Digital Signal Processing
Lecture 8 - Filter Design - IIR

Electrical Engineering and Computer Science
University of Tennessee, Knoxville
Introduction

Discrete-time signals and systems - LTI systems

Unit sample response $h[n]$ uniquely characterizes an LTI system

Frequency response: $H(e^{j\omega})$

LTI constant-coefficient difference equation

Impulse invariance

Bilinear Trans. Example

Impulse

Recap

Sampling and Reconstruction
Inspection, power series, partial fraction expansion:

$z \cdot p_1 - u z(z)X \left\{ \frac{1}{1 - z^{-1}} \right\} = [u]x$

The inverse Z-transform, $x[n] = \mathcal{Z}^{-1}(z)$

The significance of zeros

Properties of the Z-transform

System function, $H(z)$

Region of convergence - the Z-plane

$u^{-z}[u]x \sum_{n=\infty}^{-u} = (z)X(z)$

The Z-transform

Z transform

$z$-transform domain analysis - $nwz$

Properties of the $z$-transform

The significance of zeros

System function, $H(z)$

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$z$-transform domain analysis - $nwz$
Review - Design structures

- Different representations of causal LTI systems
  - LCDE with initial rest condition
  - $H(z)$ with $|z| > R_+$ and starts at $n = 0$
- Block diagram vs. Signal flow graph and how to determine system function (or unit sample response) from the graphs
- Design structures
  - Direct form I (zeros first)
  - Direct form II (poles first) - Canonic structure
  - Transposed form (zeros first)
- IIR: cascade form, parallel form, feedback in IIR (computable vs. noncomputable)
- FIR: direct form, cascade form, parallel form, linear phase
- Metric: computational resource and precision
- Sources of errors: coefficient quantization error, input quantization error, product quantization error, limit cycles
  - Pole sensitivity of 2nd-order structures: coupled form
  - Coefficient quantization examples: direct form vs. cascade form
Recap

Introduction

CT->DT

Impulse

Invariance

Bilinear Trans.

Example

Rationale

- Review of complex exponentials as eigenfunctions of the LTI
  - \( x[n] = e^{j\omega_0 n} \rightarrow y[n] = H(e^{j\omega_0})e^{j\omega_0 n} \)
  - or \( x[n] = \cos(\omega_0 n) \rightarrow y[n] = |H(e^{j\omega_0})| \cos(\omega_0 n + \theta) \)

- Separation of signal when they occupy different frequency bands — choose the system function that is unity at selective frequencies

- Given a set of specifications, design a rational transfer function that approximates the ideal filter maintaining specifications of \( \delta_p, \delta_s, \) and the transition band.
Stages of digital filter design

- The specification of the desired properties of the system
  - application-dependent
  - usually done in the frequency domain
- The approximation of the specifications using a causal discrete-time system
- The realization of the system
  - e.g., DSP board
Practical frequency-selective filters

- Approximate ideal filters by a rational function or LCDE

- Factors that affect the filter performance
  - the maximum tolerable passband ripple, $20 \log_{10} \delta_p$
  - the maximum tolerable stopband ripple, $20 \log_{10} \delta_s$
  - the passband edge frequency $\omega_p$
  - the stopband edge frequency $\omega_s$
  - $M$ and $N$: order of the LCDE
Example

- Design a discrete-time lowpass filter to filter a continuous-time signal with the following specs (with a sampling rate of $10^4$ samples/s):
  - The gain should be within $\pm 0.01$ of unity in the frequency band $0 \leq \Omega \leq 2\pi(2000)$
  - The gain should be no greater than 0.001 in the frequency band $2\pi(3000) \leq \Omega$

- Parameter setup
  - $\delta_{p1} = \delta_{p2} = 0.01$, $\delta_s = 0.001$
  - $\omega_p = 2\pi(2000)/10^4$, $\omega_s = 2\pi(3000)/10^4$
  - Ideal passband gain in decibels?
  - maximum passband gain in decibels?
  - maximum stopband gain in decibels?
Design techniques for IIR filters

- Analytical — closed-form solution of transfer function
- Continuous-time $\rightarrow$ Discrete-time
- Algorithmic
General guidelines for CT->DT

- **continuous** $\rightarrow$ **discrete**
  
  $H_a(s) \rightarrow H(z)$
  
  $h_a(t) \rightarrow h[n]$

- $j\Omega$-axis (s-plane) $\rightarrow$ unit circle (z-plane)

- if $H_a(s)$ is stable $\rightarrow$ $H(z)$ is stable
Different CT->DT approaches

- Mapping differentials to differences
  - $z = 1 + sT$
  - the $j\Omega$-axis is NOT mapped to the unit circle
  - stable poles might not be mapped to inside the unit circle
- Impulse invariance
- Bilinear transformation
Impulse invariance

\[ h[n] = T_d h_c(nT_d) \quad (1) \]

\[ H(e^{j\omega}) = \sum_{k=-\infty}^{\infty} H_c \left[ \frac{j\omega}{T_d} + \frac{j2\pi k}{T_d} \right] \quad (2) \]

- Preserve good time-domain characteristics
- Linear scaling of frequency axis, \( \omega = \Omega T \)
- Existence of aliasing
- Impulse invariance doesn’t imply step invariance
Impulse invariance (cont’)

\[ H_c(s) = \sum_{k=1}^{N} \frac{A_k}{s - s_k} \rightarrow H(z) = \sum_{k=1}^{N} \frac{T_d A_k}{1 - e^{s_k T_d} z^{-1}} \]

- Mapping poles

\[ s = s_k \rightarrow z = e^{s_k T_d} \]

- Preserve residues

- \( s = j\Omega \rightarrow z = e^{j\Omega T_d} = e^{j\omega}, \) the unit circle

