Chapter 1

Section 1-3: Traveling Waves

HW1:P1

Problem 1.1 A 2-kHz sound wave traveling in the x-direction in air was observed to have a differential pressure $p(x,t) = 10$ N/m^2 at $x = 0$ and $t = 50$ μs. If the reference phase of $p(x,t)$ is 36°, find a complete expression for $p(x,t)$. The velocity of sound in air is 330 m/s.

Solution: The general form is given by Eq. (1.17),

$$p(x,t) = A\cos\left(\frac{2\pi t}{T} - \frac{2\pi x}{\lambda} + \phi_0\right),$$

or, you could use ω, β.

where it is given that $\phi_0 = 36°$. From Eq. (1.26), $T = 1/f = 1/(2\times10^3) = 0.5$ ms. From Eq. (1.27),

$$\lambda = \frac{u_p}{f} = \frac{330\text{ m/s}}{2\times10^3\text{ /s}} = 0.165 \text{ m}.$$

$\omega = 2\pi f$

$u_p = \frac{\omega}{\beta} \Rightarrow \beta = \frac{\omega}{u_p}$

Also, since

$$p(x=0,\ t=50\ \mu\text{s}) = 10\ (\text{N/m}^2) = A\cos\left(\frac{2\pi\times50\times10^{-6}}{5\times10^{-4}} + 36°\frac{\pi\text{ rad}}{180°}\right)$$
$$= A\cos(1.26\text{ rad}) = 0.31A,$$

good practice to carry the units in equations

in italic font, signifying it is a variable (unit already included)

it follows that $A = 10/0.31 = 32.36$ N/m^2. So, with t in (s) and x in (m),

$$p(x,t) = 32.36\cos\left(2\pi\times10^6\frac{t}{500} - 2\pi\times10^3\frac{x}{165} + 36°\right) \quad (\text{N/m}^2)$$
$$= 32.36\cos(4\pi\times10^3 t - 12.12\pi x + 36°) \quad (\text{N/m}^2).$$

HW1:P2

Problem 1.2 For the pressure wave described in Example 1-1, plot

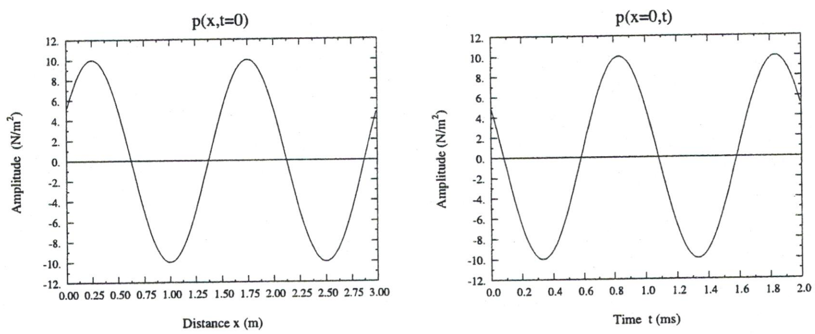
(a) $p(x,t)$ versus x at $t = 0$,

(b) $p(x,t)$ versus t at $x = 0$.

Be sure to use appropriate scales for x and t so that each of your plots covers at least two cycles.

Solution: Refer to Fig. P1.2(a) and Fig. P1.2(b).

(on next page)



(a) (b)

Figure P1.2: (a) Pressure wave as a function of distance at $t = 0$ and (b) pressure wave as a function of time at $x = 0$.

HW1:P3

Problem 1.3 A harmonic wave traveling along a string is generated by an oscillator that completes 180 vibrations per minute. If it is observed that a given crest, or maximum, travels 300 cm in 10 s, what is the wavelength?

Solution:

$$f = \frac{180}{60} = 3 \text{ Hz.}$$

$$u_p = \frac{300 \text{ cm}}{10 \text{ s}} = 0.3 \text{ m/s.}$$

$$\lambda = \frac{u_p}{f} = \frac{0.3}{3} = 0.1 \text{ m} = 10 \text{ cm.}$$

HW1:P4

Two waves, $y_1(t)$ and $y_2(t)$, have identical amplitudes and oscillate at the same frequency, but $y_2(t)$ leads $y_1(t)$ by a phase angle of 60°. If

$$y_1(t) = 4\cos(2\pi \times 10^3 t),$$

write down the expression appropriate for $y_2(t)$ and plot both functions over the time span from 0 to 2 ms.

