CHAPTER 4

Section 4-9: Boundary Conditions

Problem 4.43 With reference to Fig. 4-19, find \( E_1 \) if \( E_2 = \hat{x} 3 - \hat{y} 2 + \hat{z} 2 \) (V/m), \( \varepsilon_1 = 2\varepsilon_0 \), \( \varepsilon_2 = 18\varepsilon_0 \), and the boundary has a surface charge density \( \rho_s = 3.54 \times 10^{-11} \) (C/m\(^2\)). What angle does \( E_2 \) make with the \( z \)-axis?

Solution: We know that \( E_{1t} = E_{2t} \) for any 2 media. Hence, \( E_{1t} = E_{2t} = \hat{x} 3 - \hat{y} 2 \).
Also, \( (D_1 - D_2) \cdot \hat{n} = \rho_s \) (from Table 4.3). Hence, \( \varepsilon_1 (E_1 \cdot \hat{n}) - \varepsilon_2 (E_2 \cdot \hat{n}) = \rho_s \), which gives
\[
E_{1t} = \rho_s + \varepsilon_2 E_{2t} \frac{\varepsilon_1}{\varepsilon_1} = \frac{3.54 \times 10^{-11} + 18(2)}{2\varepsilon_0} = \frac{3.54 \times 10^{-11}}{2 \times 8.85 \times 10^{-12}} + 18 = 20 \text{ (V/m)}.
\]
Hence, \( E_1 = \hat{x} 3 - \hat{y} 2 + 200 \) (V/m). Finding the angle \( E_2 \) makes with the \( z \)-axis:
\[
E_2 \cdot \hat{z} = |E_2| \cos \theta, \quad 2 = \sqrt{9 + 4 + 4 \cos \theta}, \quad \theta = \cos^{-1} \left( \frac{2}{\sqrt{17}} \right) = 61^\circ.
\]

Problem 4.44 An infinitely long conducting cylinder of radius \( a \) has a surface charge density \( \rho_s \). The cylinder is surrounded by a dielectric medium with \( \varepsilon_r = 4 \) and contains no free charges. If the tangential component of the electric field in the region \( r \geq a \) is given by \( E_r = -\hat{\phi} \cos^2 \phi / r^2 \), find \( \rho_s \).

Solution: Let the conducting cylinder be medium 1 and the surrounding dielectric medium be medium 2. In medium 2,
\[
E_2 = \hat{r} E_r - \frac{1}{r^2} \cos^2 \phi \hat{\phi},
\]
with \( E_r \), the normal component of \( E_2 \), unknown. The surface charge density is related to \( E_r \). To find \( E_r \), we invoke Gauss's law in medium 2:
\[
\nabla \cdot \mathbf{D}_2 = 0,
\]
or
\[
\frac{1}{r} \frac{\partial}{\partial r} \left( r E_r \right) + \frac{1}{r \sin \phi} \frac{\partial}{\partial \phi} \left( \frac{1}{r^2} \cos^2 \phi \right) = 0,
\]
which leads to
\[
\frac{\partial}{\partial r} \left( r E_r \right) = \frac{\partial}{\partial \phi} \left( \frac{1}{r^2} \cos^2 \phi \right) = -\frac{2}{r^2} \sin \phi \cos \phi.
\]
Integrating both sides with respect to \( r \),
\[
\int \frac{\partial}{\partial r} \left( r E_r \right) \, dr = -2 \sin \phi \cos \phi \int \frac{1}{r^2} \, dr
\]
\[
r E_r = \frac{2}{r} \sin \phi \cos \phi,
\]
Solution: From Eq. (4.131),

$$F = -2\varepsilon_0 \frac{E^2}{2} = -2\varepsilon_0 (5 \times 10^{-4}) \left( \frac{50}{0.02} \right)^2 = -255.3 \times 10^{-9} \text{ (N)}.$$ 

**Problem 4.49** Dielectric breakdown occurs in a material whenever the magnitude of the field \( E \) exceeds the dielectric strength anywhere in that material. In the coaxial capacitor of Example 4-12,

(a) At what value of \( r \) is \( |E| \) maximum?

(b) What is the breakdown voltage if \( a = 1 \text{ cm} \), \( b = 2 \text{ cm} \), and the dielectric material is mica with \( \varepsilon_r = 6 \)?

**Solution:**

(a) From Eq. (4.114), \( E = -\varepsilon_0 \varepsilon_r \) for \( a < r < b \). Thus, it is evident that \( |E| \) is maximum at \( r = a \).

