plot the amplitudes squared than the amplitudes themselves.

But how does the plot of $|\widetilde{V}(z)|$ look?

It looks like this:



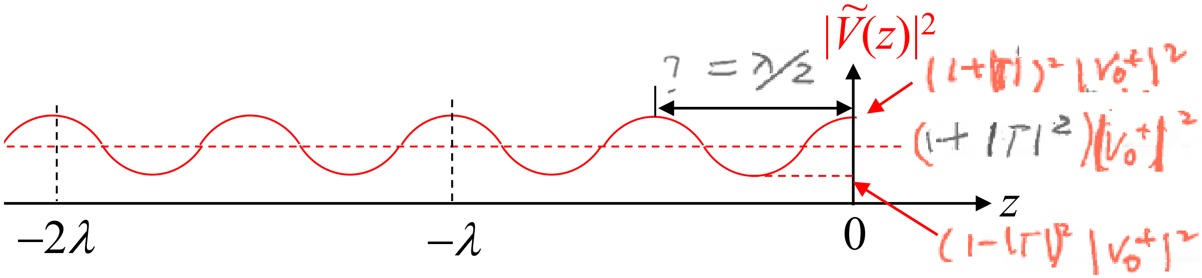
(overlaying the curve for the short circuit case for comparison)

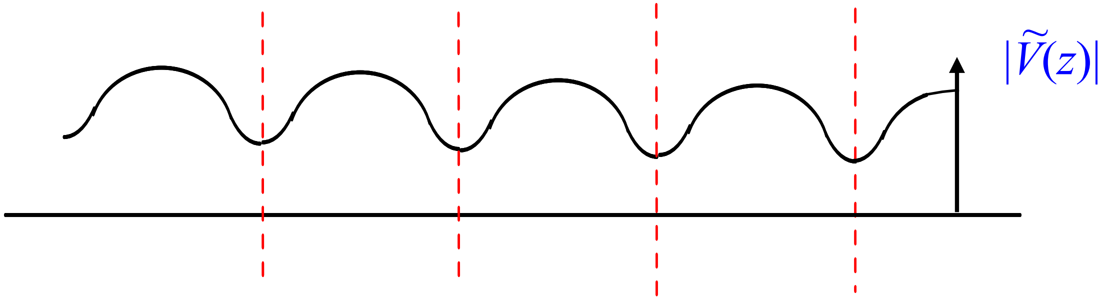
$$|\widetilde{V}(z)| = |V_0^+|\sqrt{(\cdots)(\cdots)}$$

$$= |V_0^+|\sqrt{1+|\Gamma|^2 + \underbrace{2|\Gamma|\cos(2\beta z+\theta_r)}_{\text{Interference term}}}$$

Work out the max and min values of $|\widetilde{V}(z)|$.

Notice the important difference between its shape and that of $|\widetilde{V}(z)|^2$.





$$|\tilde{V}|_{max} = |V_0^+|\sqrt{1+|\Gamma|^2+2|\Gamma|} = |V_0^+|(1+|\Gamma|)$$

Constructive, reflection added to incident.

$$|\tilde{V}|_{min} = |V_0^+|\sqrt{1+|\Gamma|^2-2|\Gamma|} = |V_0^+|(1-|\Gamma|)$$

Destructive, reflection subtracted from incident.

Now we define the voltage standing wave ratio (VSWR), or simply standing wave ratio (SWR)

$$\boxed{S = \frac{|\tilde{V}|_{max}}{|\tilde{V}|_{min}} = \frac{1+|\Gamma|}{1-|\Gamma|}}$$

Special (extreme) cases:

$|\Gamma| = 1$, $S = \infty$ ⇨ All standing wave. $|\tilde{V}|_{min} = 0$

(Recall short & open. Other such cases to be discussed)

$|\Gamma| = 0$, $S = 1$ ⇨ All traveling wave. No reflection.

(What's the condition for this? How does the plot look?)

Slotted line

A tool to measure impedance. See in the textbook, Fig. 2-16 (pp. 71 in 8/E, pp. 74 in 7/E, pp. 73 in 6/E, or pp. 60 in 5/E). Based on the one-to-one mapping between z_L and Γ.

The detector measures the local field (proportional to voltage) as a function of longitudinal position z.

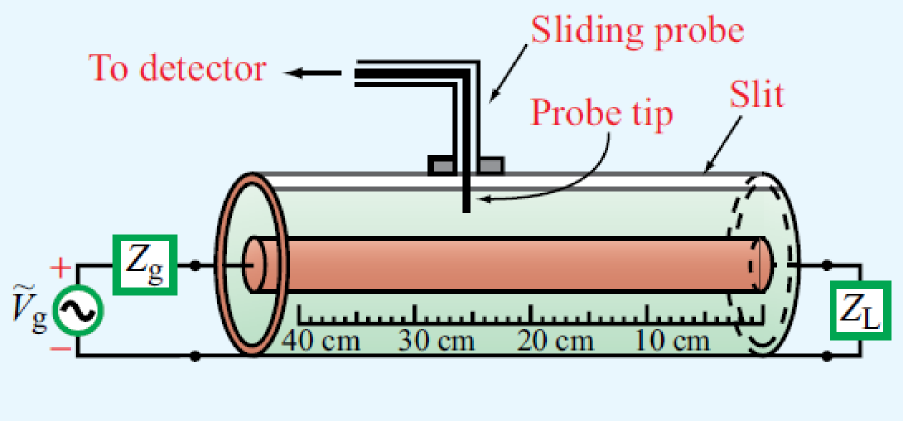
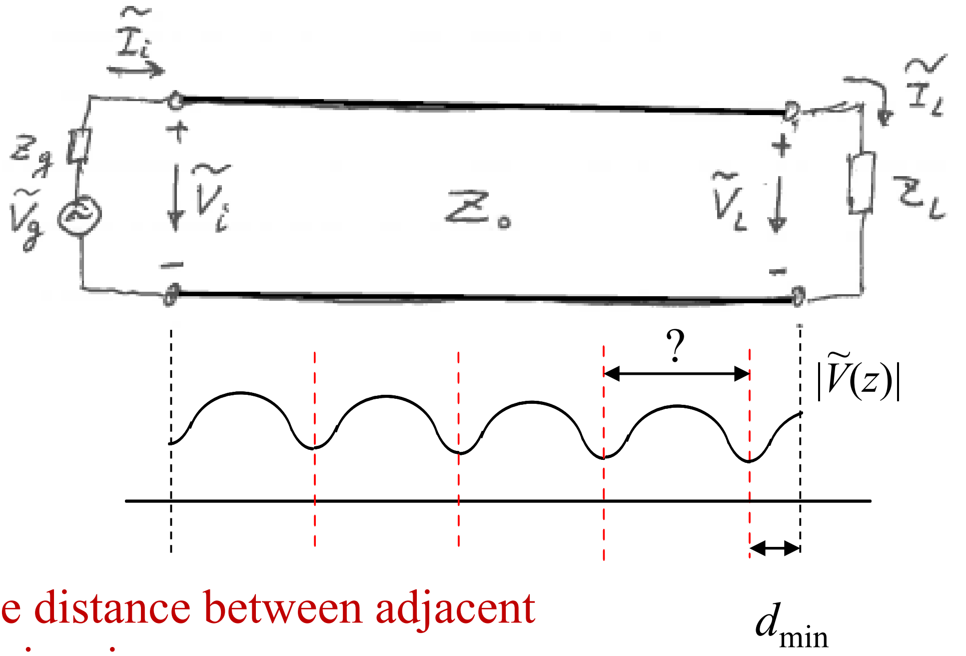


Figure 2-16 Slotted coaxial line (Example 2-6).



Sliding the detector, you find the voltage (amplitude or ac voltage) maxima and minima.

The distance between adjacent minima is ______

Review/preview textbook Section 2-6.

We stopped here on Thu 9/15/2022

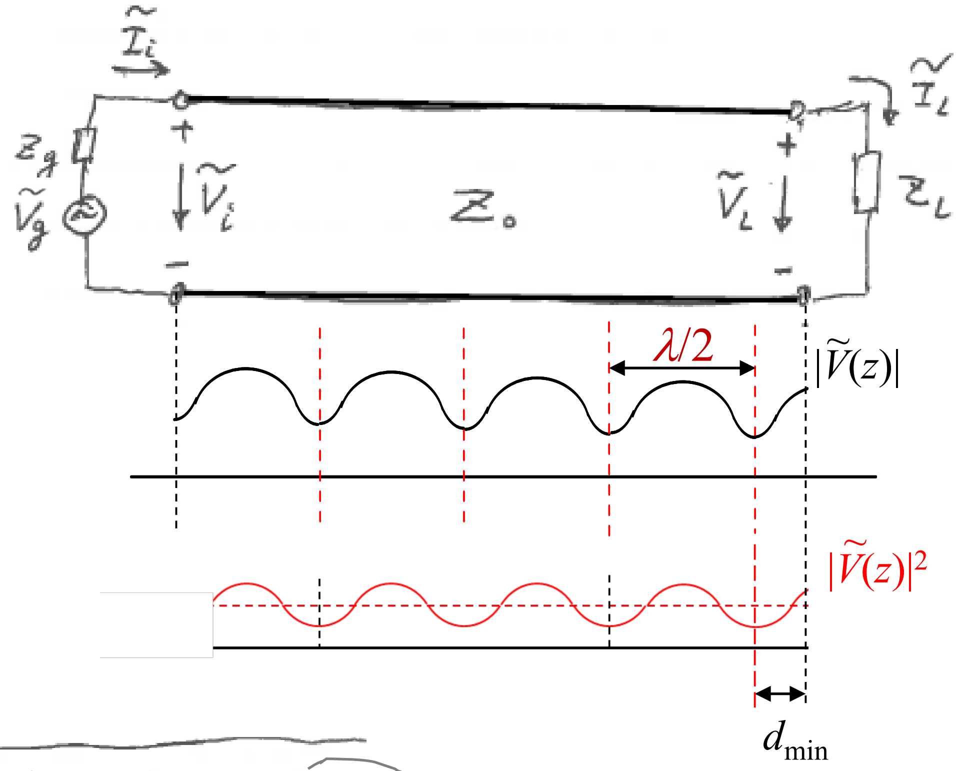
Slotted line

A tool to measure impedance. See in the textbook, Fig. 2-16 (pp. 71 in 8/E, pp. 74 in 7/E, pp. 73 in 6/E, or pp. 60 in 5/E). Based on the one-to-one mapping between z_L and Γ.

The detector measures the local field (proportional to voltage) as a function of longitudinal position z.

Sliding the detector, you find the voltage maxima and minima.

The distance between adjacent minima is $\lambda/2$.



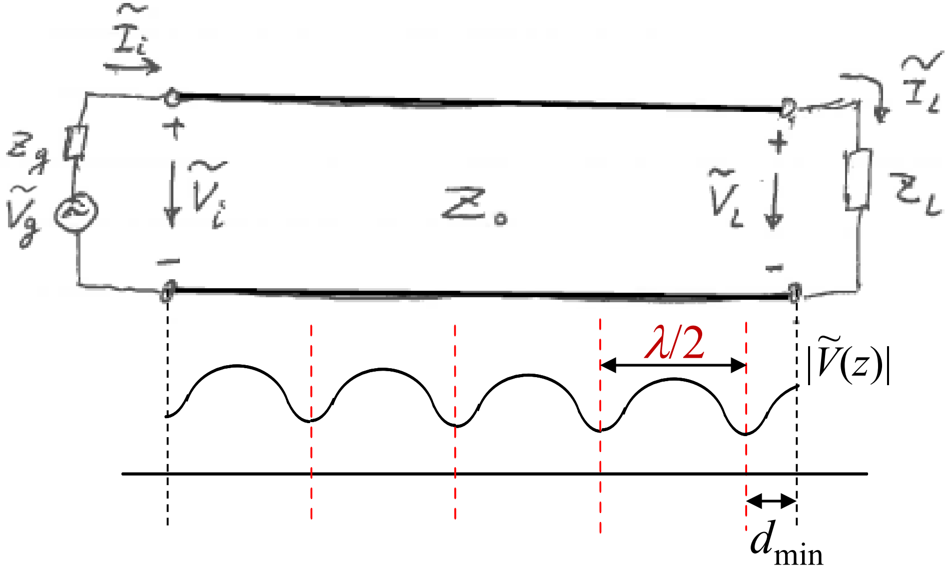
$$|\tilde{V}(z)| = |V_0^+|\sqrt{1+|\Gamma|^2 + \underbrace{2|\Gamma|\cos(2\beta z+\theta_r)}_{\text{Interference term}}}$$

corresponds to d_{min}

Slotted line

A tool to measure impedance. See in the textbook, Fig. 2-16 (pp. 71 in 8/E, pp. 74 in 7/E, pp. 73 in 6/E, or pp. 60 in 5/E). Based on the one-to-one mapping between z_L and Γ.