Solution:

$$y_2(t) = 4\cos(2\pi \times 10^3 t + 60^\circ).$$

Notice that the angular frequency is in the unit of rad/s while the phase is given in degrees. Not the best practice.

Note: We often call a sinusoidal function of time a "wave." This is not in a strict sense, as a wave is a function of both time and position. Better to say a "waveform" (as you can see with an oscilloscope) or a "signal."

To "lead" means to be ahead of, thus the + sign. It is as if y2 started earlier: at t = 0, y1 is at phase zero while y2 is already at phase 60 deg. As the signals are single-frequency, the phase difference simply translates to a time shift: phase difference = angular frequency * time shift.

y2 "leads" y1 by 1/6 ms, which, at this particular frequency,
corresponds to 60 deg, i.e. pi/3

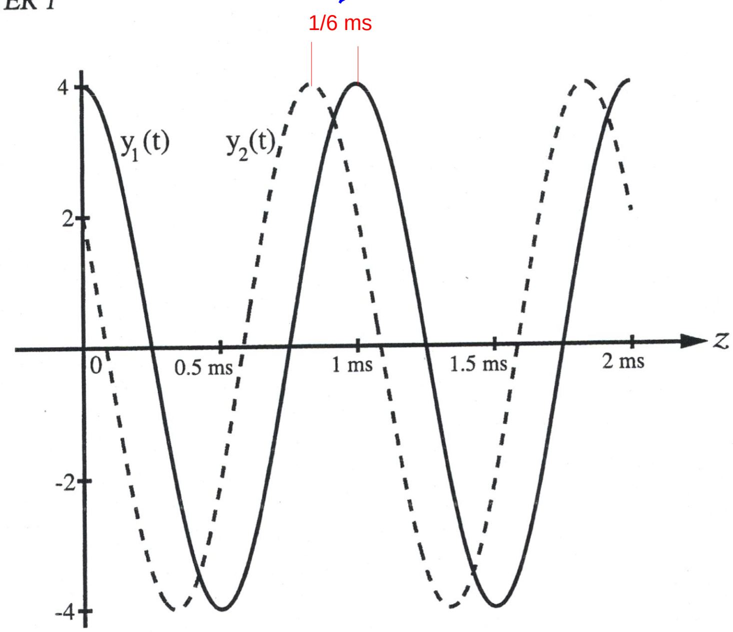


Figure P1.4: Plots of $y_1(t)$ and $y_2(t)$.

HW1:P5

The height of an ocean wave is described by the function

$$y(x,t) = 1.5\sin(0.5t - 0.6x) \quad \text{(m)}.$$

Determine the phase velocity and the wavelength and then sketch $y(x,t)$ at $t = 2$ s over the range from $x = 0$ to $x = 2\lambda$.

Solution: The given wave may be rewritten as a cosine function:

$$y(x,t) = 1.5\cos(0.5t - 0.6x - \pi/2). \text{ m}$$

You don't have to do this.
There no such thing as a "standard" way.

By comparison of this wave with Eq. (1.32),

$$y(x,t) = A\cos(\omega t - \beta x + \phi_0),$$

we deduce that

$$\omega = 2\pi f = 0.5 \text{ rad/s}, \qquad \beta = \frac{2\pi}{\lambda} = 0.6 \text{ rad/m},$$

$$u_p = \frac{\omega}{\beta} = \frac{0.5}{0.6} = 0.83 \text{ m/s}, \qquad \lambda = \frac{2\pi}{\beta} = \frac{2\pi}{0.6} = 10.47 \text{ m}.$$

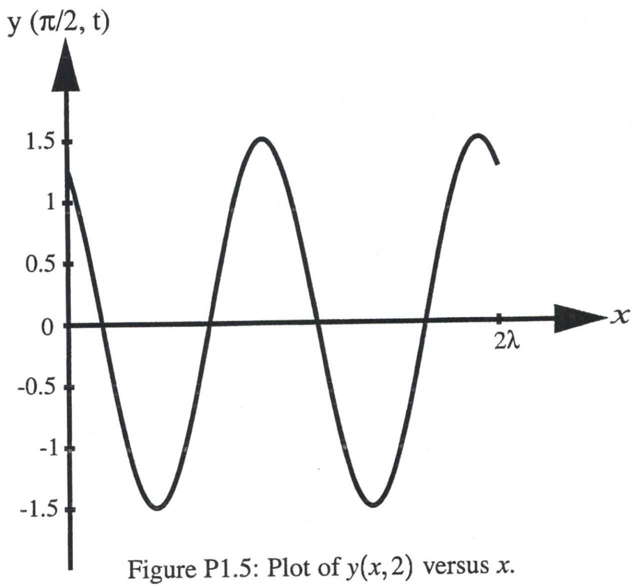


Figure P1.5: Plot of $y(x,2)$ versus x.