(b) The dielectric breaks down when \( |E| > 200 \text{ (MV/m)} \) (see Table 4-2), or

$$|E| = \frac{p_i}{2\pi \varepsilon_0} = \frac{p_i}{2\pi (6.854 \times 10^{-12})} = 200 \text{ (MV/m)},$$

which gives \( p_i = (200 \text{ MV/m})(2\pi)(6.854 \times 10^{-12})(0.01) = 667.6 \text{ (\muC/m)} \).

From Eq. (4.115), we can find the voltage corresponding to that charge density,

$$V = \frac{p_i}{2\pi \varepsilon} \ln \left( \frac{b}{a} \right) = \frac{667.6 \text{ (\muC/m)}}{2\pi (8.854 \times 10^{-12} \text{ F/m})} \ln(2) = 1.39 \text{ (MV)}.$$

Thus, \( V = 1.39 \text{ (MV)} \) is the breakdown voltage for this capacitor.

**Problem 4.50** An electron with charge \( Q_e = -1.6 \times 10^{-19} \text{ C} \) and mass \( m_e = 9.1 \times 10^{-31} \text{ kg} \) is injected at a point adjacent to the negatively charged plate in the region between the plates of an air-filled parallel-plate capacitor with separation of 1 cm and rectangular plates each 10 cm\(^2 \) in area Fig. 4-33 (P4.50). If the voltage across the capacitor is 10 V, find

(a) the force acting on the electron,

(b) the acceleration of the electron, and

(c) the time it takes the electron to reach the positively charged plate, assuming that it starts from rest.

**Solution:**

(a) The electric force acting on a charge \( Q_e \) is given by Eq. (4.14) and the electric field in a capacitor is given by Eq. (4.112). Combining these two relations, we have

$$F = Q_e E = Q_e \frac{V}{d} = -1.6 \times 10^{-19} \frac{10}{0.01} = -1.6 \times 10^{-16} \text{ (N)}.$$
Solution: Electrostatic potential energy is given by Eq. (4.124),

\[
W_e = \frac{1}{2} \int_\Omega \varepsilon |E|^2 d\Omega = \frac{\varepsilon_0}{2} \int_{z=0}^{3} \int_{y=0}^{2} \int_{x=-1}^{1} \left[ (x^2 + 2z)^2 + x^4 + (y+z)^2 \right] dx \, dy \, dz
\]

\[
= \frac{4\varepsilon_0}{2} \left( \left[ \left( \frac{2}{5} x^5 y z + \frac{2}{3} x^3 z y + \frac{4}{5} x^3 y y + \frac{1}{12} (y+z)^4 x \right) \right]_{x=-1}^{1} \right)_{y=0}^{2} \right)_{z=0}^{3} \]

\[
= \frac{4\varepsilon_0}{2} \left( \frac{1304}{5} \right) = 4.62 \times 10^{-9} \text{ (J)}.
\]

Problem 4.52 Figure 4-34a (P4.52(a)) depicts a capacitor consisting of two parallel, conducting plates separated by a distance \( d \). The space between the plates

![Diagram of capacitor](attachment:image.png)

(a)

![Equivalent circuit](attachment:image.png)

(b)

Figure P4.52: (a) Capacitor with parallel dielectric section, and (b) equivalent circuit.

contains two adjacent dielectrics, one with permittivity \( \varepsilon_1 \) and surface area \( A_1 \) and another with \( \varepsilon_2 \) and \( A_2 \). The objective of this problem is to show that the capacitance \( C \) of the configuration shown in Fig. 4-34a (P4.52(a)) is equivalent to two capacitances in parallel, as illustrated in Fig. 4-34b (P4.52(b)), with

\[
C = C_1 + C_2,
\]

(4.132)
CHAPTER 4

where

\[ C_1 = \frac{\varepsilon_1 A_1}{d}, \quad (4.133) \]

\[ C_2 = \frac{\varepsilon_2 A_2}{d}. \quad (4.134) \]

To this end, you are asked to proceed as follows:

(a) Find the electric fields \( E_1 \) and \( E_2 \) in the two dielectric layers.

(b) Calculate the energy stored in each section and use the result to calculate \( C_1 \) and \( C_2 \).

(c) Use the total energy stored in the capacitor to obtain an expression for \( C \). Show that Eq. (4.132) is indeed a valid result.

Solution:

Figure P4.52: (c) Electric field inside of capacitor.

