Chapter 1

Section 1-3: Traveling Waves

Problem 1.1 A 2-kHz sound wave traveling in the x-direction in air was observed to have a differential pressure \( p(x,t) = 10 \text{ N/m}^2 \) at \( x = 0 \) and \( t = 50 \mu s \). If the reference phase of \( p(x,t) \) is 36°, find a complete expression for \( p(x,t) \). The velocity of sound in air is 330 m/s.

Solution: The general form is given by Eq. (1.17),

\[
p(x,t) = A \cos \left( \frac{2\pi}{T} - \frac{2\pi x}{\lambda} + \phi_0 \right),
\]

or, you could use \( \omega, \beta \).

where it is given that \( \phi_0 = 36^\circ \). From Eq. (1.26), \( T = 1/f = 1/(2 \times 10^3) = 0.5 \) ms.

From Eq. (1.27),

\[
\lambda = \frac{v_p}{f} = \frac{330}{2 \times 10^3} = 0.165 \text{ m}.
\]

Also, since

\[
p(x = 0, \ t = 50 \mu s) = 10 \text{ (N/m}^2) = A \cos \left( \frac{2\pi \times 5 \times 10^{-6}}{5 \times 10^{-4}} + 36^\circ \frac{\pi \text{ rad}}{180^\circ} \right)
\]

\[
= A \cos(1.26 \text{ rad}) = 0.31A,
\]

it follows that \( A = 10/0.31 = 32.36 \text{ N/m}^2 \). So, with \( t \) in (s) and \( x \) in (m),

\[
p(x,t) = 32.36 \cos \left( 2\pi \times 10^3 \frac{t}{500} - 2\pi \times 10^3 \frac{x}{165} + 36^\circ \right) \text{ (N/m}^2) = 32.36 \cos(4\pi \times 10^3 t - 12.12\pi x + 36^\circ) \text{ (N/m}^2).
\]

Problem 1.2 For the pressure wave described in Example 1-1, plot

(a) \( p(x,t) \) versus \( x \) at \( t = 0 \),

(b) \( p(x,t) \) versus \( t \) at \( x = 0 \).

Be sure to use appropriate scales for \( x \) and \( t \) so that each of your plots covers at least two cycles.

Solution: Refer to Fig. P1.2(a) and Fig. P1.2(b).
10 should be scaled to 32.36.

Figure P1.2: (a) Pressure wave as a function of distance at \( t = 0 \) and (b) pressure wave as a function of time at \( x = 0 \).

Problem 1.3 A harmonic wave traveling along a string is generated by an oscillator that completes 180 vibrations per minute. If it is observed that a given crest, or maximum, travels 300 cm in 10 s, what is the wavelength?

Solution:

\[
\begin{align*}
  f &= \frac{180}{60} = 3 \text{ Hz.} \\
  \nu_p &= \frac{300 \text{ cm}}{10 \text{ s}} = 0.3 \text{ m/s}. \\
  \lambda &= \frac{\nu_p}{f} = \frac{0.3}{3} = 0.1 \text{ m} = 10 \text{ cm}.
\end{align*}
\]

HW1: P3

Two waves, \( y_1(t) \) and \( y_2(t) \), have identical amplitudes and oscillate at the same frequency, but \( y_2(t) \) leads \( y_1(t) \) by a phase angle of 60°. If

\[
y_1(t) = 4 \cos(2\pi \times 10^3 t),
\]

write down the expression appropriate for \( y_2(t) \) and plot both functions over the time span from 0 to 2 ms.

Solution:

\[
y_2(t) = 4 \cos(2\pi \times 10^3 t + 60^\circ).
\]

Note: We often call a sinusoidal function of time a "wave." This is not in a strict sense, as a wave is a function of both time and position. Better to say a "waveform" (as you can see with an oscilloscope) or a "signal."