The detector measures the local field (proportional to voltage) as a function of longitudinal position z.



Sliding the detector, you find the voltage maxima and minima.

The distance between adjacent minima is $\lambda/2$.

You also get the max/min ratio

$$S = \frac{|\tilde{V}|_{max}}{|\tilde{V}|_{min}} = \frac{1+|\Gamma|}{1-|\Gamma|}$$

(You only care about the ratio, not the actual values.)

Solving $S = \frac{1+|\Gamma|}{1-|\Gamma|}$, you get $|\Gamma|$.

But this is not Γ yet!

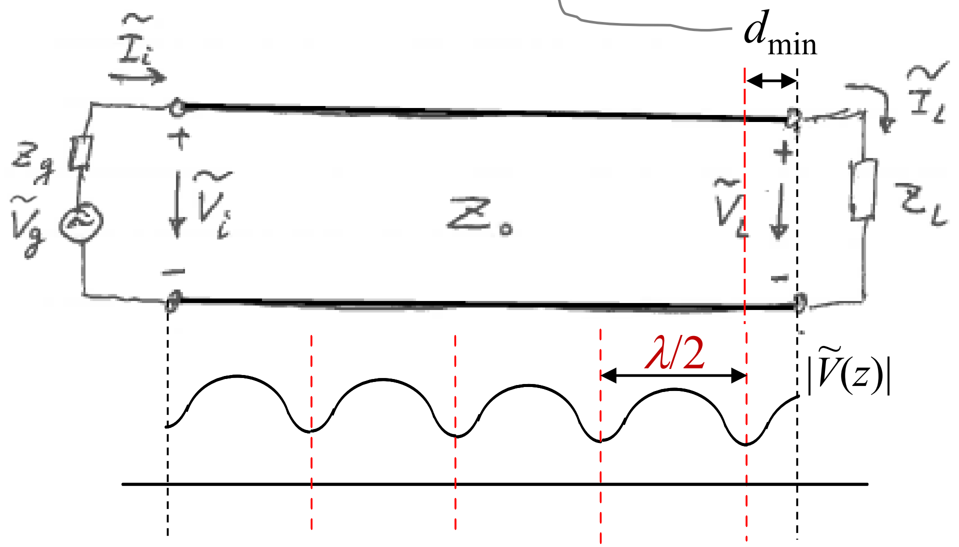
The hope is: If you know Γ, you get z_L using the one-to-one mapping between the two. (Recall that.) You know Z_0, thus you can find Z_L.

The hope is: If you know Γ, you get z_L using the one-to-one mapping between the two. (Recall that.) You know Z_0, thus you can find Z_L.

$$\Gamma = |\Gamma| e^{j\theta_r}$$

We already know $|\Gamma|$. Just need to find θ_r.

$$|\tilde{V}(z)| = |V_0^+| \sqrt{1 + |\Gamma|^2 + \underline{2|\Gamma|\cos(2\beta z + \theta_r)}}$$



We know $\frac{\lambda}{2} \xrightarrow{\beta = \frac{2\pi}{\lambda}} \beta$

We know $z_{min} = -d_{min}$

$$2\beta z_{min} + \theta_r = \boxed{-\pi}$$

$$-2\beta d_{min} + \theta_r = -\pi$$

$$2\beta d_{min} - \theta_r = \pi$$

So you find θ_r.

$$\Gamma = |\Gamma| e^{j\theta_r} \Rightarrow z_L \Rightarrow Z_L$$

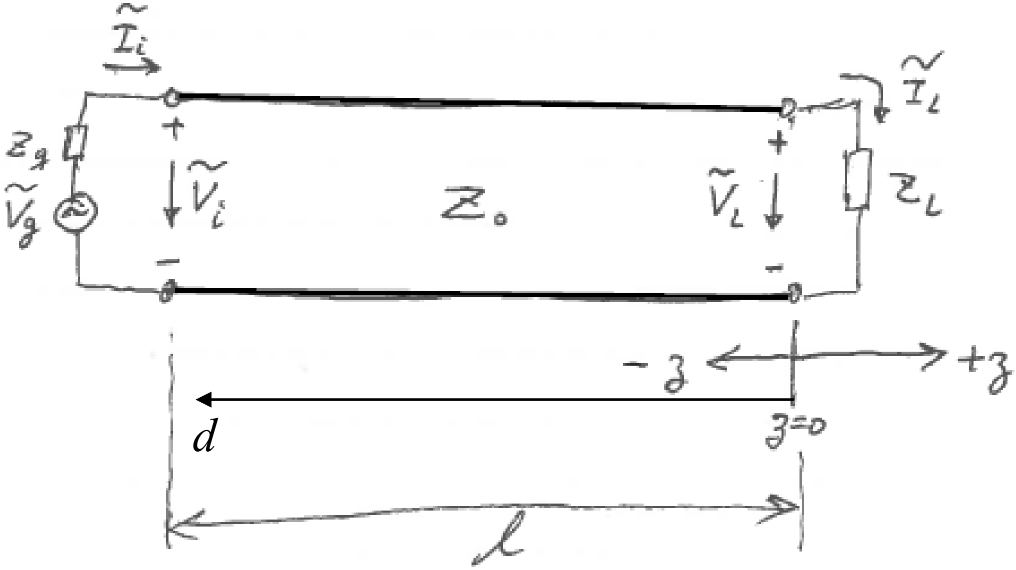
Question:

In principle, we can also obtain the result by measuring positions of maxima. But, in practice, we prefer minima. Why?

We have so far always dealt with negative *z*, because we draw the transmission line to the left of the load.

We don't like to always carry the negative sign.

So we define $d = -z$, the distance from the load.



$$\tilde{V}(z) = \underbrace{V_0^+ e^{-j\beta z}}_{\text{Incident}} + \underbrace{\Gamma V_0^+ e^{j\beta z}}_{\text{Reflected}}$$

$$\tilde{V}(z) = V_0^+ e^{-j\beta z} + \Gamma V_0^+ e^{j\beta z}$$

$$\Rightarrow \tilde{V}(d) = V_0^+ e^{j\beta d} + \Gamma V_0^+ e^{-j\beta d}$$

Pay attention to signs.

$$\tilde{I}(z) = I_0^+ e^{-j\beta z} - \Gamma I_0^+ e^{j\beta z}$$

$$\Rightarrow \tilde{I}(d) = I_0^+ e^{j\beta d} - \Gamma I_0^+ e^{-j\beta d}$$

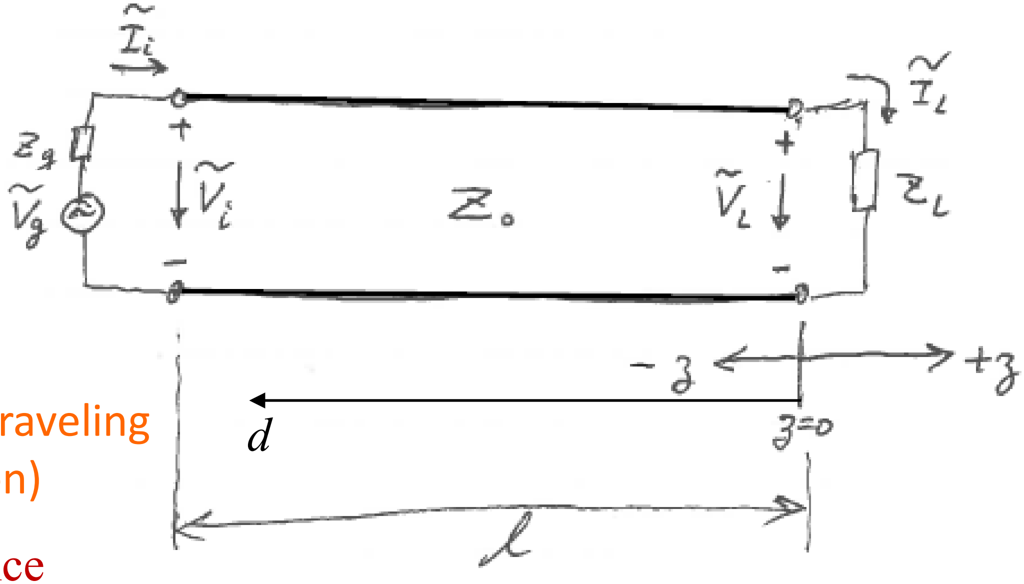
This sign is the most asked about.
Let's focus on the present discussion now.
I will give a better explanation later.

$$\tilde{V}(d) = V_0^+ e^{j\beta d} + \Gamma V_0^+ e^{-j\beta d}$$

$$\tilde{I}(d) = I_0^+ e^{j\beta d} - \Gamma I_0^+ e^{-j\beta d}$$

$$\boxed{\frac{\tilde{V}^+(d)}{\tilde{I}^+(d)} = \frac{V_0^+}{I_0^+} = Z_0}$$

(always holds for a traveling wave in one direction)



Now let's consider the equivalent impedance looking into the transmission line at a distance *d* from the load:

Important concept:

equivalent impedance at distance *d*

$$z(d) = \frac{\tilde{V}(d)}{\tilde{I}(d)} = \frac{\cancel{V_0^+}(e^{j\beta d} + \Gamma e^{-j\beta d})}{\cancel{V_0^+}(e^{j\beta d} - \Gamma e^{-j\beta d})} Z_0$$

$$= Z_0 \cdot \frac{1 + \Gamma e^{-2j\beta d}}{1 - \Gamma e^{-2j\beta d}} \equiv Z_0 \cdot \frac{1 + \Gamma_d}{1 - \Gamma_d}$$

$$\Gamma_d = \Gamma e^{-2j\beta d}$$

Compare to $\frac{Z_L}{Z_0} = \frac{1+\Gamma}{1-\Gamma}$, thus the definition.

$z(d)$-Γ_d one-to-on correspondence exactly same as z_L-Γ

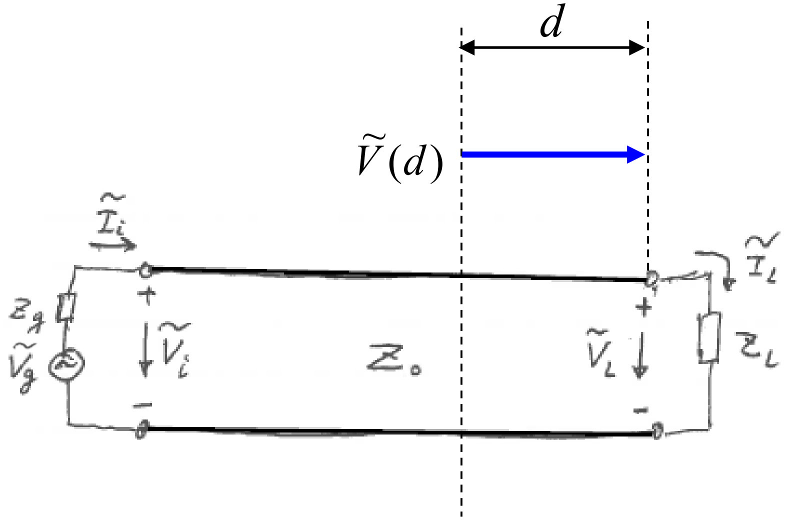
$z(d) = Z_0 \cdot \frac{1+\Gamma_d}{1-\Gamma_d} \Leftrightarrow \Gamma_d = \Gamma e^{-2j\beta d}$ (equivalent reflection coefficient at *d*)

How to interpret this (from a wave point of view)?

Say, the incident wave voltage is $\tilde{V}(d)$ at *d* from Z_L.

At the load, *d* away in the propagation direction, the incident wave is __________.

