Standing Wave

Interference between the incident & reflected waves → Standing wave

A string with one end fixed on a wall

Incident: \( y_1(z, t) = Y_0^+ \cos(\omega t - \beta z) \)

\[ \tilde{Y}_1(z) = Y_0^+ e^{-j \beta z} \]

(Set the incident wave’s phase to be 0, i.e., \( Y_0^+ \) real & positive.)

Reflected: \( y_2(z, t) = Y_0^- | \cos(\omega t + \beta z + \phi) \)

\[ \tilde{Y}_2(z) = Y_0^- e^{j \beta z}, \text{ where } Y_0^- = |Y_0^-| e^{j \phi} = |Y_0^-| \angle \phi \]

The total displacement

\[ \tilde{Y}(z) = \tilde{Y}_1(z) + \tilde{Y}_2(z) = Y_0^+ e^{-j \beta z} + Y_0^- e^{j \beta z} \]

We must have \( \tilde{Y}(0) = 0 \) \( \Rightarrow \) \( Y_0^+ + Y_0^- = 0 \) i.e. \( Y_0^- = -Y_0^+ \)

\[ \tilde{Y}(z) = Y_0^+ (e^{-j \beta z} - e^{j \beta z}) \]
\( \widetilde{Y}(z) = Y_0^+ (e^{-j\beta z} - e^{j\beta z}) \)

Recall that \( e^{j\theta} - e^{-j\theta} = 2j \sin \theta \) \( \Rightarrow \) \( \widetilde{Y}(z) = -2jY_0^+ \sin(\beta z) \)

\[ y(z, t) = \text{Re}[-2jY_0^+ \sin(\beta z)] = 2Y_0^+ \sin(\beta z) \sin(\omega t) \quad \text{Why sin?} \]

See Wikipedia Standing Wave animation to get visual picture:

Harmonic oscillation at each \( z \), with amplitude following \( \sin(\beta z) \)

\[ \Gamma = \frac{Z_L - Z_0}{Z_L + Z_0} = -1 \]

Similarly, shorted transmission line:

By definition of “short circuit, \( \widetilde{V}(0) = 0 \) \( \Rightarrow \) \( V_0^+ + V_0^- = 0 \)

\( V_0^- = -V_0^+ \) \( \Gamma = -1 \)

\( \widetilde{V}(z) = -2jV_0^+ \sin(\beta z) \)

\[ v(z, t) = \text{Re}[-2jV_0^+ \sin(\beta z)] = 2V_0^+ \sin(\beta z) \sin(\omega t) \]

Like a mirror. What property of a mirror makes it a mirror?

But, \( I_0^- = I_0^+ \quad \frac{I_0^+}{I_0^-} = -\Gamma = 1 \)

Find \( \widetilde{i}(z) \) and \( i(z, t) \) on your own.
What if the transmission line is terminated in open circuit?

Note: open ended ≠ open circuit for high frequencies!
(You will see how to make an open circuit later.)

\[ V_0^- = V_0^+ \quad \Gamma = \frac{Z_L - Z_0}{Z_L + Z_0} = 1 \]

\[ \tilde{V}(z) = V_0^+ (e^{-j\beta z} + e^{j\beta z}) = 2V_0^+ \cos(\beta z) \]

Find \( v(z, t) \) on your own.

\[ I_0^- = -I_0^+ \quad \text{(since total current is 0, by definition of open circuit)} \]

Find \( \tilde{i}(z) \) and \( i(z, t) \) on your own.

In all the above examples, \( \Gamma = \pm 1 \). Completely reflected.
At very high frequencies, we often can only measure the amplitude or power \( \propto \) amplitude squared, but not the instantaneous values or the waveform.

The following example is for the short circuit. The open circuit is similar (just with a shift of origin).

The amplitude of the voltage wave \( v(z, t) \) at position \( z \) is

\[
|\tilde{V}(z)| = \sqrt{\tilde{V}(z)\tilde{V}^*(z)}
\]

The “complex amplitude” containing the phase

\[
|\tilde{V}(z)| = \sqrt{\tilde{V}(z)\tilde{V}^*(z)}
\]

\[
= | -2 j V_0^+ \sin(\beta z) |
\]

\[
= 2 | V_0^+ | | \sin(\beta z) |
\]

\[
|\tilde{V}(z)|^2 = 4 | V_0^+ |^2 \sin^2(\beta z) = 2 | V_0^+ |^2 \left[ 1 - 2 \cos(2\beta z) \right]
\]

**Question:** What’s the spatial period of the standing wave?

The open circuit: Just a shift of the origin

\[
|\tilde{V}(z)| = 2 | V_0^+ | | \cos(\beta z) |
\]

\[
|\tilde{V}(z)|^2 = 4 | V_0^+ |^2 \cos^2(\beta z)
\]

\[
= 2 | V_0^+ |^2 \left[ 1 + 2 \cos(2\beta z) \right]
\]
For both the short circuit (SC) and open circuit (OC),

\[
|\tilde{V}(z)|_{\text{max}} = 2 |V_0^+| \quad \text{Constructive}
\]
\[
|\tilde{V}(z)|_{\text{min}} = 0 \quad \text{Destructive}
\]
\[
|\Gamma| = 1
\]

Complete reflection. Completely a standing wave.