To "lead" means to be ahead of, thus the + sign. It is as if \( y_2 \) started earlier: at \( t = 0 \), \( y_1 \) is at phase zero while \( y_2 \) is already at phase 60 deg. As the signals are single-frequency, the phase difference simply translates to a time shift: phase difference = angular frequency \( \times \) time shift. Here
The height of an ocean wave is described by the function
\[ y(x, t) = 1.5 \sin(0.5t - 0.6x) \] (m).

Determine the phase velocity and the wavelength and then sketch \( y(x, t) \) at \( t = 2 \text{ s} \) over the range from \( x = 0 \) to \( x = 2\lambda \).

**Solution.** The given wave may be rewritten as a cosine function:
\[ y(x, t) = 1.5 \cos(0.5t - 0.6x - \pi/2). \]

By comparison of this wave with Eq. (1.32),
\[ y(x, t) = A \cos(\omega x - \beta x + \phi_0), \]
we deduce that
\[ \omega = 2\pi f = 0.5 \text{ rad/s}, \quad \beta = \frac{2\pi}{\lambda} = 0.6 \text{ rad/m}, \]
\[ u_p = \frac{\omega}{\beta} = \frac{0.5}{0.6} = 0.83 \text{ m/s}, \quad \lambda = \frac{2\pi}{\beta} = \frac{2\pi}{0.6} = 10.47 \text{ m}. \]
At $t = 2$ s, $y(x, 2) = 1.5 \sin(1 - 0.6x)$ (m), with the argument of the cosine function given in radians. Plot is shown in Fig. P1.5.

A wave traveling along a string in the +x-direction is given by

$$y_1(x, t) = A \cos(\omega t - \beta x),$$

where $x = 0$ is the end of the string, which is tied rigidly to a wall, as shown in Fig. 1-21 (P1.6). When wave $y_1(x, t)$ arrives at the wall, a reflected wave $y_2(x, t)$ is generated. Hence, at any location on the string, the vertical displacement $y_s$ will be the sum of the incident and reflected waves:

$$y_s(x, t) = y_1(x, t) + y_2(x, t).$$

(a) Write down an expression for $y_2(x, t)$, keeping in mind its direction of travel and the fact that the end of the string cannot move.

(b) Generate plots of $y_1(x, t)$, $y_2(x, t)$ and $y_s(x, t)$ versus $x$ over the range $-2\lambda \leq x \leq 0$ at $\omega \tau = \pi/4$ and at $\omega \tau = \pi/2$.

Solution:

(a) Since wave $y_2(x, t)$ was caused by wave $y_1(x, t)$, the two waves must have the same angular frequency $\omega$, and since $y_2(x, t)$ is traveling on the same string as $y_1(x, t)$,
the two waves must have the same phase constant $\beta$. Hence, with its direction being in the negative $x$-direction, $y_2(x,t)$ is given by the general form

$$y_2(x,t) = B \cos(\omega x + \beta x + \phi_0),$$  \hspace{1cm} (1)$$

where $B$ and $\phi_0$ are yet-to-be-determined constants. The total displacement is

$$y_s(x,t) = y_1(x,t) + y_2(x,t) = A \cos(\omega x - \beta x) + B \cos(\omega x + \beta x + \phi_0).$$

Since the string cannot move at $x = 0$, the point at which it is attached to the wall, $y_s(0,t) = 0$ for all $t$. Thus,

$$y_s(0,t) = A \cos \omega t + B \cos(\omega t + \phi_0) = 0. \hspace{1cm} (2)$$

(i) Easy Solution: The physics of the problem suggests that a possible solution for (2) is $B = A$ and $\phi_0 = 0$, in which case we have

$$y_2(x,t) = -A \cos(\omega t + \beta x). \hspace{1cm} (3)$$

(ii) Rigorous Solution: By expanding the second term in (2), we have

$$A \cos \omega t + B(\cos \omega t \cos \phi_0 - \sin \omega t \sin \phi_0) = 0,$$

or

$$(A + B \cos \phi_0) \cos \omega t - (B \sin \phi_0) \sin \omega t = 0. \hspace{1cm} (4)$$