-- just a phase shift.



Question: What is the phase difference between $v(z, t)$ and $v(z + \Delta z, t)$?
What is the phase difference between$v(d, t)$ and $v(d - \Delta z, t)$?

$$z(d) = Z_0 \cdot \frac{1+\Gamma_d}{1-\Gamma_d} \Leftrightarrow \Gamma_d = \Gamma e^{-2j\beta d}$$ (equivalent reflection coefficient at *d*)

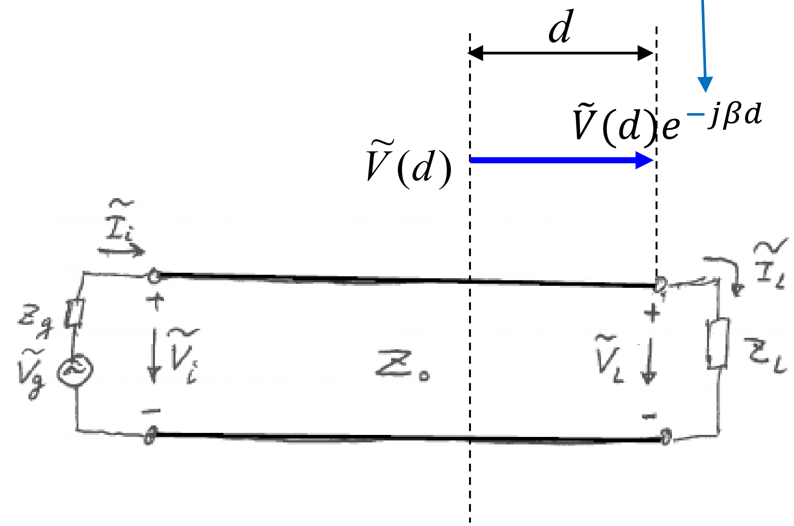
How to interpret this (from a wave point of view)?

Say, the incident wave voltage is $\tilde{V}(d)$ at *d* from Z_L.

At the load, *d* away in the propagation direction, the incident wave is $\tilde{V}(d)e^{-j\beta d}$.

-- just a phase shift.

Notice this sign. Phase shift always negative when wave travels on in our convention.



Note: This means the phase difference at any time is $-\beta d$.

Question: What is the phase difference between $v(d,t)$ and $v\left(0, t - d/v_p\right)$?

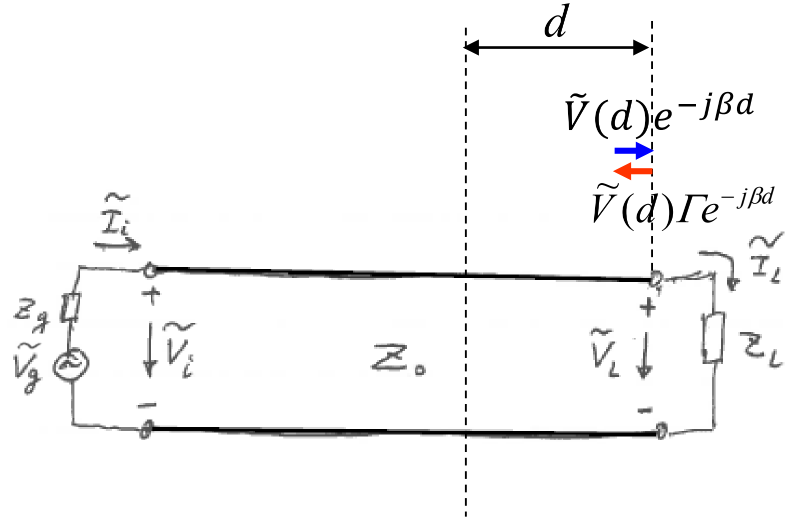
$Z(d) = Z_0 \cdot \frac{1+\Gamma_d}{1-\Gamma_d} \Leftrightarrow \Gamma_d = \Gamma e^{-2j\beta d}$ (equivalent reflection coefficient at *d*)

How to interpret this (from a wave point of view)?

Say, the incident wave voltage is $\tilde{V}(d)$ at d from Z_L.

At the load, *d* away in the propagation direction,
the incident wave is $\tilde{V}(d)e^{-j\beta d}$.
-- just a phase shift.

At the load, the reflected wave is $\tilde{V}(d)\Gamma e^{-j\beta d}$



$$Z(d) = Z_0 \cdot \frac{1+\Gamma_d}{1-\Gamma_d} \Leftrightarrow \Gamma_d = \Gamma e^{-2j\beta d}$$ (equivalent reflection coefficient at *d*)

How to interpret this (from a wave point of view)?

Say, the incident wave voltage is $\tilde{V}(d)$ at *d* from Z_L.

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-- just a phase shift.

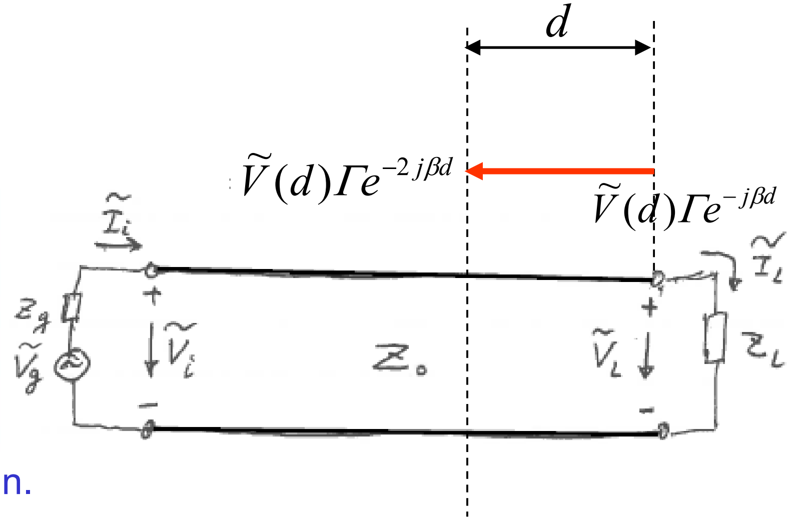
At the load, the reflected wave is $\tilde{V}(d)\Gamma e^{-j\beta d}$

Back at the point *d* away from the load, in the propagation direction (of the reflection), the reflected wave is

$$\tilde{V}(d)\Gamma e^{-j\beta d} e^{-j\beta d} = \tilde{V}(d)\Gamma e^{-2j\beta d}$$

Notice this sign.

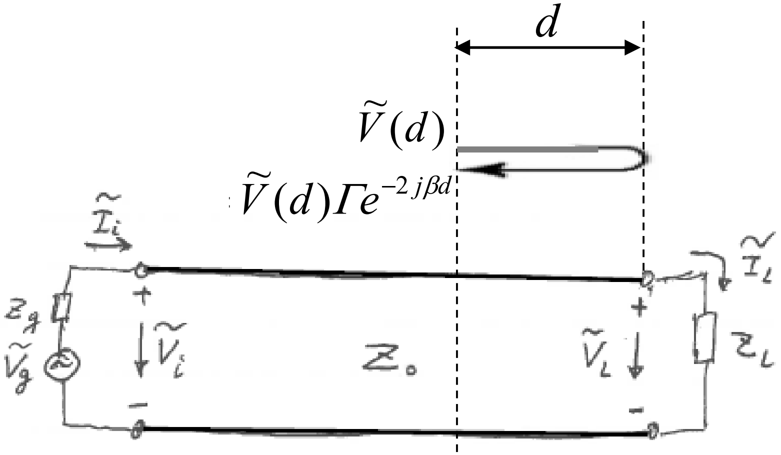
-- just another phase shift.



$$z(d) = Z_0 \cdot \frac{1+\Gamma_d}{1-\Gamma_d} \Leftrightarrow \Gamma_d = \Gamma e^{-2j\beta d}$$ (equivalent reflection coefficient at *d*)

How to interpret this (from a wave point of view)?

Say, the incident wave voltage is $\tilde{V}(d)$ at d from Z_L.



At the load, d away in the propagation direction, the incident wave is $\tilde{V}(d)e^{-j\beta d}$.

-- just a phase shift.

At the load, the reflected wave is $\tilde{V}(d)\Gamma e^{-j\beta d}$

Back at the point d away from the load, in the propagation direction (of the reflection), the reflected wave is

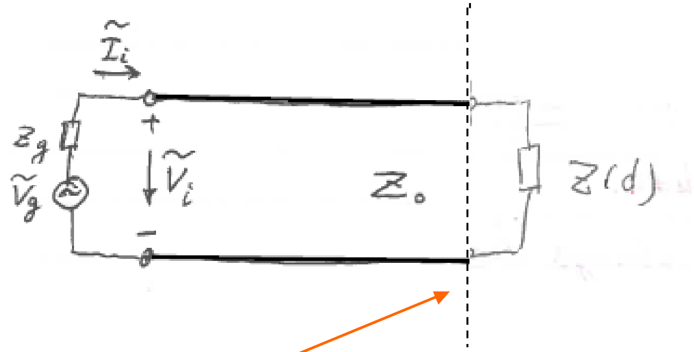
$$\tilde{V}(d)\Gamma e^{-j\beta d} e^{-j\beta d} = \tilde{V}(d)\Gamma e^{-2j\beta d}$$

Notice this sign.

-- just another phase shift.

Thus the equivalent reflection coefficient at d is

$$\Gamma_d = \Gamma e^{-2j\beta d}$$

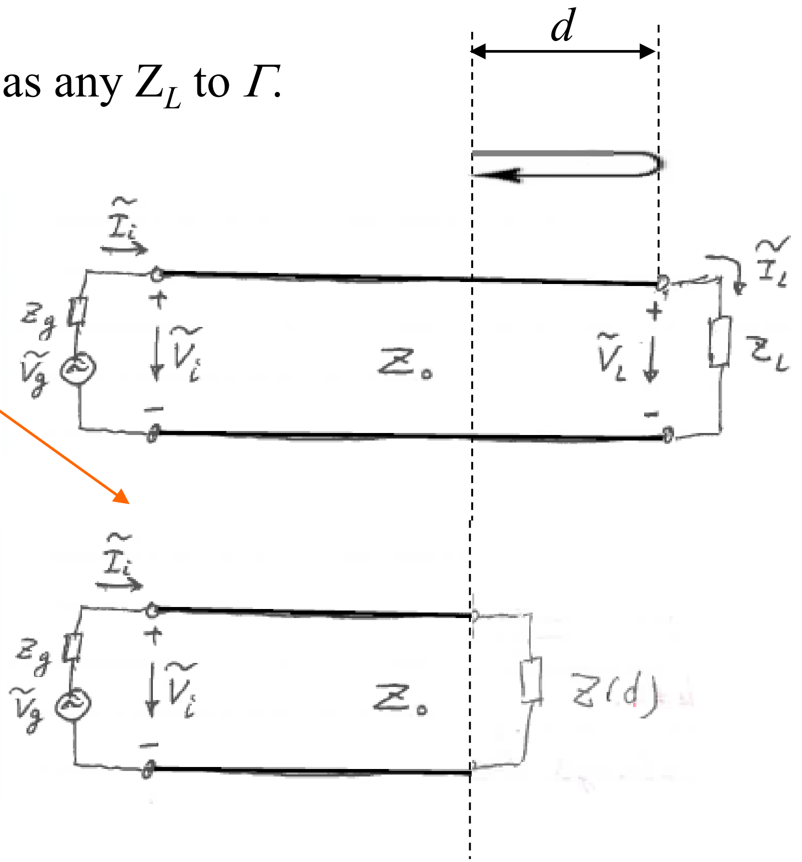


Just imagine the interface is at d. We have the equivalent circuit.