There are cases where $|\Gamma| = 1$ but $\Gamma \neq \pm 1$.
Also complete reflection. We’ll talk about those cases later.

What if $|\Gamma| \neq 1$?

Partially standing, partially traveling.

Now, let’s look at the maxima and minima of this combination of a standing wave and a traveling wave.
Recall that, in general, $\Gamma$ is a complex number:

\[
\Gamma = |\Gamma|e^{j\theta_r}
\]
\[ \tilde{V}(z) = V_0^+ e^{-j\beta z} + V_0^- e^{j\beta z} = V_0^+ e^{-j\beta z} + \Gamma V_0^+ e^{j\beta z} = V_0^+ e^{-j\beta z} + |\Gamma| e^{i\theta_r} V_0^+ e^{i\beta z} \]

Incident  Reflected

\[ |\tilde{V}(z)| = \sqrt{\tilde{V}(z) \tilde{V}^*(z)} = \sqrt{\tilde{V}_0^+(z) \tilde{V}_0^+(z)^*} \]

Notice that \( V_0^+ \) is a complex amplitude

\[ |\tilde{V}(z)| = |V_0^+| \sqrt{\ldots \ldots \ldots} \]

\[ = |V_0^+| \sqrt{1 + |\Gamma|^2 - 2|\Gamma| \cos(2\beta z + \theta_r)} \]

Interference term

Similarly, \[ |\tilde{I}(z)| = |I_0^+| \sqrt{\ldots \ldots \ldots} \]

\[ = |I_0^+| \sqrt{1 + |\Gamma|^2 - 2|\Gamma| \cos(2\beta z + \theta_r)} \]

It's more convenient to plot \( |\tilde{V}(z)|^2 \) and \( |\tilde{I}(z)|^2 \) than the amplitudes.

\[ 2/\beta \gamma = 2\pi \]

\[ \gamma = \frac{2\pi}{2/\beta} = \frac{\lambda}{2} \]

\[ z = \gamma / z \]

\[ |\tilde{V}(z)|^2 = (1 + |\Gamma|^2) |V_0^+|^2 \]

\[ |\tilde{I}(z)|^2 = (1 - |\Gamma|^2) |V_0^+|^2 \]
\[ |\vec{V}(z)| = |V_0^+| \sqrt{1 + |\Gamma|^2 + 2|\Gamma| \cos (2\beta z + \theta_r)} \]

Interference term

Constructive interference
\[
(\vec{V} |_{\text{max}} = |V_0^+| \sqrt{1 + |\Gamma|^2 + 2|\Gamma|} = |V_0^+| (1 + |\Gamma|)
\]

Destructive interference
\[
(\vec{V} |_{\text{min}} = |V_0^+| \sqrt{1 + |\Gamma|^2 - 2|\Gamma|} = |V_0^+| (1 - |\Gamma|)
\]

Pay attention to the max, min, and average values

In this plot, we have assumed a special case \( \theta_r = 0 \). Can you think of a kind of load that leads to \( \theta_r = 0 \)? Question: In general, what’s the condition for \( \theta_r = 0 \)?
We stated that it’s more convenient to plot the amplitudes squared than the amplitudes themselves.
But how does the plot of $|\tilde{V}(z)|$ look?

It looks like this:

Work out the max and min values.
Notice the important difference between its shape and that of $|\tilde{V}(z)|$. 
Now we define the voltage standing wave ratio (VSWR), or simply standing wave ratio (SWR)

\[
|\tilde{V}|_{\text{max}} = |V_0^+| \sqrt{1 + |\Gamma|^2 + 2|\Gamma|} \\
|\tilde{V}|_{\text{min}} = |V_0^+| \sqrt{1 + |\Gamma|^2 - 2|\Gamma|}
\]

Special (extreme) cases:

\[|\Gamma| = 1, \quad S = \infty \quad \Rightarrow \quad \text{All standing wave. } |\tilde{V}|_{\text{min}} = 0\]

(Recall short & open. Other such cases to be discussed)

\[|\Gamma| = 0, \quad S = 1 \quad \Rightarrow \quad \text{All traveling wave. No reflection.} \]

(What’s the condition for this? How does the plot look?)
Slotted line

A tool to measure impedance. See in the textbook, Fig. 2-16 (pp. 74 in 7/E, pp. 73 in 6/E, or pp. 60 in 5/E)