At $t = 2$ s, $y(x,2) = 1.5\sin(1-0.6x)$ (m), with the argument of the cosine function given in radians. Plot is shown in Fig. P1.5.

HW1:P6

Important! Make sure you really understand the solution.

A wave traveling along a string in the $+x$-direction is given by

$$y_1(x,t) = A\cos(\omega t - \beta x),$$

where $x = 0$ is the end of the string, which is tied rigidly to a wall, as shown in Fig. 1-21 (P1.6). When wave $y_1(x,t)$ arrives at the wall, a reflected wave $y_2(x,t)$ is generated. Hence, at any location on the string, the vertical displacement y_s will be the sum of the incident and reflected waves:

$$y_s(x,t) = y_1(x,t) + y_2(x,t).$$

(a) Write down an expression for $y_2(x,t)$, keeping in mind its direction of travel and the fact that the end of the string cannot move.

(b) Generate plots of $y_1(x,t)$, $y_2(x,t)$ and $y_s(x,t)$ versus x over the range $-2\lambda \leq x \leq 0$ at $\omega t = \pi/4$ and at $\omega t = \pi/2$.

Solution:

(a) Since wave $y_2(x,t)$ was caused by wave $y_1(x,t)$, the two waves must have the same angular frequency ω, and since $y_2(x,t)$ is traveling on the same string as $y_1(x,t)$,

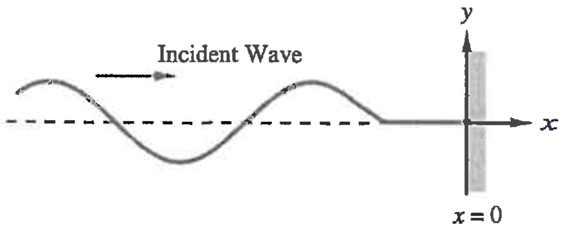


Figure P1.6: Wave on a string tied to a wall at $x = 0$ (Problem 1.6).

the two waves must have the same phase constant β. Hence, with its direction being in the negative x-direction, $y_2(x,t)$ is given by the general form

$$y_2(x,t) = B\cos(\omega t + \beta x + \phi_0), \tag{1}$$

where B and ϕ_0 are yet-to-be-determined constants. The total displacement is

$$y_s(x,t) = y_1(x,t) + y_2(x,t) = A\cos(\omega t - \beta x) + B\cos(\omega t + \beta x + \phi_0).$$

Since the string cannot move at $x = 0$, the point at which it is attached to the wall, $y_s(0,t) = 0$ for all t. Thus,

$$y_s(0,t) = A\cos\omega t + B\cos(\omega t + \phi_0) = 0. \tag{2}$$

(i) Easy Solution: The physics of the problem suggests that a possible solution for (2) is $B = -A$ and $\phi_0 = 0$, in which case we have

$$y_2(x,t) = -A\cos(\omega t + \beta x). \tag{3}$$

(ii) Rigorous Solution: By expanding the second term in (2), we have

$$A\cos\omega t + B(\cos\omega t\cos\phi_0 - \sin\omega t\sin\phi_0) = 0,$$

or

$$(A + B\cos\phi_0)\cos\omega t - (B\sin\phi_0)\sin\omega t = 0. \tag{4}$$

This equation has to be satisfied for all values of t. At $t = 0$, it gives

$$A + B\cos\phi_0 = 0, \tag{5}$$

The "rigorous solution" is not necessary. It just proves the uniqueness of the solution (meaning A = -B is the only solution) from a mathematical point of view.

and at $\omega t = \pi/2$, (4) gives

$$B \sin \phi_0 = 0. \tag{6}$$

Equations (5) and (6) can be satisfied simultaneously only if

$$A = B = 0 \tag{7}$$

or

$$A = -B \quad \text{and} \quad \phi_0 = 0. \tag{8}$$

Clearly (7) is not an acceptable solution because it means that $y_1(x,t) = 0$, which is contrary to the statement of the problem. The solution given by (8) leads to (3).