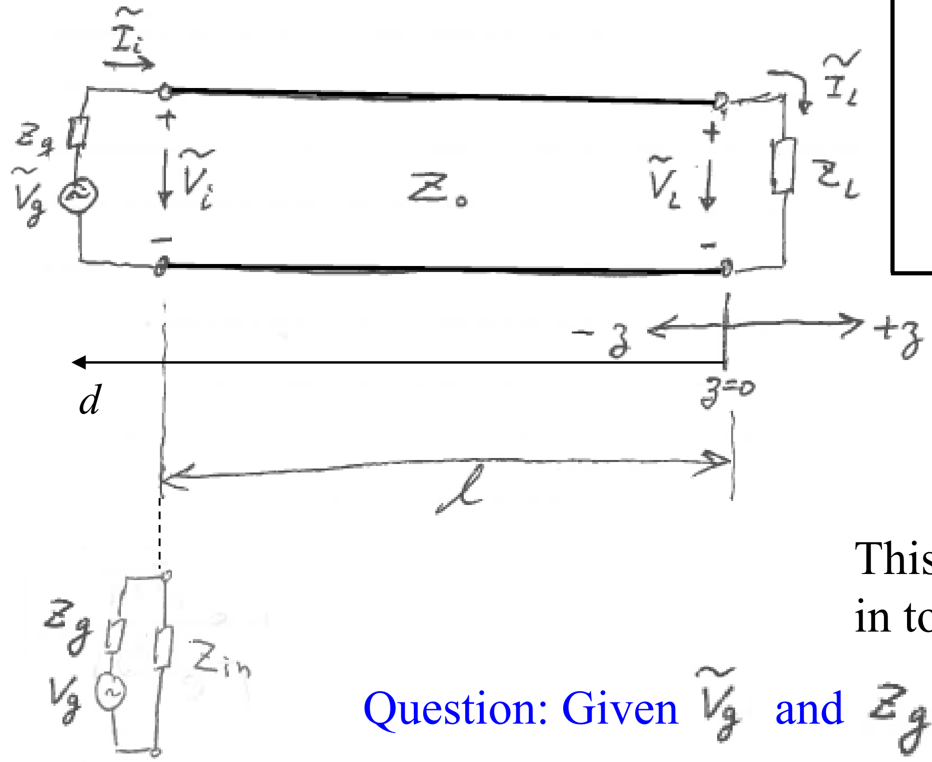
$z(d)$ corresponds to Γ_d in exactly the same manner as any z_L to Γ.

$Z(d)$ corresponds to Γ_d in exactly the same manner as any Z_L to Γ.

Therefore the equivalent circuit.



You can have such an equivalent circuit at any d, all the way up to l for the entire transmission line:



At the input end of the transmission line,

$$Z_{in} = \underset{d=\ell}{Z(\ell)} = Z_0 \frac{1+\Gamma_\ell}{1-\Gamma_\ell}$$

This way, you turn the transmission line problem in to a simple circuit problem.

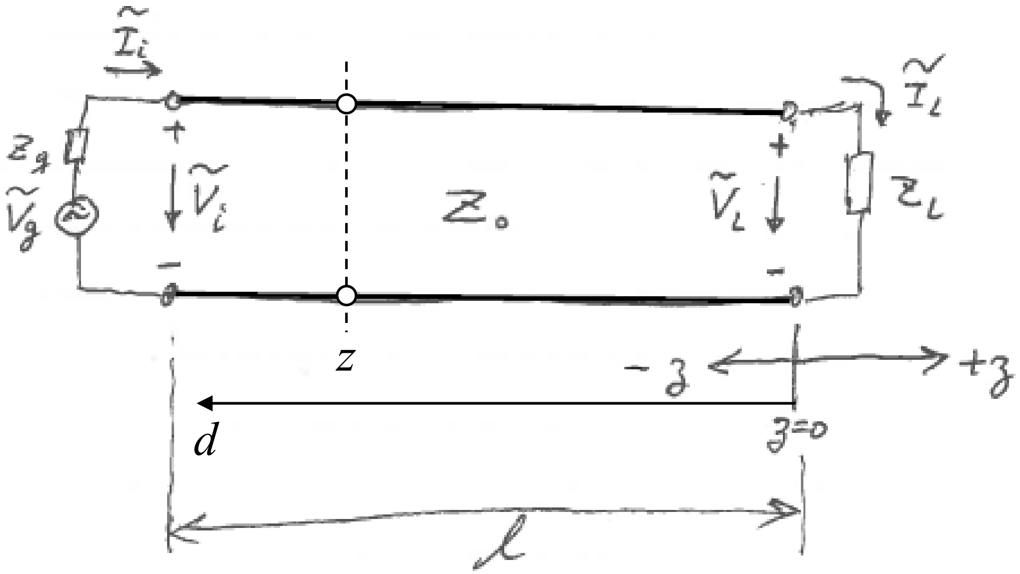
Question: Given $\tilde{V}_g$ and Z_g, how do you find $\tilde{V}_i$?

About That Negative Sign

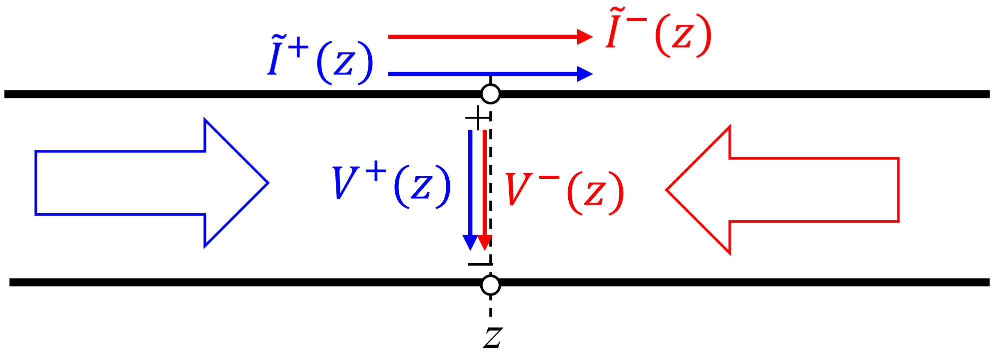
(read offline)

$$\tilde{V}(z) = \underbrace{V_0^+ e^{-j\beta z}}_{\text{Incident}} + \underbrace{\Gamma V_0^+ e^{j\beta z}}_{\text{Reflected}}$$

$$\tilde{I}(z) = I_0^+ e^{-j\beta z} - \Gamma I_0^+ e^{j\beta z}$$



More generally, for a traveling wave going towards +*z* and another one going –*z*:
(not necessarily the incident and the reflected)



We have learned

$$\frac{\tilde{V}^+(z)}{\tilde{I}^+(z)} = \frac{V_0^+ e^{-j\beta z}}{I_0^+ e^{-j\beta z}} = \frac{V_0^+}{I_0^+} = Z_0$$

The traveling wave going towards –*z* must follow the same physics:

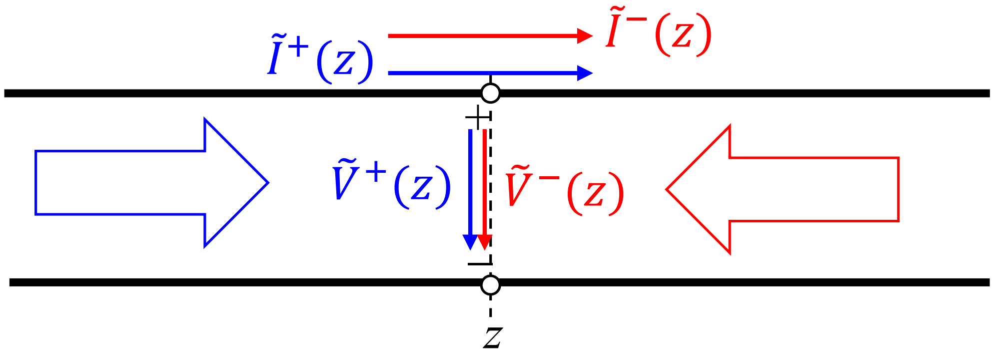
$$\frac{\tilde{V}^-(z)}{-\tilde{I}^-(z)} = \frac{V_0^- e^{j\beta z}}{-I_0^- e^{j\beta z}} = \frac{V_0^-}{-I_0^-} = Z_0$$

This negative sign is due to the way we define the polarity of $\tilde{I}^-$

$$\frac{\tilde{V}^-(z)}{\tilde{I}^-(z)} = \frac{V_0^- e^{j\beta z}}{I_0^- e^{j\beta z}} = \frac{V_0^-}{I_0^-} = -Z_0$$

Our convention for a traveling wave going towards +*z* and another one going −*z*:

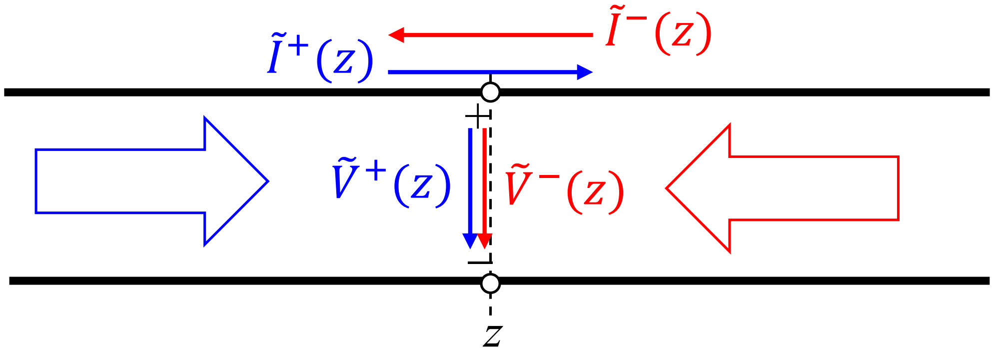
(not necessarily the incident and the reflected)



$$\frac{\tilde{V}^+(z)}{\tilde{I}^+(z)} = \frac{V_0^+ \, e^{-j\beta z}}{I_0^+ e^{-j\beta z}} = \frac{V_0^+}{I_0^+} = Z_0$$

$$\frac{\tilde{V}^-(z)}{-\tilde{I}^-(z)} = \frac{V_0^- \, e^{j\beta z}}{-I_0^- e^{j\beta z}} = \frac{V_0^-}{-I_0^-} = Z_0 \quad \Longleftrightarrow \quad \frac{\tilde{V}^-(z)}{\tilde{I}^-(z)} = \frac{V_0^- \, e^{j\beta z}}{I_0^- e^{j\beta z}} = \frac{V_0^-}{I_0^-} = -Z_0$$

If we *wanted* to be fair with the two waves, we could use a different convention:

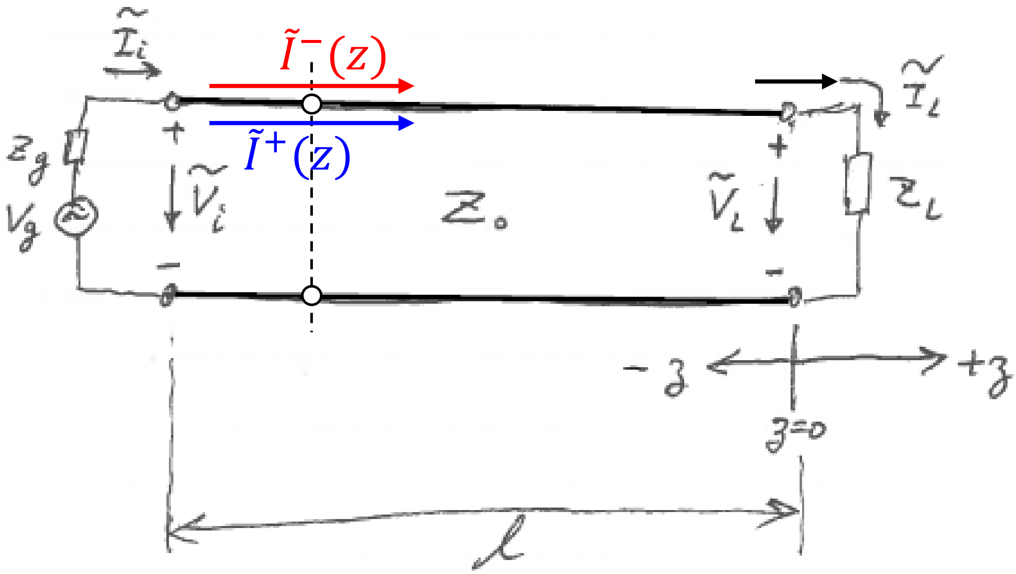


$$\frac{\tilde{V}^+(z)}{\tilde{I}^+(z)} = \frac{V_0^+ \, e^{-j\beta z}}{I_0^+ e^{-j\beta z}} = \frac{V_0^+}{I_0^+} = Z_0$$

$$\frac{\tilde{V}^-(z)}{\tilde{I}^-(z)} = \frac{V_0^- \, e^{j\beta z}}{I_0^- e^{j\beta z}} = \frac{V_0^-}{I_0^-} = Z_0$$

Both conventions give us the same $Z(z)/Z_0$ (or $Z(d)/Z_0$ with $d = -z$).