This equation has to be satisfied for all values of $t$. At $t = 0$, it gives

$$A + B \cos \phi_0 = 0, \hspace{1cm} (5)$$
The "rigorous solution" is not necessary. It just proves the uniqueness of the solution (meaning \( A = -B \) is the only solution) from a mathematical point of view.

and at \( \omega t = \pi/2 \), (4) gives

\[ B \sin \phi_0 = 0. \]  

(6)

Equations (5) and (6) can be satisfied simultaneously only if

\[ A = B = 0 \]  

(7)

or

\[ A = -B \quad \text{and} \quad \phi_0 = 0. \]  

(8)

Clearly (7) is not an acceptable solution because it means that \( y_1(x,t) = 0 \), which is contrary to the statement of the problem. The solution given by (8) leads to (3).

(b) At \( \omega t = \pi/4 \),

\[ y_1(x,t) = A \cos(\pi/4 - \beta x) = A \cos \left( \frac{\pi}{4} - \frac{2\pi x}{\lambda} \right), \]

\[ y_2(x,t) = -A \cos(\omega t + \beta x) = -A \cos \left( \frac{\pi}{4} + \frac{2\pi x}{\lambda} \right). \]

Plots of \( y_1 \), \( y_2 \), and \( y_3 \) are shown in Fig. P1.6(b).

Figure P1.6: (b) Plots of \( y_1 \), \( y_2 \), and \( y_3 \) versus \( x \) at \( \omega t = \pi/4 \).

At \( \omega t = \pi/2 \),

\[ y_1(x,t) = A \cos(\pi/2 - \beta x) = A \sin \beta x = A \sin \frac{2\pi x}{\lambda}, \]
\[ y_2(x, t) = -A \cos(\pi/2 + \beta x) = A \sin \beta x = A \sin \frac{2\pi x}{\lambda} \, . \]

Plots of \( y_1, y_2, \) and \( y_3 \) are shown in Fig. P1.6(c).

\[ \omega t = \pi/2 \]

Figure P1.6: (c) Plots of \( y_1, y_2, \) and \( y_3 \) versus \( x \) at \( \omega t = \pi/2 \).

Two waves on a string are given by the following functions:

\[ y_1(x, t) = 4 \cos(20t - 30x) \, \text{(cm)}, \]
\[ y_2(x, t) = -4 \cos(20t + 30x) \, \text{(cm)}, \]

where \( x \) is in centimeters. The waves are said to interfere constructively when their superposition \( |y_4| = |y_1 + y_2| \) is a maximum and they interfere destructively when \( |y_4| \) is a minimum.

(a) What are the directions of propagation of waves \( y_1(x, t) \) and \( y_2(x, t) \)?

(b) At \( t = (\pi/50) \, \text{s} \), at what location \( x \) do the two waves interfere constructively, and what is the corresponding value of \( |y_4| \)?

(c) At \( t = (\pi/50) \, \text{s} \), at what location \( x \) do the two waves interfere destructively, and what is the corresponding value of \( |y_4| \)?

Solution:

(a) \( y_1(x, t) \) is traveling in positive \( x \)-direction. \( y_2(x, t) \) is traveling in negative \( x \)-direction.
(b) At \( t = (\pi/50) \) s, \( y_s = y_1 + y_2 = 4[\cos(0.4\pi - 30x) - \cos(0.4\pi + 3x)] \). Using the formulas from Appendix C,

\[
2\sin x \sin y = \cos(x - y) - (\cos x + y),
\]

we have

\[
y_s = 8\sin(0.4\pi) \sin 30x = 7.61 \sin 30x.
\]

Hence,

\[
|y_s|_{\text{max}} = 7.61
\]

and it occurs when \( \sin 30x = 1 \), or \( 30x = \frac{\pi}{2} + 2n\pi \), or \( x = \left( \frac{\pi}{60} + \frac{2n\pi}{30} \right) \) cm, where \( n = 0, 1, 2, \ldots \).

(c) \( |y_s|_{\text{min}} = 0 \) and it occurs when \( 30x = n\pi \), or \( x = \frac{n\pi}{30} \) cm.