- if \( s_k \) is stable, i.e., region of \( s_k \) is less than 0,
  \( \rightarrow |z_k| < 1 \rightarrow \) digital filter is stable
Impulse invariance - An example

- Find the system function of the digital filter mapped from the analog filter with a system function
  \[ H_c(s) = \frac{s+a}{(s+a)^2 + b^2} \]. Compare magnitude of the frequency response and pole-zero distributions in the s- and z-plane

- Sol: \[ H(z) = \frac{1 - (e^{-aT} \cos bT)z^{-1}}{(1-e^{-(a+jb)T}z^{-1})(1-e^{-(a-jb)T}z^{-1})} \]

- Note that zeros are not mapped. Also note that \( |H_s(j\Omega)| \) is not periodic but \( |H(e^{j\omega})| \) is.
Bilinear transformation

- Mapping from s-plane to z-plane by relating s and z according to a bilinear transformation. \( H_c(s) \rightarrow H(z) \)

\[
s = \frac{2}{T} \left( \frac{1 - z^{-1}}{1 + z^{-1}} \right), \quad \text{or} \quad z = \frac{1 + \frac{sT}{2}}{1 - \frac{sT}{2}}
\]

- Two guidelines
  - Preserves the frequency characteristics? I.e., maps the \( j\Omega \)-axis to the unit circle?
  - Stable analog filter mapped to stable digital filter?

- Important properties of bilinear transformation
  - Left-side of the s-plane \( \rightarrow \) interior of the unit circle; Right-side of the s-plane \( \rightarrow \) exterior of the unit circle.
  - Therefore, stable analog filters \( \rightarrow \) stable digital filters.
  - The \( j\Omega \)-axis gets mapped exactly once around the unit circle.
    - No aliasing
    - The \( j\Omega \)-axis is infinitely long but the unit circle isn’t \( \rightarrow \) nonlinear distortion of the frequency axis.
Bilinear transformation - Mappings

Image of $s = j\Omega$ (unit circle)

Image of left half-plane

$\omega = 2 \arctan \left( \frac{\Omega T_d}{2} \right)$
Bilinear transformation - How to tolerate distortions?

- **Prewarp** the digital cutoff frequency to an analog cutoff frequency through $\Omega = \frac{2}{T} \tan \frac{\omega}{2}$

- Better used to approximate piecewise constant filters which will be mapped as constant as well

- Can’t be used to obtain digital lowpass filter with linear-phase

- Avoid aliasing at the price of distortion of the frequency axis
The class of analog filters

- **Butterworth filter**
  - \[ |H_c(j\Omega)|^2 = \frac{1}{1+(\frac{j\Omega}{\Omega_c})^{2N}} \]
  - Note about the butterworth circle with radius \( \Omega_c \)
  - \( \Omega_c \) is also called the 3dB-cutoff frequency when
    \[ -10\log_{10}|H_c(j\Omega)|^2|_{\Omega=\Omega_c} \approx 3 \]
  - Monotonic function in both passband and stopband
  - Matlab functions: `buttord`, `butter`

- **Chebyshev filter**
  - Type I Chebyshev has an equiripple freq response in
    the passband and varies monotonically in the stopband,
    \[ |H_c(j\Omega)|^2 = \frac{1}{1+\epsilon^2 T_N^2(\Omega/\Omega_p)} \]
  - Type II Chebyshev is monotonic in the passband and
    equiripple in the stopband, \( |H_c(j\Omega)|^2 = \frac{1}{1+\epsilon^2 [T_N(\Omega_s/\Omega_p)]^2} \)
  - Matlab functions: `cheb1ord`, `cheby1`, `cheb2ord`, `cheby2`
The class of analog filters (cont’d)

- Elliptic filter
  - $|H_c(j\Omega)^2 = \frac{1}{1+\epsilon^2 R_N^2(\Omega/\Omega_p)}$ where $R_N(\Omega)$ is a rational function of order $N$ satisfying the property $R_N(1/\Omega) = 1/R_N(\Omega)$ with the roots of its numerator lying within the interval $0 < \Omega < 1$ and the roots of its denominator lying in the interval $1 < \Omega < \infty$
  - Equiripple in both the passband and the stopband
  - Matlab functions: `ellipord`, `ellip`
Example

- Specs of the discrete-time filter: passband gain between 0dB and -1dB, and stopband attenuation of at least -15dB.

\[
1 - \delta_p \geq -1dB, \delta_s \leq -15dB
\]

\[
20 \log_{10} |H(e^{j0.2\pi})| \geq -1 \rightarrow |H(e^{j0.2\pi})| \geq 10^{-0.05} = 0.8913
\]  

(3)

\[
20 \log_{10} |H(e^{j0.3\pi})| \leq -15 \rightarrow |H(e^{j0.3\pi})| \leq 10^{-0.75} = 0.1778
\]  

(4)
Example (cont’d)

- **Impulse invariance**
  - Round up to the next integer of \( N \)
  - Due to aliasing problem, meet the passband exactly with exceeded stopband

\[
1 + \left( \frac{j^{0.2\pi}}{j\Omega_c} \right)^{2N} = 10^{0.1} \tag{5}
\]

\[
1 + \left( \frac{j^{0.3\pi}}{j\Omega_c} \right)^{2N} = 10^{1.5} \tag{6}
\]

- **Bilinear transformation**
  - Round up to the next integer of \( N \)
  - By convention, choose to meet the stopband exactly with exceeded passband

\[
1 + \left( \frac{j2\tan(0.1\pi)}{j\Omega_c} \right)^{2N} = 10^{0.1} \tag{7}
\]

\[
1 + \left( \frac{j2\tan(0.15\pi)}{j\Omega_c} \right)^{2N} = 10^{1.5} \tag{8}
\]
Example - Comparison