Now, in the context of reflection



What if $Z_L \neq Z_0$?

The load says $\frac{\tilde{V}(0)}{\tilde{I}(0)} = \frac{\tilde{V}_L}{\tilde{I}_L} = Z_L$

If there were only the incident wave, $\frac{\tilde{V}^+(0)}{\tilde{I}^+(0)} = \frac{V_0^+}{I_0^+} = Z_0$

Something has to happen to resolve this "conflict." That something is reflection.

$$\tilde{V}_L = \tilde{V}(z=0) = V_0^+ + V_0^-$$

$$\tilde{I}_L = \tilde{I}(z=0) = I_0^+ + I_0^- = \frac{V_0^+}{Z_0} - \frac{V_0^-}{Z_0}$$

Sign due to convention

$$\frac{V_0^-}{I_0^-} = -Z_0$$

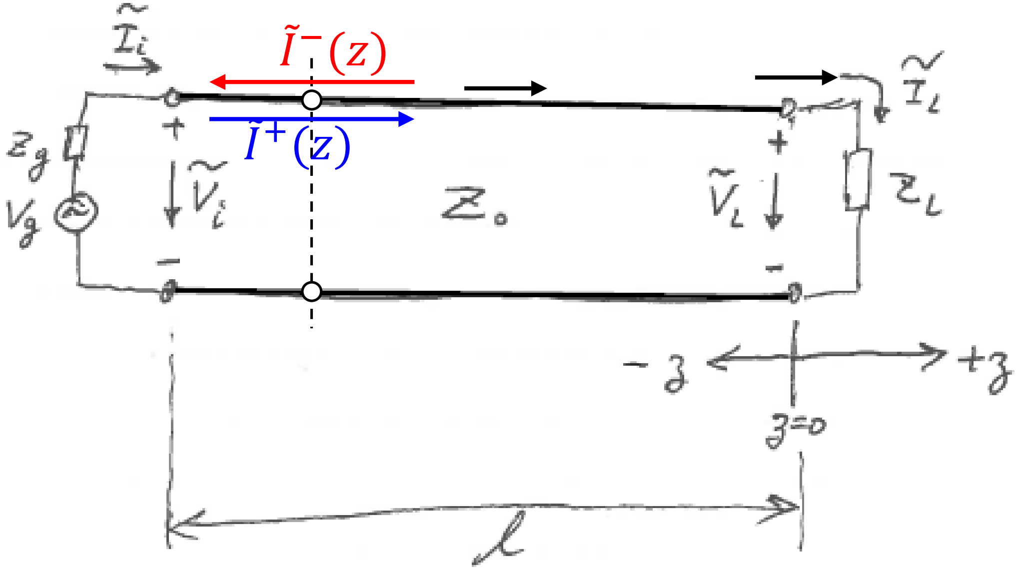
This sounds like the reflection is just due to our sign convention, doesn't it? If we used the fair alternative convention, would there be no reflection?

By definition, $Z_L = \frac{\tilde{V}_L}{\tilde{I}_L} = \left(\frac{V_0^+ + V_0^-}{V_0^+ - V_0^-}\right) Z_0$

Solve it and we have $V_0^- = \frac{Z_L - Z_0}{Z_L + Z_0} V_0^+$

In the *fair* alternative convention

$$\tilde{I}(z) = \tilde{I}^{+}(z) \underset{\substack{\Big\Uparrow \\ !!!}}{-} \tilde{I}^{-}(z)$$



The load says $\dfrac{\tilde{V}(0)}{\tilde{I}(0)} = \dfrac{\tilde{V}_L}{\tilde{I}_L} = Z_L$

If there were only the incident wave, $\dfrac{\tilde{V}^{+}(0)}{\tilde{I}^{+}(0)} = \dfrac{V_0^{+}}{I_0^{+}} = Z_0$

Something has to happen to resolve this "conflict." That something is reflection.

$$\tilde{V}_L = \tilde{V}(z=0) = V_0^{+} + V_0^{-}$$

$$\tilde{I}_L = \tilde{I}(z=0) = I_0^{+} - I_0^{-} = \frac{V_0^{+}}{Z_0} - \frac{V_0^{-}}{Z_0}$$

In the *fair* convention $\dfrac{V_0^{-}}{I_0^{-}} = Z_0$

By definition, $Z_L = \dfrac{\tilde{V}_L}{\tilde{I}_L} = \left(\dfrac{V_0^{+} + V_0^{-}}{V_0^{+} - V_0^{-}}\right) Z_0$

We end up with exactly the same thing!

Solve it and we have $V_0^{-} = \dfrac{Z_L - Z_0}{Z_L + Z_0} V_0^{+}$

Review textbook Sections 2-6, 2-7.
Do HW2 up to Problem 9.

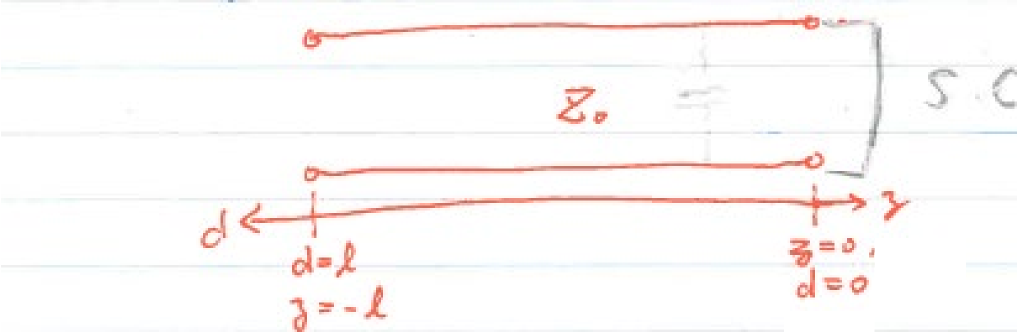
Notice that we take a different approach than in the textbook (again).
We started from special cases: short and open circuit terminations.
Then we moved on to the general case.
Now we are going back to the special cases, but not that special.

Short circuit $\Gamma = -1$. Open circuit $\Gamma = 1$.
There are cases where $|\Gamma| = 1$ but $\Gamma \neq \pm 1$.
Also complete reflection, but neither short nor open.

What loads make those cases?

All cases of $|\Gamma| = 1$

There are cases where $|\Gamma| = 1$ but $\Gamma \neq \pm 1$. Also complete reflection, but neither short nor open.



A quarter wavelength away from a short, the equivalent circuit is an open.
(See next slide – an old one)

At very high frequencies, we often can only measure the amplitude or power ($\propto$ amplitude squared), but not the instantaneous values or the waveform.

The amplitude of the voltage wave $v(z, t)$ at position z is $|\tilde{V}(z)| = \sqrt{\tilde{V}(z)\tilde{V}^*(z)}$

The “complex amplitude” containing the phase

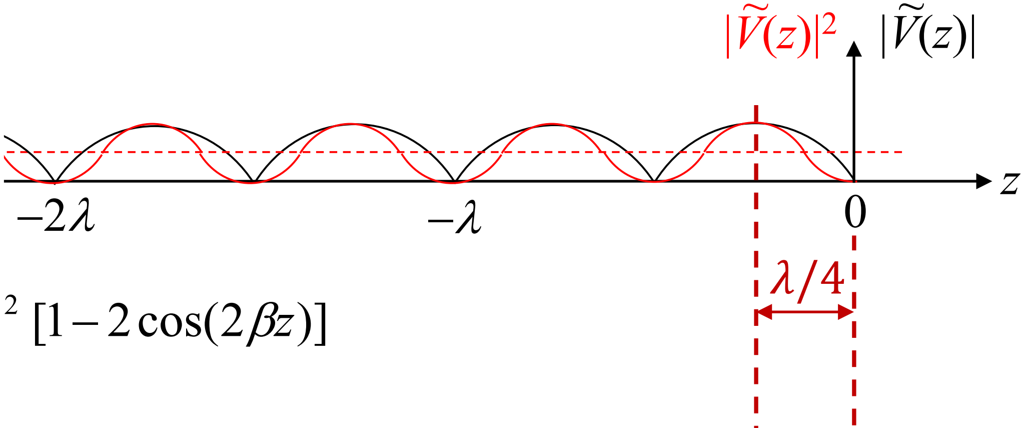
Short

$$\tilde{V}(z) = -2jV_0^+ \sin(\beta z)$$

$$\Rightarrow \quad |\tilde{V}(z)| = |-2jV_0^+ \sin(\beta z)|$$

$$= 2|V_0^+||\sin(\beta z)|$$

$$|\tilde{V}(z)|^2 = 4|V_0^+|^2 \sin^2(\beta z) = 2|V_0^+|^2 [1 - 2\cos(2\beta z)]$$

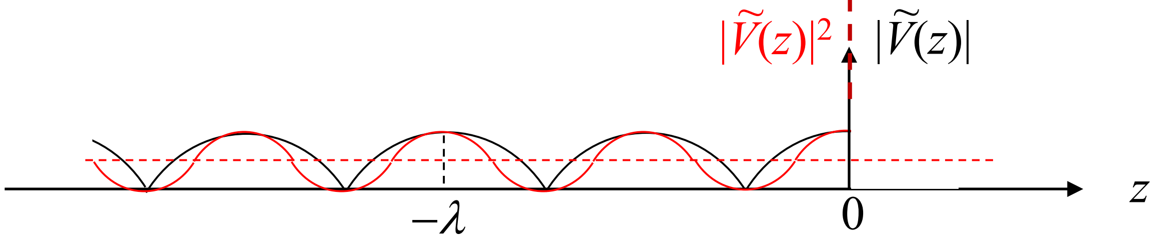


Open

$$|\tilde{V}(z)| = 2|V_0^+||\cos(\beta z)|$$

$$|\tilde{V}(z)|^2 = 4|V_0^+|^2 \cos^2(\beta z)$$

$$= 2|V_0^+|^2 [1 + 2\cos(2\beta z)]$$



Between short and open: Just a shift of the origin by $\lambda/4$.

All cases of $|\Gamma| = 1$

There are cases where $|\Gamma| = 1$ but $\Gamma \neq \pm 1$. Also complete reflection, but neither short nor open.

A quarter wavelength away from a short, the equivalent circuit is an open.

What is the equivalent impedance anywhere in between?