Give expressions for \( y(x,t) \) for a sinusoidal wave traveling along a string in the negative \( x \)-direction, given that \( y_{\text{max}} = 40 \) cm, \( \lambda = 30 \) cm, \( f = 10 \) Hz, and

(a) \( y(x,0) = 0 \) at \( x = 0 \),

(b) \( y(x,0) = 0 \) at \( x = 7.5 \) cm. Notice difference from problem in textbook

**Solution:** For a wave traveling in the negative \( x \)-direction, we use Eq. (1.17) with \( \omega = 2\pi f = 20\pi \) (rad/s), \( \beta = 2\pi/\lambda = 2\pi/0.3 = 20\pi/3 \) (rad/s), \( A = 40 \) cm, and \( x \) assigned a positive sign:

\[
y(x,t) = 40 \cos \left( 20\pi t + \frac{20\pi}{3}x + \phi_0 \right) \text{ (cm)},
\]

with \( x \) in meters.

(a) \( y(0,0) = 0 = 40 \cos \phi_0 \). Hence, \( \phi_0 = \pm \pi/2 \), and

\[
y(x,t) = 40 \cos \left( 20\pi t + \frac{20\pi}{3}x \pm \frac{\pi}{2} \right)
\]

\[
= \begin{cases} 
-40 \sin \left( 20\pi t + \frac{20\pi}{3}x \right) \text{ (cm)}, & \text{if } \phi_0 = \pi/2, \\
40 \sin \left( 20\pi t + \frac{20\pi}{3}x \right) \text{ (cm)}, & \text{if } \phi_0 = -\pi/2.
\end{cases}
\]

(b) At \( x = 7.5 \) cm = \( 7.5 \times 10^{-2} \) m, \( y = 0 = 40 \cos(\pi/2 + \phi_0) \). Hence, \( \phi_0 = 0 \) or \( \pi \), and

3.75 cm in textbook

\[Pi/4 or -3Pi/4\]

\[
y(x,t) = \begin{cases} 
40 \cos \left( 20\pi t + \frac{20\pi}{3}x \right) \text{ (cm)}, & \text{if } \phi_0 = 0, \\
-40 \cos \left( 20\pi t + \frac{20\pi}{3}x \right) \text{ (cm)}, & \text{if } \phi_0 = \pi.
\end{cases}
\]

Change accordingly for Phi_0 values of Pi/4, -3Pi/4
Hence, $y_2(t)$ lags $y_1(t)$ by $54^\circ$.

**Problem 1.12** The voltage of an electromagnetic wave traveling on a transmission line is given by $V(z, t) = 5e^{-\alpha z}$ $\sin(4\pi \times 10^9 t - 20\pi z)$ (V), where $z$ is the distance in meters from the generator.

(a) Find the frequency, wavelength, and phase velocity of the wave.
(b) At $z = 2$ m, the amplitude of the wave was measured to be 1 V. Find $\alpha$.

**Solution:**

(a) This equation is similar to that of Eq. (1.28) with $\omega = 4\pi \times 10^9$ rad/s and $\beta = 20\pi$ rad/m. From Eq. (1.29a), $f = \omega/2\pi = 2 \times 10^9$ Hz = 2 GHz; from Eq. (1.29b), $\lambda = 2\pi/\beta = 0.1$ m. From Eq. (1.30),

$$u_p = \frac{\omega}{\beta} = 2 \times 10^8 \text{ m/s}.$$  

(b) Using just the amplitude of the wave,

$$1 = 5e^{-\alpha z}, \quad \alpha = \frac{1}{2} \ln \left( \frac{1}{5} \right) = 0.81 \text{ Np/m}.$$  

A certain electromagnetic wave traveling in sea water was observed to have an amplitude of 98.02 (V/m) at a depth of 10 m and an amplitude of 81.87 (V/m) at a depth of 100 m. What is the attenuation constant of sea water?