(a) Within each dielectric section, \( E \) will point from the plate with positive voltage to the plate with negative voltage, as shown in Fig. P4.52(c). From \( V = Ed \),

\[ E_1 = E_2 = \frac{V}{d}. \]

(b)

\[ W_{e_1} = \frac{1}{2} \varepsilon_1 E_1^2 \cdot V = \frac{1}{2} \varepsilon_1 \frac{V^2}{d^2} \cdot A_1 d = \frac{1}{2} \varepsilon_1 V^2 \frac{A_1}{d}. \]

But, from Eq. (4.121),

\[ W_{e_1} = \frac{1}{2} C_1 V^2. \]

Hence \( C_1 = \varepsilon_1 \frac{A_1}{d} \). Similarly, \( C_2 = \varepsilon_2 \frac{A_2}{d} \).

(c) Total energy is

\[ W_e = W_{e_1} + W_{e_2} = \frac{1}{2} \frac{V^2}{d} \left( \varepsilon_1 A_1 + \varepsilon_2 A_2 \right) = \frac{1}{2} CV^2. \]
Hence,

\[ C = \frac{\varepsilon_1 A_1}{d} + \frac{\varepsilon_2 A_2}{d} = C_1 + C_2. \]

**Problem 4.53** Use the result of Problem 4.52 to determine the capacitance for each of the following configurations:

(a) conducting plates are on top and bottom faces of rectangular structure in Fig. 4-35(a) (P4.53(a)),

(b) conducting plates are on front and back faces of structure in Fig. 4-35(a) (P4.53(a)),

(c) conducting plates are on top and bottom faces of the cylindrical structure in Fig. 4-35(b) (P4.53(b)).

**Solution:**

(a) The two capacitors share the same voltage; hence they are in parallel.

\[
C_1 = \varepsilon_1 \frac{A_1}{d} = 2\varepsilon_0 \frac{(5 \times 1) \times 10^{-4}}{2 \times 10^{-2}} = 5\varepsilon_0 \times 10^{-2},
\]

\[
C_2 = \varepsilon_2 \frac{A_2}{d} = 4\varepsilon_0 \frac{(5 \times 3) \times 10^{-4}}{2 \times 10^{-2}} = 30\varepsilon_0 \times 10^{-2},
\]

\[ C = C_1 + C_2 = (5\varepsilon_0 + 30\varepsilon_0) \times 10^{-2} = 0.35\varepsilon_0 = 3.1 \times 10^{-12} \text{ F}. \]

(b)

\[
C_1 = \varepsilon_1 \frac{A_1}{d} = 2\varepsilon_0 \frac{(2 \times 1) \times 10^{-4}}{5 \times 10^{-2}} = 0.8\varepsilon_0 \times 10^{-2},
\]

\[
C_2 = \varepsilon_2 \frac{A_2}{d} = 4\varepsilon_0 \frac{(3 \times 2) \times 10^{-4}}{5 \times 10^{-2}} = \frac{24}{5} \varepsilon_0 \times 10^{-2},
\]

\[ C = C_1 + C_2 = 0.5 \times 10^{-12} \text{ F}. \]

(c)

\[
C_1 = \varepsilon_1 \frac{A_1}{d} = 8\varepsilon_0 \frac{\left(\frac{\pi r_2^2}{2}\right)}{10^{-2}} = \frac{4\pi\varepsilon_0}{10^{-2}} (2 \times 10^{-3})^2 = 0.04 \times 10^{-12} \text{ F},
\]

\[
C_2 = \varepsilon_2 \frac{A_2}{d} = 4\varepsilon_0 \frac{\left(\frac{\pi (r_2^2 - r_1^2)}{2}\right)}{10^{-2}} = \frac{8\pi\varepsilon_0}{10^{-2}} [(4 \times 10^{-3})^2 - (2 \times 10^{-3})^2] = 0.06 \times 10^{-12} \text{ F},
\]

\[
C_3 = \varepsilon_3 \frac{A_3}{d} = 2\varepsilon_0 \frac{\left(\frac{\pi (r_2^2 - r_3^2)}{2}\right)}{10^{-2}} = \frac{\pi\varepsilon_0}{10^{-2}} [(8 \times 10^{-3})^2 - (4 \times 10^{-3})^2] = 0.12 \times 10^{-12} \text{ F},
\]

\[ C = C_1 + C_2 + C_3 = 0.22 \times 10^{-12} \text{ F}. \]
Figure P4.53: Dielectric sections for Problems 4.53 and 4.55.
Problem 4.54 The capacitor shown in Fig. 4-36 (P4.54) consists of two parallel dielectric layers. We wish to use energy considerations to show that the equivalent capacitance of the overall capacitor, $C$, is equal to the series combination of the capacitances of the individual layers, $C_1$ and $C_2$, namely