(b) At $\omega t = \pi/4$,

$$y_1(x,t) = A\cos(\pi/4 - \beta x) = A\cos\left(\frac{\pi}{4} - \frac{2\pi x}{\lambda}\right),$$

$$y_2(x,t) = -A\cos(\omega t + \beta x) = -A\cos\left(\frac{\pi}{4} + \frac{2\pi x}{\lambda}\right).$$

Plots of y_1, y_2, and y_s are shown in Fig. P1.6(b).

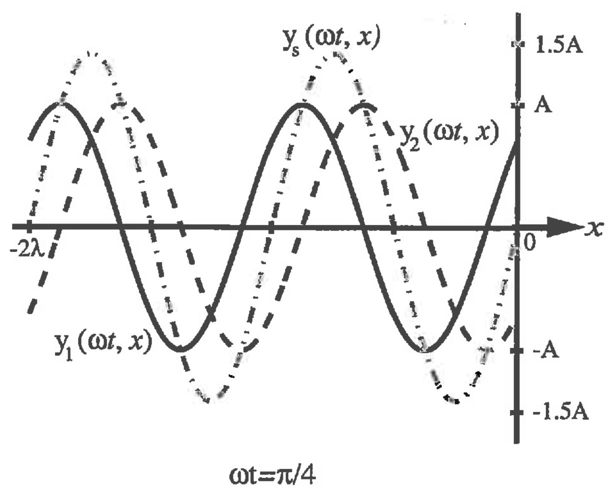


Figure P1.6: (b) Plots of y_1, y_2, and y_s versus x at $\omega t = \pi/4$.

At $\omega t = \pi/2$,

$$y_1(x,t) = A\cos(\pi/2 - \beta x) = A\sin\beta x = A\sin\frac{2\pi x}{\lambda},$$

$$y_2(x,t) = -A\cos(\pi/2+\beta x) = A\sin\beta x = A\sin\frac{2\pi x}{\lambda}.$$

Plots of y_1, y_2, and y_3 are shown in Fig. P1.6(c).

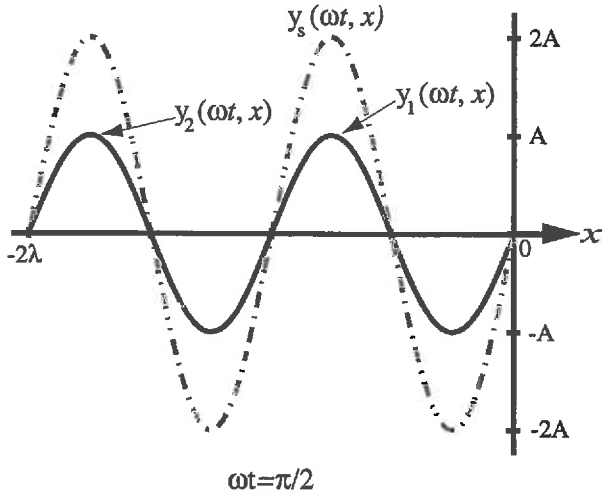


Figure P1.6: (c) Plots of y_1, y_2, and y_s versus x at $\omega t = \pi/2$.

HW1:P7

Two waves on a string are given by the following functions:

$$y_1(x,t) = 4\cos(20t - 30x) \quad \text{(cm)},$$
$$y_2(x,t) = -4\cos(20t + 30x) \quad \text{(cm)},$$

where x is in centimeters. The waves are said to interfere constructively when their superposition $|y_s| = |y_1 + y_2|$ is a maximum and they interfere destructively when $|y_s|$ is a minimum.

(a) What are the directions of propagation of waves $y_1(x,t)$ and $y_2(x,t)$?

(b) At $t = (\pi/50)$ s, at what location x do the two waves interfere constructively, and what is the corresponding value of $|y_s|$?

(c) At $t = (\pi/50)$ s, at what location x do the two waves interfere destructively, and what is the corresponding value of $|y_s|$?

Solution:

(a) $y_1(x,t)$ is traveling in positive x-direction. $y_2(x,t)$ is traveling in negative x-direction.