**Solution:** The amplitude has the form $Ae^{\alpha z}$. At $z = 10$ m,

$$Ae^{-10\alpha} = 98.02$$

and at $z = 100$ m,

$$Ae^{-100\alpha} = 81.87$$

The ratio gives

$$\frac{e^{-10\alpha}}{e^{-100\alpha}} = \frac{98.02}{81.87} = 1.20$$

or

$$e^{-10\alpha} = 1.2e^{-100\alpha}.$$  

Taking the natural log of both sides gives

$$\ln(e^{-10\alpha}) = \ln(1.2e^{-100\alpha}),$$

$$-10\alpha = \ln(1.2) - 100\alpha,$$

$$90\alpha = \ln(1.2) = 0.18.$$  

Hence,

$$\alpha = \frac{0.18}{90} = 2 \times 10^{-3} \text{ (Np/m)}. $$
CHAPTER 1

Section 1-6: Phasors

Problem 1.21 A voltage source given by $v_s(t) = 25 \cos(2\pi \times 10^3 t - 30^\circ)$ (V) is connected to a series RC load as shown in Fig. 1-19. If $R = 1$ M\(\Omega\) and $C = 200$ pF, obtain an expression for $v_c(t)$, the voltage across the capacitor.

Solution: In the phasor domain, the circuit is a voltage divider, and

$$
\bar{V}_c = \bar{V}_s \frac{1/j\omega C}{R + 1/j\omega C} = \frac{\bar{V}_s}{(1 + j\omega RC)}.
$$

Now $\bar{V}_s = 25e^{-j30^\circ}$ V with $\omega = 2\pi \times 10^3$ rad/s, so

$$
\bar{V}_c = \frac{25e^{-j30^\circ}}{1 + j(2\pi \times 10^3 \text{ rad/s}) \times (10^6 \Omega) \times (200 \times 10^{-12} \text{ F})}
$$

$$
= \frac{25e^{-j30^\circ}}{1 + j(2\pi \times 81.5^\circ)} = 15.57e^{-j81.5^\circ} \text{ V.}
$$

Converting back to an instantaneous value,

$$
v_c(t) = \Re\{\bar{V}_c e^{j\omega t}\} = 15.57 \cos(2\pi \times 10^3 t - 81.5^\circ) \text{ V,}
$$

where $t$ is expressed in seconds.

---

Find the phasors of the following time functions:

(a) $v(t) = 3\cos(\omega t - \pi/3)$ (V),
(b) $v(t) = 12\sin(\omega t + \pi/4)$ (V),
(c) $i(t) = -2e^{-3t} \sin(\omega t + \pi/6)$ (A),
(d) $i(t) = -2\cos(\omega t + 3\pi/4)$ (A),
(e) $i(t) = 4\sin(\omega t + \pi/3) + 3\cos(\omega t - \pi/6)$ (A).

Solution:

(a) $\bar{V} = 3e^{-j\pi/3}$ V.
(b) $v(t) = 12\sin(\omega t + \pi/4) = 12\cos(\pi/2 - (\omega t + \pi/4)) = 12\cos(\omega t - \pi/4)$ V, $\bar{V} = 12e^{-j\pi/4}$ V.
(c) $i(t) = 2e^{-3t} \sin(\omega t + \pi/6)$ A = $2e^{-3t} \cos(\pi/2 - (\omega t + \pi/6))$ A = $2e^{-3t} \cos(\omega t - \pi/3)$ A, $\bar{i} = 2e^{-3} e^{-j\pi/3}$ A.

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HW1:P10

1.26 in 6/E

Amplitude is 9 in 7/E and 6/E of textbook
Again, amplitude different in newer versions
(d) \[ i(t) = -2 \cos(\omega t + 3\pi/4), \]
\[ \tilde{I} = -2e^{j3\pi/4} = 2e^{-j\pi/4} \]
\[ 2e^{-j\pi/4} = 2e^{-j\pi/4} \text{ A}. \]