Let's have a closer look at the short circuit.

$$\tilde{V}_{sc}(d) = V_0^+(e^{j\beta d} - e^{-j\beta d})$$

$$= 2jV_0^+ \sin\beta d$$

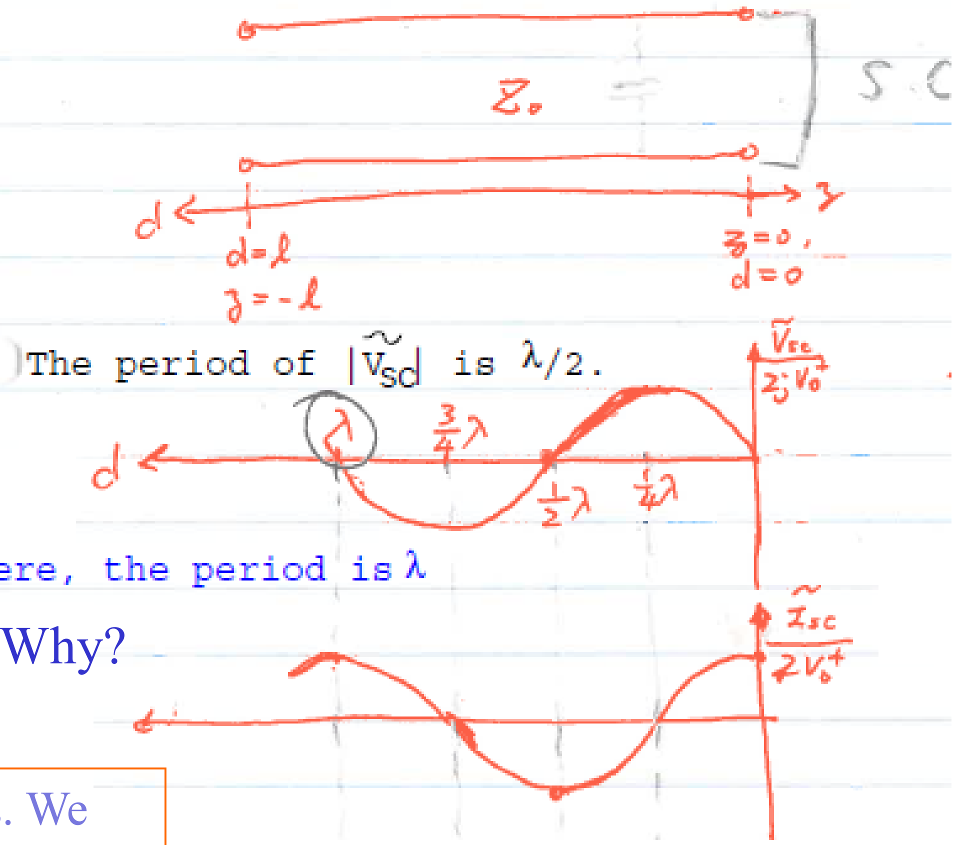
$$\tilde{I}_{sc}(d) = \frac{V_0^+}{Z_0}(e^{j\beta d} + e^{-j\beta d}) = \frac{2V_0^+}{Z_0}\cos\beta d$$

$$Z_{sc}(d) = \frac{\tilde{V}_{sc}(d)}{\tilde{I}_{sc}(d)}$$

$$= \boxed{j} Z_0 \underline{\tan\beta d}$$

Pay attention to signs. We are using *d* now.

Compare these Equations to those in first 4 slides of this ppt.



The period of $|\tilde{V}_{sc}|$ is $\lambda/2$.

Here, the period is λ

Why?

To visualize $Z(d)$, we now plot $\tilde{V}(d)$ and $\tilde{I}(d)$.

Previously, we plotted either $|\tilde{V}(d)|^2$ and $|\tilde{I}(d)|^2$ or $|\tilde{V}(d)|$ and $|\tilde{I}(d)|$ because they are measurable.

All cases of $|\Gamma| = 1$

There are cases where $|\Gamma| = 1$ but $\Gamma \neq \pm 1$. Also complete reflection, but neither short nor open.

A quarter wavelength away from a short, the equivalent circuit is an open.

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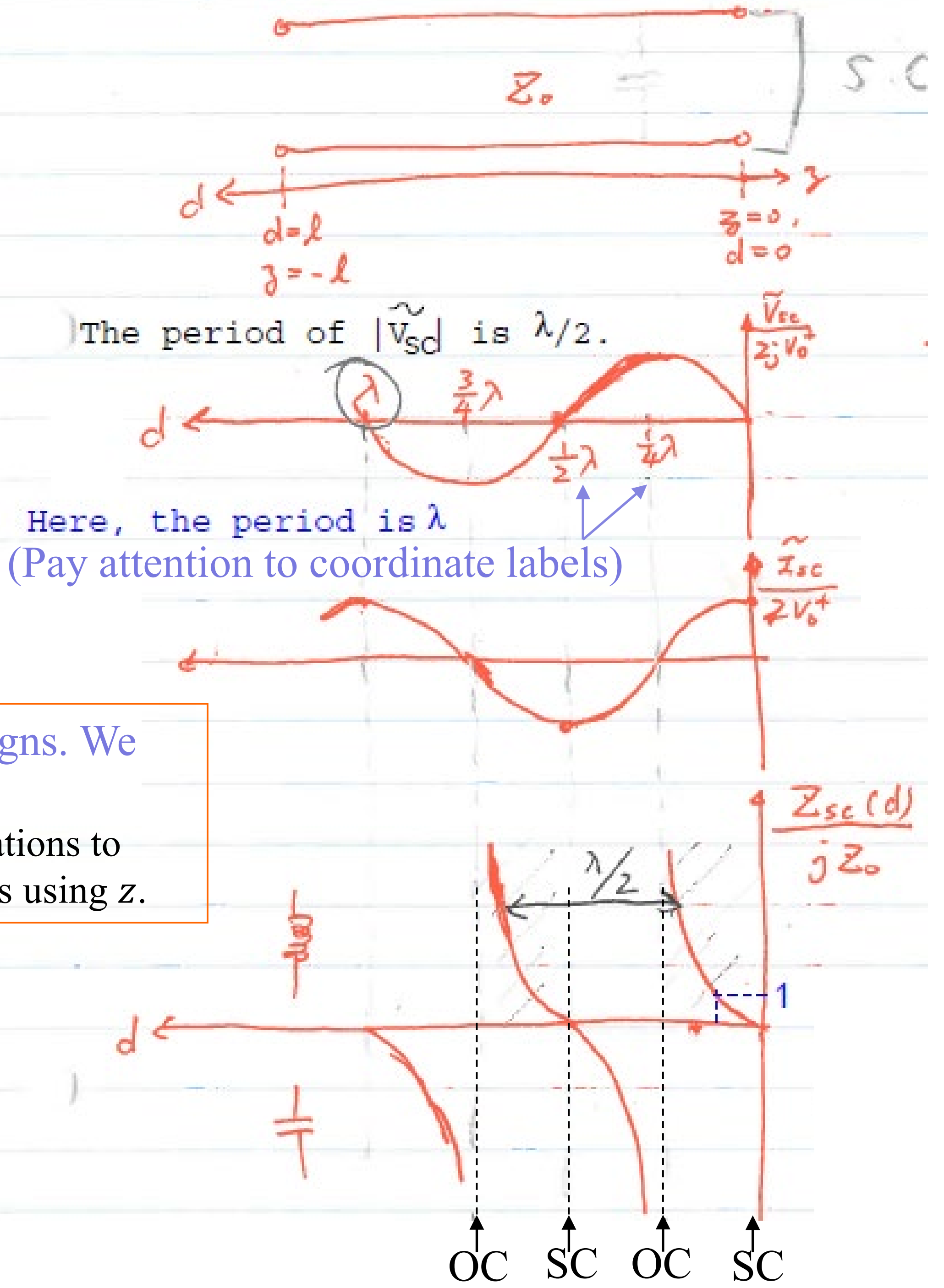
$$\tilde{I}_{sc}(d) = \frac{V_0^+}{Z_0}(e^{j\beta d} + e^{-j\beta d}) = \frac{2V_0^+}{Z_0}\cos\beta d$$

$$Z_{sc}(d) = \frac{\tilde{V}_{sc}(d)}{\tilde{I}_{sc}(d)}$$

$$= \boxed{j} Z_0 \underline{\tan\beta d}$$

Pay attention to signs. We are using *d* now.

Compare these Equations to those in earlier slides using *z*.



The period of $|\tilde{V}_{sc}|$ is $\lambda/2$.

Here, the period is λ

(Pay attention to coordinate labels)

$$Z_{sc}(d) = \frac{\tilde{V}_{sc}(d)}{\tilde{I}_{sc}(d)} = \boxed{j} Z_0 \underbrace{\tan\beta d}$$

Understand this from a physics point of view:

Reactive loads don't dissipate power.
Thus complete reflection.
The difference is just in the phase.

Equivalent impedance

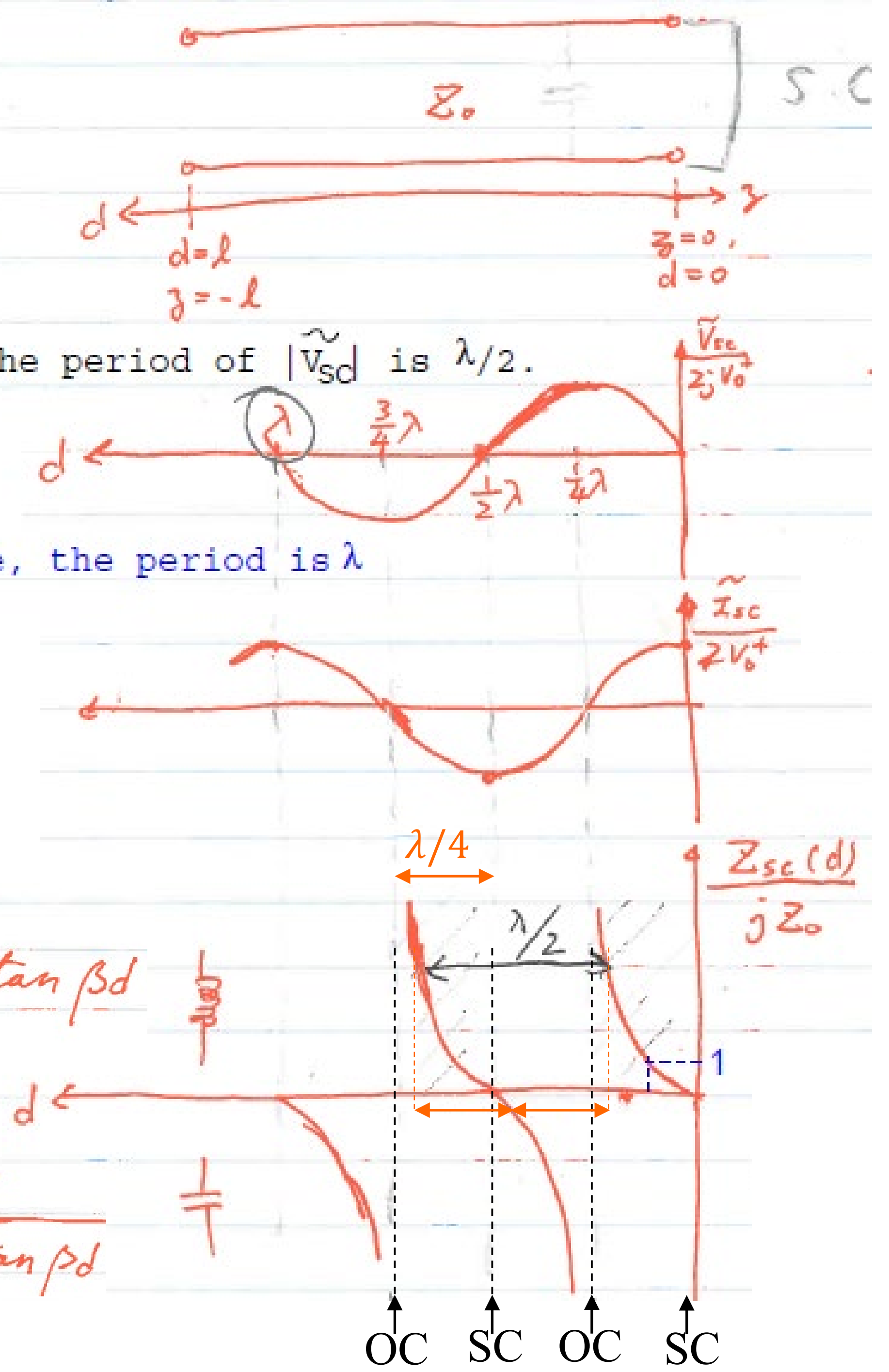
For $\tan\beta d > 0$

$$j\omega L_{eq} = jZ_0 \tan\beta d \Rightarrow L_{eq} = \frac{Z_0}{\omega}\tan\beta d$$

For $\tan\beta d < 0$

$$\frac{1}{j\omega C_{eq}} = jZ_0 \tan\beta d \Rightarrow C_{eq} = -\frac{1}{\omega Z_0 \tan\beta d}$$

Notice frequency dependence.