(b) At $t = (\pi/50)$ s, $y_s = y_1 + y_2 = 4[\cos(0.4\pi - 30x) - \cos(0.4\pi + 3x)]$. Using the formulas from Appendix C,

$$2 \sin x \sin y = \cos(x - y) - (\cos x + y),$$

we have

$$y_s = 8 \sin(0.4\pi) \sin 30x = 7.61 \sin 30x.$$ Missing unit; not good practice

Hence,

$$|y_s|_{max} = 7.61 \text{ cm}$$

and it occurs when $\sin 30x = 1$, or $30x = \frac{\pi}{2} + 2n\pi$, or $x = \left(\frac{\pi}{60} + \frac{2n\pi}{30}\right)$ cm, where $n = 0, 1, 2, \ldots$.

(c) $|y_s|_{min} = 0$ and it occurs when $30x = n\pi$, or $x = \frac{n\pi}{30}$ cm.

HW1:P8 (1.9 in 6/E)

Give expressions for $y(x,t)$ for a sinusoidal wave traveling along a string in the negative x-direction, given that $y_{max} = 40$ cm, $\lambda = 30$ cm, $f = 10$ Hz, and

(a) $y(x,0) = 0$ at $x = 0$,

(b) $y(x,0) = 0$ at $x = 7.5$ cm. Notice difference from problem in textbook

Solution: For a wave traveling in the negative x-direction, we use Eq. (1.17) with $\omega = 2\pi f = 20\pi$ (rad/s), $\beta = 2\pi/\lambda = 2\pi/0.3 = 20\pi/3$ (rad/s), $A = 40$ cm, and x assigned a positive sign:

$$y(x,t) = 40 \cos\left(20\pi t + \frac{20\pi}{3}x + \phi_0\right) \quad \text{(cm)},$$

with x in meters.

(a) $y(0,0) = 0 = 40 \cos \phi_0$. Hence, $\phi_0 = \pm\pi/2$, and

$$y(x,t) = 40 \cos\left(20\pi t + \frac{20\pi}{3}x \pm \frac{\pi}{2}\right)$$
$$= \begin{cases} -40 \sin\left(20\pi t + \frac{20\pi}{3}x\right) \text{ (cm)}, & \text{if } \phi_0 = \pi/2, \\ 40 \sin\left(20\pi t + \frac{20\pi}{3}x\right) \text{ (cm)}, & \text{if } \phi_0 = -\pi/2. \end{cases}$$

(b) At $x = 7.5 \text{ cm} = 7.5 \times 10^{-2}$ m, $y = 0 = 40 \cos(\pi/2 + \phi_0)$. Hence, $\phi_0 = 0$ or π, and

3.75 cm in textbook

Pi/4 or -3Pi/4

$$y(x,t) = \begin{cases} 40 \cos\left(20\pi t + \frac{20\pi}{3}x\right) \text{ (cm)}, & \text{if } \phi_0 = 0, \\ -40 \cos\left(20\pi t + \frac{20\pi}{3}x\right) \text{ (cm)}, & \text{if } \phi_0 = \pi. \end{cases}$$

Change accordingly for Phi_0 values of Pi/4, -3Pi/4

Hence, $y_2(t)$ lags $y_1(t)$ by 54°.

Problem 1.12 The voltage of an electromagnetic wave traveling on a transmission line is given by $v(z,t) = 5e^{-\alpha z}\sin(4\pi \times 10^9 t - 20\pi z)$ (V), where z is the distance in meters from the generator.

(a) Find the frequency, wavelength, and phase velocity of the wave.

(b) At $z = 2$ m, the amplitude of the wave was measured to be 1 V. Find α.

Solution:

(a) This equation is similar to that of Eq. (1.28) with $\omega = 4\pi \times 10^9$ rad/s and $\beta = 20\pi$ rad/m. From Eq. (1.29a), $f = \omega/2\pi = 2 \times 10^9$ Hz $= 2$ GHz; from Eq. (1.29b), $\lambda = 2\pi/\beta = 0.1$ m. From Eq. (1.30),

$$u_p = \omega/\beta = 2 \times 10^8 \text{ m/s}.$$

(b) Using just the amplitude of the wave,

$$1 = 5e^{-\alpha 2}, \qquad \alpha = \frac{-1}{2\text{ m}} \ln\left(\frac{1}{5}\right) = 0.81 \text{ Np/m}.$$

HW1:P9 (1.14 in 6/E)

A certain electromagnetic wave traveling in sea water was observed to have an amplitude of 98.02 (V/m) at a depth of 10 m and an amplitude of 81.87 (V/m) at a depth of 100 m. What is the attenuation constant of sea water?