(e) \[ i(t) = 4 \sin(\omega t + \pi/3) + 3 \cos(\omega t - \pi/6) \]
\[ = 4 \cos(\pi/2 - (\omega t + \pi/3)) + 3 \cos(\omega t - \pi/6) \]
\[ = 4 \cos(-\omega t + \pi/6) + 3 \cos(\omega t - \pi/6) \]
\[ = 4 \cos(\omega t - \pi/6) + 3 \cos(\omega t - \pi/6) = 7 \cos(\omega t - \pi/6), \]
\[ \tilde{I} = 7e^{-j\pi/6} \text{ A}. \]

Find the instantaneous time sinusoidal functions corresponding to the following phasors:

(a) \[ \tilde{V} = -5e^{j\pi/3} \text{ (V)}, \]
(b) \[ \tilde{V} = j6e^{-j\pi/4} \text{ (V)}, \]
(c) \[ \tilde{I} = (6 + j8) \text{ (A)}, \]
(d) \[ \tilde{I} = -3 + j2 \text{ (A)}, \]
(e) \[ \tilde{I} = j \text{ (A)}, \]
(f) \[ \tilde{I} = 2e^{j\pi/6} \text{ (A)}. \]

Solution:

(a) \[ \tilde{V} = -5e^{j\pi/3} \text{ V} = 5e^{j(\pi/3 - \pi)} \text{ V} = 5e^{-j2\pi/3} \text{ V}, \]
\[ v(t) = 5 \cos(\omega t - 2\pi/3) \text{ V}. \]

(b) \[ \tilde{V} = j6e^{-j\pi/4} \text{ V} = 6e^{-j(\pi/4 + \pi/2)} \text{ V} = 6e^{j3\pi/4} \text{ V}, \]
\[ v(t) = 6 \cos(\omega t + \pi/4) \text{ V}. \]

(c) \[ \tilde{I} = (6 + j8) \text{ A} = 10e^{j31.1^\circ} \text{ A}, \]
\[ i(t) = 10 \cos(\omega t + 53.1^\circ) \text{ A}. \]

(d) \[ \tilde{I} = -3 + j2 = 3.61e^{j46.31^\circ}, \]
\[ i(t) = 9 \Re \{3.61e^{j46.31^\circ}e^{j\omega t}\} = 3.61 \cos(\omega t + 146.31^\circ) \text{ A}. \]
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(e) 
\[ \tilde{I} = j e^{j\pi/2}, \]
\[ i(t) = \Re \{ e^{j\pi/2} e^{j\omega t} \} = \cos(\omega t + \pi/2) = -\sin \omega t \text{ A}. \]

(f) 
\[ \tilde{I} = 2 e^{j\pi/6}, \]
\[ i(t) = \Re \{ 2 e^{j\pi/6} e^{j\omega t} \} = 2\cos(\omega t + \pi/6) \text{ A}. \]

Problem 1.24 A series RLC circuit is connected to a generator with a voltage \( v_s(t) = V_0 \cos(\omega t + \pi/3) \) (V).

(a) Write down the voltage loop equation in terms of the current \( i(t), R, L, C, \) and \( v_s(t). \)

(b) Obtain the corresponding phasor-domain equation.

(c) Solve the equation to obtain an expression for the phasor current \( \tilde{I}. \)

\[ \begin{align*}
V_s(t) & \quad R \quad L \quad i \\
| & & & | \\
\text{C} & & & \\
\end{align*} \]

Figure P1.24: RLC circuit.

Solution:

(a) \( v_s(t) = Ri + L \frac{di}{dt} + \frac{1}{C} \int i dt. \)

(b) In phasor domain: \( \tilde{V}_S = R\tilde{I} + j\omega L\tilde{I} + \frac{\tilde{I}}{j\omega C}. \)

(c) \( \tilde{I} = \frac{\tilde{V}_S}{R + j(\omega L - 1/\omega C)} = \frac{\omega CV_0 e^{j\pi/3}}{R + j(\omega L - 1/\omega C)} = \frac{\omega CV_0 e^{j\pi/3}}{\omega RC + j(\omega^2 LC - 1)}. \)

Problem 1.25 A wave traveling along a string is given by

\[ \gamma(x,t) = 2\sin(4\pi t + 10\pi x) \text{ (cm)} \]