$$C = \frac{C_1 C_2}{C_1 + C_2}, \quad (4.136)$$

where

$$C_1 = \varepsilon_1 \frac{A}{d_1}, \quad C_2 = \varepsilon_2 \frac{A}{d_2}.$$

(a) Let $V_1$ and $V_2$ be the electric potentials across the upper and lower dielectrics, respectively. What are the corresponding electric fields $E_1$ and $E_2$? By applying the appropriate boundary condition at the interface between the two dielectrics, obtain explicit expressions for $E_1$ and $E_2$ in terms of $\varepsilon_1$, $\varepsilon_2$, $V$, and the indicated dimensions of the capacitor.

(b) Calculate the energy stored in each of the dielectric layers and then use the sum to obtain an expression for $C$. 

Figure P4.54: (a) Capacitor with parallel dielectric layers, and (b) equivalent circuit (Problem 4.54).
(c) Show that $C$ is given by Eq. (4.136).

**Solution:**

(a) If $V_1$ is the voltage across the top layer and $V_2$ across the bottom layer, then

$$V = V_1 + V_2,$$

and

$$E_1 = \frac{V_1}{d_1}, \quad E_2 = \frac{V_2}{d_2}.$$

According to boundary conditions, the normal component of $\mathbf{D}$ is continuous across the boundary (in the absence of surface charge). This means that at the interface between the two dielectric layers,

$$D_{1n} = D_{2n}$$

or

$$\varepsilon_1 E_1 = \varepsilon_2 E_2.$$

Hence,

$$V = E_1 d_1 + E_2 d_2 = E_1 d_1 + \frac{\varepsilon_1 E_1}{\varepsilon_2} d_2,$$

which can be solved for $E_1$:

$$E_1 = \frac{V}{d_1 + \frac{\varepsilon_1}{\varepsilon_2} d_2}.$$

Similarly,

$$E_2 = \frac{V}{d_2 + \frac{\varepsilon_2}{\varepsilon_1} d_1}.$$
(b)

\[
W_{e_1} = \frac{1}{2} \varepsilon_1 E_1^2 \cdot v_1 = \frac{1}{2} \varepsilon_1 \left( \frac{V}{d_1 + \varepsilon_1 d_2} \right)^2 \cdot A d_1 = \frac{1}{2} V^2 \left[ \frac{\varepsilon_1 \varepsilon_2^2 A d_1}{(\varepsilon_2 d_1 + \varepsilon_1 d_2)^2} \right],
\]

\[
W_{e_2} = \frac{1}{2} \varepsilon_2 E_2^2 \cdot v_2 = \frac{1}{2} \varepsilon_2 \left( \frac{V}{d_2 + \varepsilon_2 d_1} \right)^2 \cdot A d_2 = \frac{1}{2} V^2 \left[ \frac{\varepsilon_2^3 \varepsilon_1 A d_2}{(\varepsilon_1 d_2 + \varepsilon_2 d_1)^2} \right],
\]

\[
W_e = W_{e_1} + W_{e_2} = \frac{1}{2} V^2 \left[ \frac{\varepsilon_1 \varepsilon_2^2 A d_1 + \varepsilon_2^3 \varepsilon_1 A d_2}{(\varepsilon_1 d_2 + \varepsilon_2 d_1)^2} \right].
\]

But \( W_e = \frac{1}{2} CV^2 \), hence,

\[
C = \frac{\varepsilon_1 \varepsilon_2^2 A d_1 + \varepsilon_2^3 \varepsilon_1 A d_2}{(\varepsilon_1 d_2 + \varepsilon_2 d_1)^2} = \varepsilon_1 \varepsilon_2 A \frac{(\varepsilon_2 d_1 + \varepsilon_1 d_2)}{(\varepsilon_2 d_1 + \varepsilon_1 d_2)^2} = \frac{\varepsilon_1 \varepsilon_2 A}{\varepsilon_2 d_1 + \varepsilon_1 d_2}.
\]

(c) Multiplying numerator and denominator of the expression for \( C \) by \( A/d_1 d_2 \), we have

\[
\frac{\varepsilon_1 \varepsilon_2 A}{d_1} \cdot \frac{A}{d_2} = \frac{C_1 C_2}{C_1 + C_2},
\]

where

\[
C_1 = \frac{\varepsilon_1 A}{d_1}, \quad C_2 = \frac{\varepsilon_2 A}{d_2}.
\]

**Problem 4.55** Use the expressions given in Problem 4.54 to determine the capacitance for the configurations in Fig. 4.35(a) (P4.55) when the conducting plates are placed on the right and left faces of the structure.