Solution: The amplitude has the form $Ae^{\alpha z}$. At $z = 10$ m,

$$Ae^{-10\alpha} = 98.02$$

and at $z = 100$ m,

$$Ae^{-100\alpha} = 81.87$$

The ratio gives

$$\frac{e^{-10\alpha}}{e^{-100\alpha}} = \frac{98.02}{81.87} = 1.20$$

or

$$e^{-10\alpha} = 1.2e^{-100\alpha}.$$

Taking the natural log of both sides gives

$$\ln(e^{-10\alpha}) = \ln(1.2e^{-100\alpha}),$$
$$-10\alpha = \ln(1.2) - 100\alpha,$$
$$90\alpha = \ln(1.2) = 0.18.$$

Hence,

$$\alpha = \frac{0.18}{90} = 2 \times 10^{-3} \quad \text{(Np/m)}.$$

Section 1-6: Phasors

Problem 1.21 A voltage source given by $v_s(t) = 25\cos(2\pi \times 10^3 t - 30°)$ (V) is connected to a series RC load as shown in Fig. 1-19. If $R = 1$ MΩ and $C = 200$ pF, obtain an expression for $v_c(t)$, the voltage across the capacitor.

Solution: In the phasor domain, the circuit is a voltage divider, and

$$\widetilde{V}_c = \widetilde{V}_s \frac{1/j\omega C}{R + 1/j\omega C} = \frac{\widetilde{V}_s}{(1 + j\omega RC)}.$$

Now $\widetilde{V}_s = 25e^{-j30°}$ V with $\omega = 2\pi \times 10^3$ rad/s, so

$$\widetilde{V}_c = \frac{25e^{-j30°} \text{ V}}{1 + j((2\pi \times 10^3 \text{ rad/s}) \times (10^6\ \Omega) \times (200 \times 10^{-12} \text{ F}))}$$
$$= \frac{25e^{-j30°} \text{ V}}{1 + j2\pi/5} = 15.57e^{-j81.5°} \text{ V}.$$

Converting back to an instantaneous value,

$$v_c(t) = \mathfrak{Re}\widetilde{V}_c e^{j\omega t} = \mathfrak{Re}15.57e^{j(\omega t - 81.5°)} \text{ V} = 15.57\cos(2\pi \times 10^3 t - 81.5°) \text{ V},$$

where t is expressed in seconds.

HW1:P10 (1.26 in 6/E)

Find the phasors of the following time functions:

(a) $v(t) = 3\cos(\omega t - \pi/3)$ (V), Amplitude is 9 in 7/E and 6/E of textbook
(b) $v(t) = 12\sin(\omega t + \pi/4)$ (V),
(c) $i(x,t) = 2e^{-3x}\sin(\omega t + \pi/6)$ (A), Again, amplitude different in newer versions
(d) $i(t) = -2\cos(\omega t + 3\pi/4)$ (A),
(e) $i(t) = 4\sin(\omega t + \pi/3) + 3\cos(\omega t - \pi/6)$ (A).

Solution:

(a) $\widetilde{V} = 3e^{-j\pi/3}$ V.

(b) $v(t) = 12\sin(\omega t + \pi/4) = 12\cos(\pi/2 - (\omega t + \pi/4)) = 12\cos(\omega t - \pi/4)$ V, $\widetilde{V} = 12e^{-j\pi/4}$ V.

(c)

$$i(t) = 2e^{-3x}\sin(\omega t + \pi/6) \text{ A} = 2e^{-3x}\cos(\pi/2 - (\omega t + \pi/6)) \text{ A}$$
$$= 2e^{-3x}\cos(\omega t - \pi/3) \text{ A},$$
$$\widetilde{I} = 2e^{-3x}e^{-j\pi/3} \text{ A}.$$

(d)

$$i(t) = -2\cos(\omega t + 3\pi/4),$$
$$\widetilde{I} = -2e^{j3\pi/4} = 2e^{-j\pi}e^{j3\pi/4} = 2e^{-j\pi/4} \text{ A}.$$