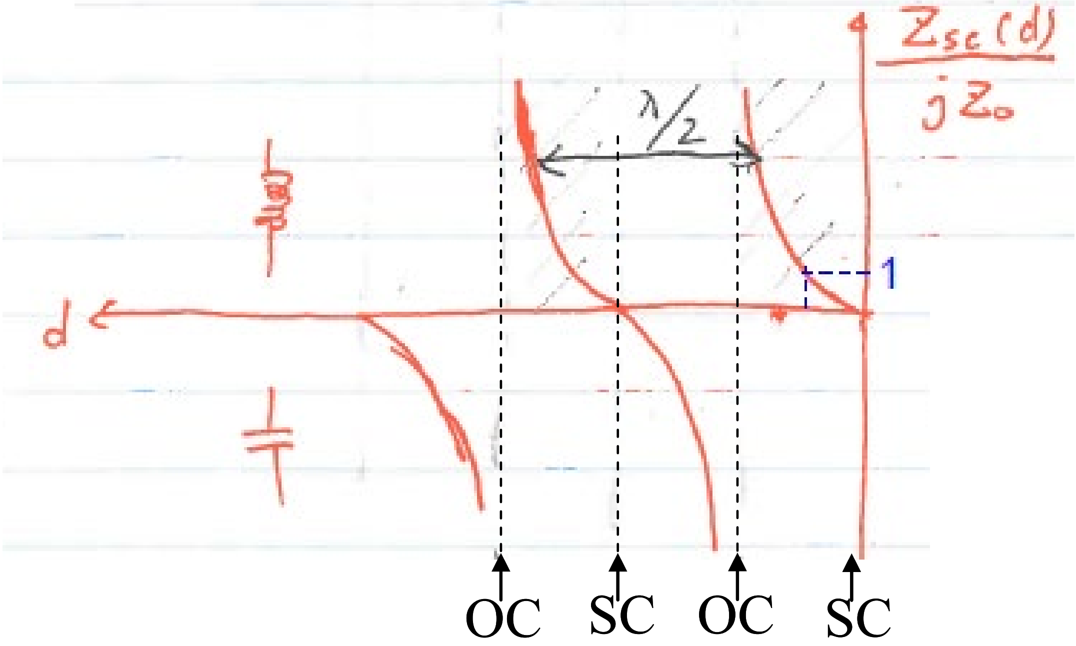
The period of $|\tilde{V}_{sc}|$ is $\lambda/2$.

Here, the period is λ

The case of open circuit termination

Now that we already know the case of short circuit termination, what's the easiest way to work out the open circuit termination case?

Leaving a transmission line open ended does not make an open circuit termination.



With a short circuit, you can make an open circuit.

(For complete solution, see Fig. 2-21 in textbook, pp. 77 in 8/E pp. 81 in 7/E or pp. 82 in 6/E)

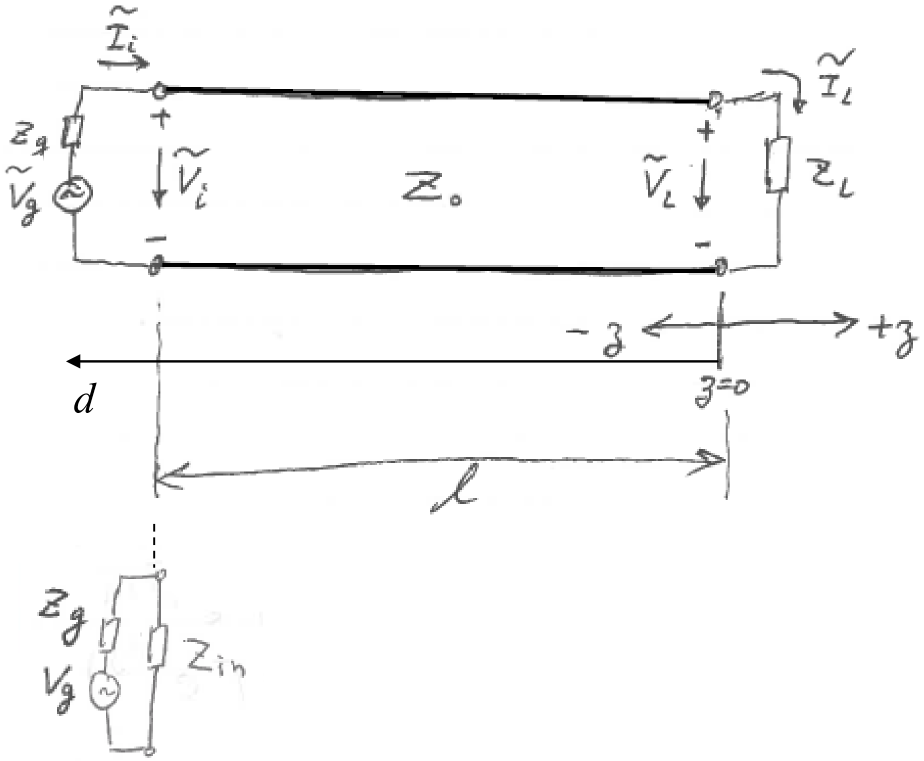
Now let's go back to the general case and look at the equivalent input impedance.

$$Z_{in} = Z(d=l) = Z_0 \cdot \frac{e^{j\beta l} + \Gamma e^{-j\beta l}}{e^{j\beta l} - \Gamma e^{-j\beta l}}$$

(make sure you understand how this is arrived at)

Insert

$$\begin{cases} \Gamma = \dfrac{Z_L - Z_0}{Z_L + Z_0} \\ e^{j\beta l} = \cos\beta l + j\sin\beta l \\ e^{-j\beta l} = \cos\beta l - j\sin\beta l \end{cases}$$



and you get

$$Z_{in} = \frac{Z_L \cos\beta l + jZ_0 \sin\beta l}{Z_0 \cos\beta l + jZ_L \sin\beta l} Z_0$$

or in the normalized form:

$$z_{in} = \frac{z_L \cos\beta l + j\sin\beta l}{\cos\beta l + jz_L \sin\beta l}$$

Recall that $z_L = \dfrac{Z_L}{Z_0}$

Questions: What is the unit of z_L? What is the unit of z_{in}?

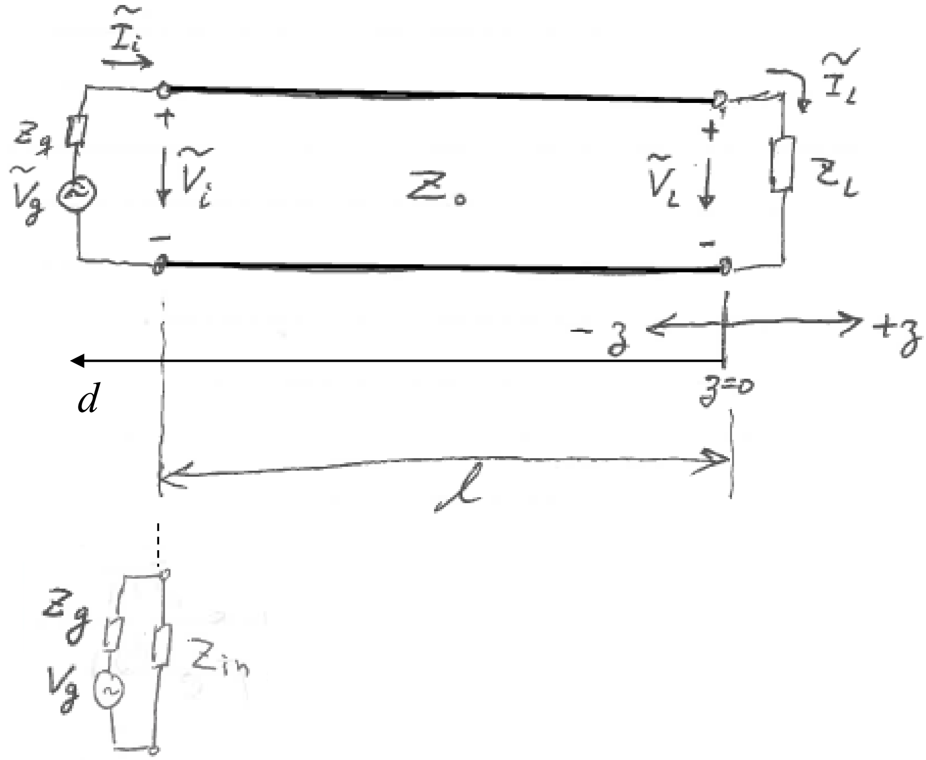
Do not confuse “input” with “incident”

$\tilde{V}_i$ or $\tilde{V}_{in}$ is the voltage at the input end. It is the sum of incident and reflected waves **there**.

$$\tilde{V}_{\underline{in}} = \frac{\tilde{V}_g Z_{in}}{Z_g + Z_{in}} = V_0^+ \left(\underbrace{e^{j\beta l}}_{\text{incident}} + \underbrace{\Gamma e^{-j\beta l}}_{\text{reflected}}\right)$$

The incident wave voltage
at the input end is

$$\underline{\tilde{V}_{inc}(l) = V_0^+ e^{j\beta l}}$$



At this point read textbook Sections 2-7, 2-8.1
2-8.2 & 2-8.3.
We changed the sequence of the contents here for easier understanding – Special/extreme cases (short & open) are often easier to understand.
Finish HW2 (P10).
Work on HW3 Problems 1-3.

The quarter wavelength magic

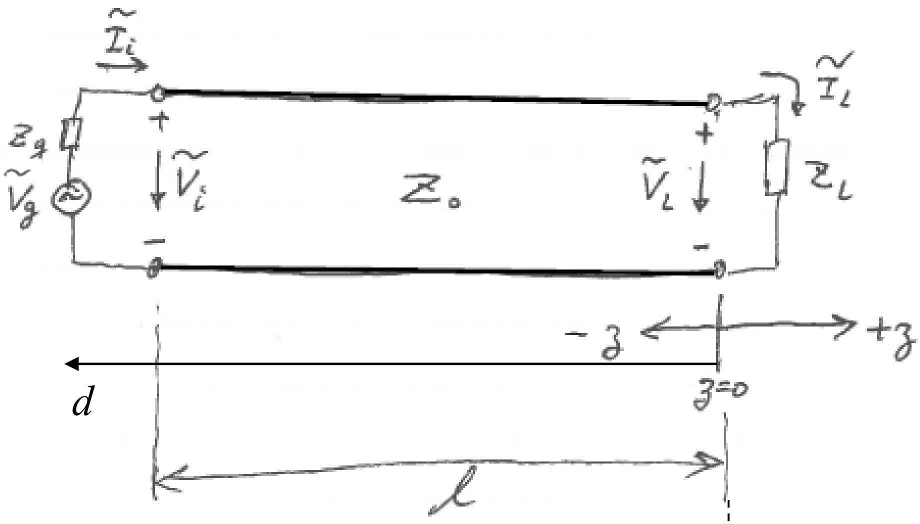
(Textbook Section 2-8.4)

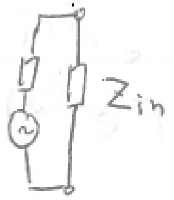
$$Z_{in} = \frac{Z_L \cos\beta l + jZ_0 \sin\beta l}{Z_0 \cos\beta l + jZ_L \sin\beta l} Z_0$$

For $\beta l = n\pi$,

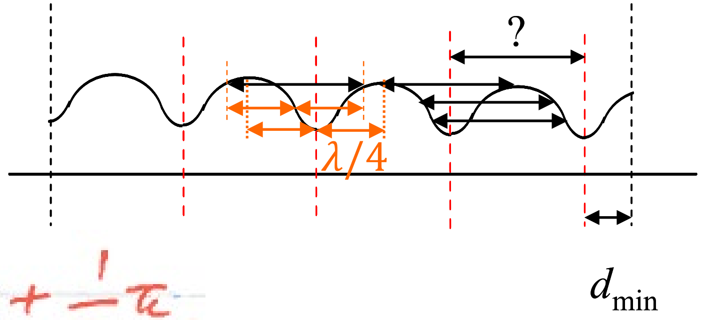
i.e. $l = n \cdot \frac{\lambda}{2}$

$$\boxed{Z_{in} = Z_L}$$





Periodic. This is for generic Z_L.
For the special case of purely reactive loads, see slides 41 & 44.