**Solution:**

\[
C_1 = \frac{\varepsilon_1 A}{d_1} = 2\varepsilon_0 \frac{(2 \times 5) \times 10^{-4}}{1 \times 10^{-2}} = 20\varepsilon_0 \times 10^{-2} = 1.77 \times 10^{-12} \text{ F},
\]

\[
C_2 = \frac{\varepsilon_2 A}{d_2} = 4\varepsilon_0 \frac{(2 \times 5) \times 10^{-4}}{3 \times 10^{-2}} = 1.18 \times 10^{-12} \text{ F},
\]

\[
C = \frac{C_1 C_2}{C_1 + C_2} = \frac{1.77 \times 1.18}{1.77 + 1.18} \times 10^{-12} = 0.71 \times 10^{-12} \text{ F}.
\]
Section 4-12: Image Method

Problem 4.56 With reference to Fig. 4-37 (P4.56), charge $Q$ is located at a distance $d$ above a grounded half-plane located in the $x$-$y$ plane and at a distance $d$ from another grounded half-plane in the $x$-$z$ plane. Use the image method to

(a) establish the magnitudes, polarities, and locations of the images of charge $Q$ with respect to each of the two ground planes (as if each is infinite in extent), and

(b) then find the electric potential and electric field at an arbitrary point $P(0, y, z)$.

Solution:

(a) The original charge has magnitude and polarity $+Q$ at location $(0, d, d)$. Since the negative $y$-axis is shielded from the region of interest, there might as well be a conducting half-plane extending in the $-y$ direction as well as the $+y$ direction. This ground plane gives rise to an image charge of magnitude and polarity $-Q$ at location $Q(0, y, d)$.
From Eq. (4.51),

\[ E = -\nabla V \]

\[ = \frac{Q}{4\pi\varepsilon} \left( \nabla \frac{1}{\sqrt{x^2 + (y - d)^2 + (z - d)^2}} - \nabla \frac{1}{\sqrt{x^2 + (y + d)^2 + (z - d)^2}} + \nabla \frac{1}{\sqrt{x^2 + (y + d)^2 + (z + d)^2}} - \nabla \frac{1}{\sqrt{x^2 + (y - d)^2 + (z + d)^2}} \right) \]

\[ = \frac{Q}{4\pi\varepsilon} \left( \frac{\mathbf{\hat{x}} x + \mathbf{\hat{y}} (y - d) + \mathbf{\hat{z}} (z - d)}{(x^2 + (y - d)^2 + (z - d)^2)^{3/2}} - \frac{\mathbf{\hat{x}} x + \mathbf{\hat{y}} (y + d) + \mathbf{\hat{z}} (z - d)}{(x^2 + (y + d)^2 + (z - d)^2)^{3/2}} + \frac{\mathbf{\hat{x}} x + \mathbf{\hat{y}} (y + d) + \mathbf{\hat{z}} (z + d)}{(x^2 + (y + d)^2 + (z + d)^2)^{3/2}} - \frac{\mathbf{\hat{x}} x + \mathbf{\hat{y}} (y - d) + \mathbf{\hat{z}} (z + d)}{(x^2 + (y - d)^2 + (z + d)^2)^{3/2}} \right) \ (V/m). \]

**Problem 4.57** Conducting wires above a conducting plane carry currents \( I_1 \) and \( I_2 \) in the directions shown in Fig. 4.38 (P4.57). Keeping in mind that the direction of a current is defined in terms of the movement of positive charges, what are the directions of the image currents corresponding to \( I_1 \) and \( I_2 \)?

**Solution:**

(a) In the image current, movement of negative charges downward = movement of positive charges upward. Hence, image of \( I_1 \) is same as \( I_1 \).

---

Figure P4.57: Currents above a conducting plane (Problem 4.57).
(b) In the image current, movement of negative charges to right = movement of positive charges to left.

Problem 4.58  Use the image method to find the capacitance per unit length of an infinitely long conducting cylinder of radius $a$ situated at a distance $d$ from a parallel conducting plane, as shown in Fig. 4-39 (P4.58).

Solution: Let us distribute charge $\rho_I$ (C/m) on the conducting cylinder. Its image cylinder at $z = -d$ will have charge density $-\rho_I$.

For the line at $z = d$, the electric field at any point $z$ (at a distance of $d - z$ from the center of the cylinder) is, from Eq. (4.33),

$$E_1 = \frac{\rho_I}{2\pi\varepsilon_0(d - z)}$$