(e)

$$\begin{aligned} i(t) &= 4\sin(\omega t + \pi/3) + 3\cos(\omega t - \pi/6) \\ &= 4\cos[\pi/2 - (\omega t + \pi/3)] + 3\cos(\omega t - \pi/6) \\ &= 4\cos(-\omega t + \pi/6) + 3\cos(\omega t - \pi/6) \\ &= 4\cos(\omega t - \pi/6) + 3\cos(\omega t - \pi/6) = 7\cos(\omega t - \pi/6), \end{aligned}$$
$$\widetilde{I} = 7e^{-j\pi/6} \text{ A}.$$

HW1:P11 Find the instantaneous time sinusoidal functions corresponding to the following phasors:

(a) $\widetilde{V} = -5e^{j\pi/3}$ (V),
(b) $\widetilde{V} = j6e^{-j\pi/4}$ (V),
(c) $\widetilde{I} = (6 + j8)$ (A),
(d) $\widetilde{I} = -3 + j2$ (A),
(e) $\widetilde{I} = j$ (A),
(f) $\widetilde{I} = 2e^{j\pi/6}$ (A).

Solution:

(a)

$$\widetilde{V} = -5e^{j\pi/3} \text{ V} = 5e^{j(\pi/3 - \pi)} \text{ V} = 5e^{-j2\pi/3} \text{ V},$$
$$v(t) = 5\cos(\omega t - 2\pi/3) \text{ V}.$$

(b)

$$\widetilde{V} = j6e^{-j\pi/4} \text{ V} = 6e^{j(-\pi/4 + \pi/2)} \text{ V} = 6e^{j\pi/4} \text{ V},$$
$$v(t) = 6\cos(\omega t + \pi/4) \text{ V}.$$

(c)

$$\widetilde{I} = (6 + j8) \text{ A} = 10e^{j53.1^\circ} \text{ A},$$
$$i(t) = 10\cos(\omega t + 53.1^\circ) \text{ A}.$$

(d)

$$\widetilde{I} = -3 + j2 = 3.61\, e^{j146.31^\circ},$$
$$i(t) = \mathfrak{Re}\{3.61\, e^{j146.31^\circ} e^{j\omega t}\} = 3.61\cos(\omega t + 146.31^\circ) \text{ A}.$$

(e)

$$\tilde{I} = j = e^{j\pi/2},$$
$$i(t) = \mathfrak{Re}\{e^{j\pi/2}e^{j\omega t}\} = \cos(\omega t + \pi/2) = -\sin\omega t \text{ A}.$$

(f)

$$\tilde{I} = 2e^{j\pi/6},$$
$$i(t) = \mathfrak{Re}\{2e^{j\pi/6}e^{j\omega t}\} = 2\cos(\omega t + \pi/6) \text{ A}.$$

Problem 1.24 A series RLC circuit is connected to a generator with a voltage $v_s(t) = V_0\cos(\omega t + \pi/3)$ (V).

(a) Write down the voltage loop equation in terms of the current $i(t)$, R, L, C, and $v_s(t)$.

(b) Obtain the corresponding phasor-domain equation.

(c) Solve the equation to obtain an expression for the phasor current $\tilde{I}$.

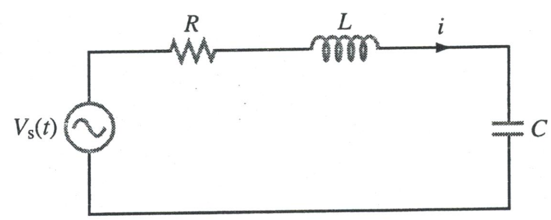


Figure P1.24: RLC circuit.

Solution:

(a) $v_s(t) = Ri + L\dfrac{di}{dt} + \dfrac{1}{C}\displaystyle\int i\,dt.$

(b) In phasor domain: $\tilde{V}_s = R\tilde{I} + j\omega L\tilde{I} + \dfrac{\tilde{I}}{j\omega C}.$

(c) $\tilde{I} = \dfrac{\tilde{V}_s}{R + j(\omega L - 1/\omega C)} = \dfrac{V_0 e^{j\pi/3}}{R + j(\omega L - 1/\omega C)} = \dfrac{\omega C V_0 e^{j\pi/3}}{\omega RC + j(\omega^2 LC - 1)}.$

Problem 1.25 A wave traveling along a string is given by

$$y(x,t) = 2\sin(4\pi t + 10\pi x) \quad \text{(cm)}$$