For $\begin{cases} \cos\beta l = 0 \\ \sin\beta l = \pm 1 \end{cases}$ ie: $\beta l = n\pi + \frac{1}{2}\pi$,

i.e. $l = n \cdot \frac{\lambda}{2} + \frac{\lambda}{4}$,

$$\boxed{Z_{in} = \frac{Z_0^2}{Z_L}}$$

l	βl
λ	2π
$\lambda/2$	π
$\lambda/4$	$\pi/2$

For $l = n \cdot \frac{\lambda}{2} + \boxed{\frac{\lambda}{4}}$,

l	βl
λ	2π
$\lambda/2$	π
$\lambda/4$	$\pi/2$

$$\boxed{Z_{in} = \frac{Z_0^2}{Z_L}}$$

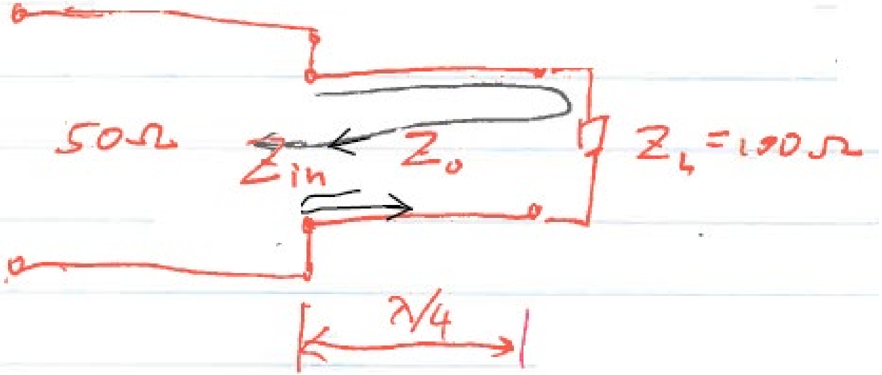
or

$$\boxed{Z_0 = \sqrt{Z_{in} \cdot Z_L}}$$

Question: What are the equivalent "normalized" forms?

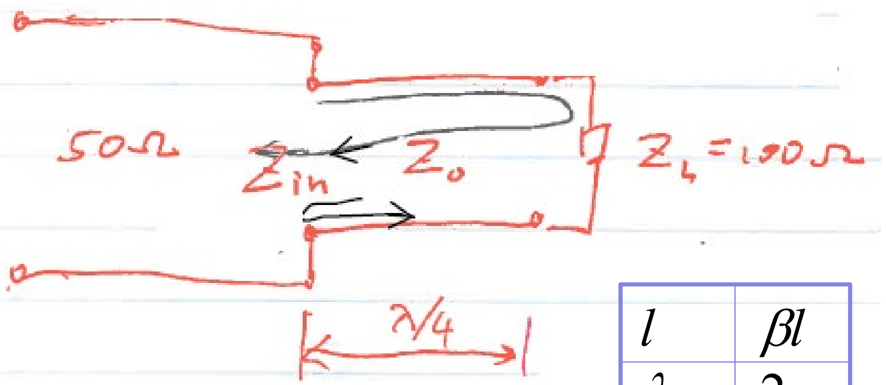
The quarter wavelength transformer

-- a method for impedance matching



The quarter wavelength transformer

$Z_{in} = \frac{Z_0^2}{Z_L}$ or $Z_0 = \sqrt{Z_{in} \cdot Z_L}$



l	βl
λ	2π
$\lambda/2$	π
$\lambda/4$	$\pi/2$

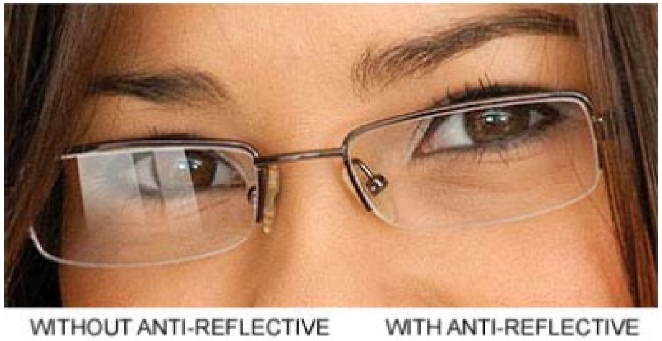
In the "normalized" forms:

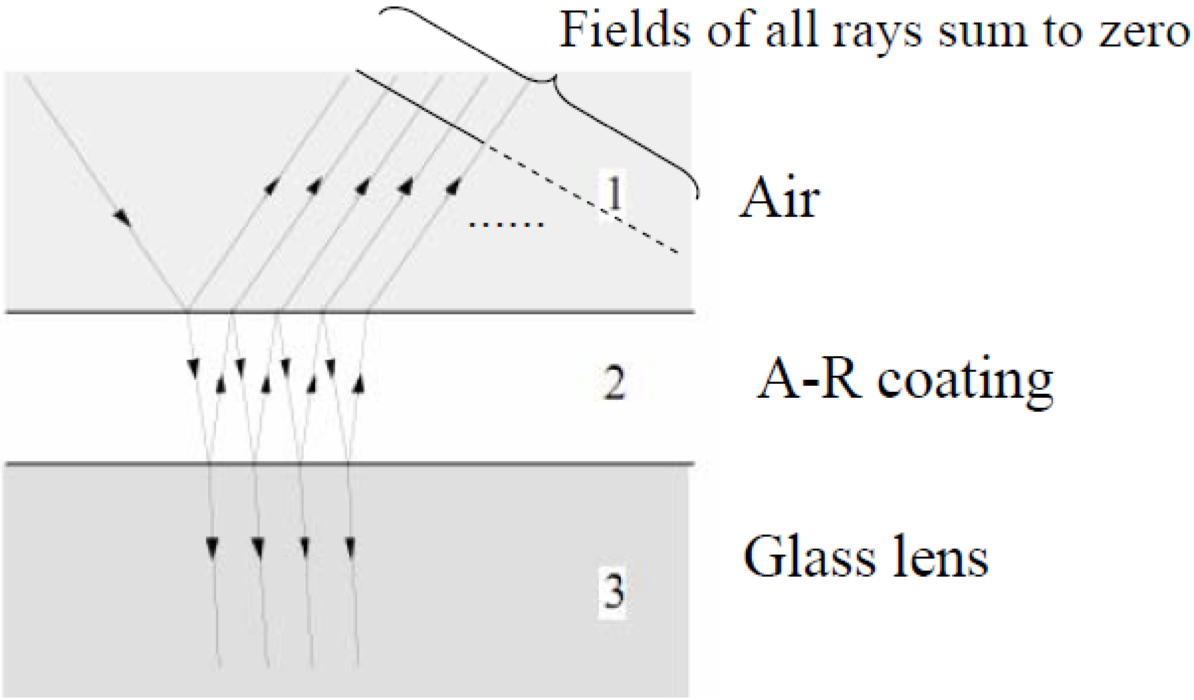
$$z_{\text{in}} = \frac{1}{z_{\text{L}}} \quad \text{or} \quad z_{\text{in}} z_{\text{L}} = 1$$

We stopped here on Thu 9/22/2022.

To better understand why it works,
let's look at its optical analog.

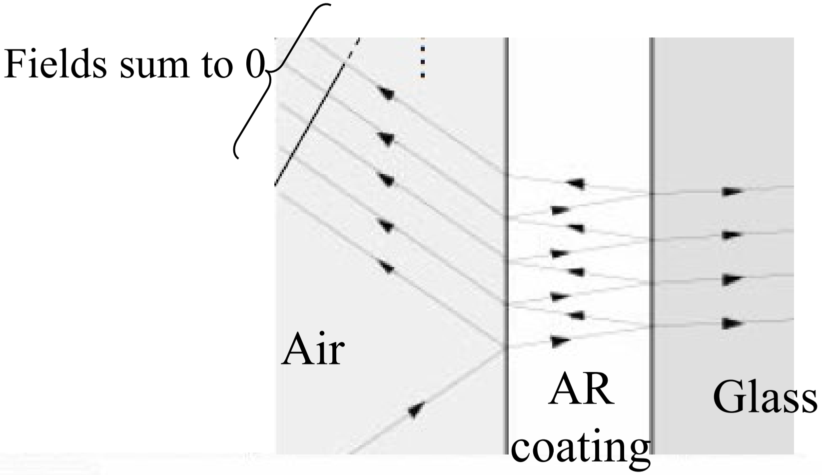
Anti-reflection coating





The quarter wavelength magic explained in the multiple reflection point of view

Optical analog: the AR coating



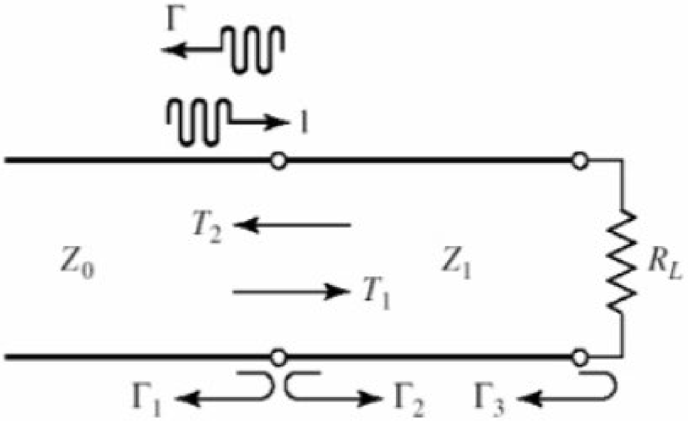
$$\Gamma_1 = \frac{Z_1 - Z_o}{Z_1 + Z_o}$$

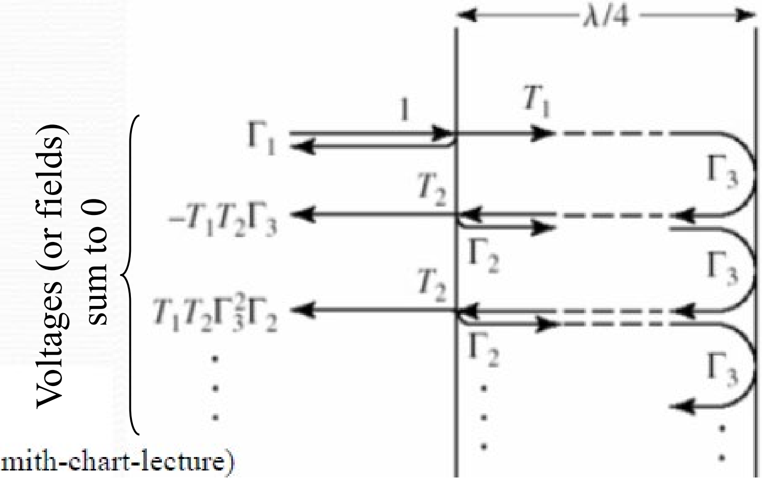
$$\Gamma_2 = \frac{Z_0 - Z_1}{Z_0 + Z_1}$$

$$\Gamma_3 = \frac{R_L - Z_1}{R_L + Z_1}$$

$$T_1 = \frac{2Z_1}{Z_1 + Z_o}$$

$$T_2 = \frac{2Z_o}{Z_1 + Z_o}$$





l	βl
λ	2π
$\lambda/2$	π
$\lambda/4$	$\pi/2$

(Adapted from: Naveed Ramzan,
http://www.slideshare.net/nramzan19/smith-chart-lecture)

Take-home messages

- Standing waves are simply due to interference between incident & reflected waves.
- The variations of real positive amplitudes (of voltage & current) and modulus squares of amplitudes ($|\widetilde{V}(d)|^2$ and $|\widetilde{I}(d)|^2$) with position (i.e. distance from load) are periodic, analogous to interference stripes in optics and are indeed one-dimensional interference patterns.
- The period of the (observed) patterns is half wavelength.
- The reflected wave amplitude is a fraction (≤ 1) of the incident, and its phase is shifted relative to the incident, right upon reflection. Thus the reflection coefficient is complex.
- In general, voltage and current of a transmission line are combinations of a traveling wave and a standing wave.
- When the load is purely reactive (including short and open), complete reflection happens. What's in common is absence of energy dissipation. Thus you can obtain any desired reactance value by terminating a transmission line in any reactive component; you only need to have the right distance from the load. This equivalence is frequency specific.
- At any distance d from the load, you have an equivalent impedance $Z(d)$, such that you feel as if the transmission line is terminated in $Z(d)$ right there.
- $Z(\lambda/4)$ and Z_L [or more generally $Z(d + \lambda/4)$ and $Z(d)$] have a special relation, which is used as a method for impedance matching. This method eliminates reflection because multiple reflections sum up to 0.

Finish reading textbook Section 2-8.
Do Homework 3 through Problem 5.

We wrapped up this slide set, went through the next one, moved on to the Smith Chart on Tue 9/27/2022.