ECE406/506 Real-Time Digital Signal Processing
Lecture 4 - Structures for Discrete-Time Systems

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Overview

Discrete-time systems

- Special properties: linearity, TI, stability, causality
- LTI systems: the unit sample response $h[n]$ uniquely characterizes an LTI system

$$y[n]=\sum_{k=-\infty}^{\infty} x[k]h[n-k]=x[n]*h[n]$$

- Frequency response: $H(e^{j\omega})$ is eigenvalues of LTI systems, complex exponentials are eigenfunctions of LTI systems, i.e., if $x[n] = e^{j\omega n}$,

$$y[n]=H(e^{j\omega})x[n]=(\sum_{k=-\infty}^{\infty} h[k]e^{-j\omega k})e^{j\omega n}$$

  - Fourier transform: Generalization of frequency response (a periodic continous function of ω)

$$X(e^{j\omega})=\sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n}, x[n]=\frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\omega})e^{j\omega n}d\omega$$

- The z-transform as a generalization to the Fourier transform, $X(z) = \sum_{n=-\infty}^{\infty} x[n]z^{-n}$, and the system function $H(z)$

Different system representations

- Using LCDE with initial rest condition

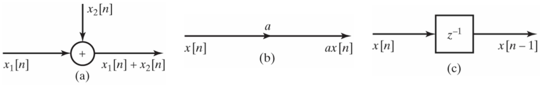
$$y[n] = \sum_{k=0}^{M} b_k x[n-k] + \sum_{k=1}^{N} a_k y[n-k]$$

- Using system function with ROC $|z| > R_+$

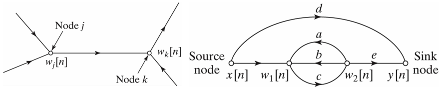
$$H(z) = \frac{\sum_{k=0}^{M} b_k z^{-k}}{1 - \sum_{k=1}^{N} a_k z^{-k}}$$

Block diagram vs. Signal flow graph

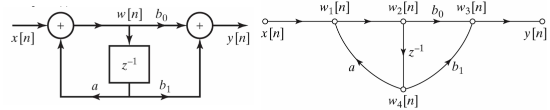
- Block diagram symbols: adder, multiplier, unit delay (memory)



- Signal flow graph: directed branches (branch gain, delay branch), nodes (source node, sink node)

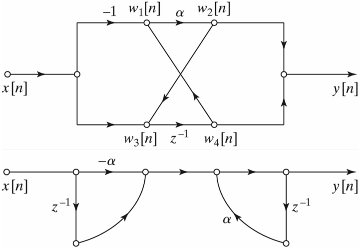


- A comparison: nodes in the flow graph represent both branching points and adders, whereas in the block diagram a special symbol is used for adders



Determination of the system function from a flow graph

- Different flow graph representations require different amounts of computational resources



Direct form I and II

$$H(z)=\frac{\sum_{k=0}^{M} b_k z^{-k}}{1-\sum_{k=1}^{N} a_k z^{-k}}$$

- Direct form I: implementing zeros first

$$H(z)=H_2(z)H_1(z)=\left(\frac{1}{1-\sum_{k=1}^{N} a_k z^{-k}}\right)\left(\sum_{k=0}^{M} b_k z^{-k}\right) \tag{1}$$

$$V(z)=H_1(z)X(z)=\left(\sum_{k=0}^{M} b_k z^{-k}\right)X(z) \tag{2}$$

$$Y(z)=H_2(z)V(z)=\left(\frac{1}{1-\sum_{k=1}^{N} a_k z^{-k}}\right)V(z) \tag{3}$$

$$y[n]=\sum_{k=0}^{M} b_k x[n-k]+\sum_{k=1}^{N} a_k y[n-k] \tag{4}$$

- Direct form II: implementing poles first

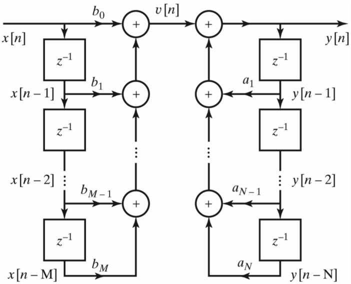
$$H(z)=H_1(z)H_2(z)=\left(\sum_{k=0}^{M} b_k z^{-k}\right)\left(\frac{1}{1-\sum_{k=1}^{N} a_k z^{-k}}\right) \tag{5}$$

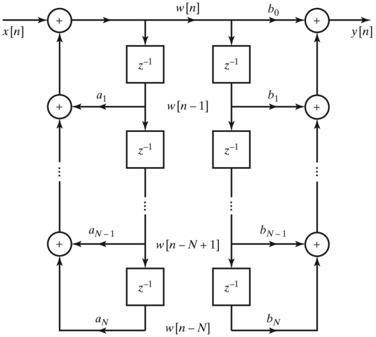
$$W(z)=H_2(z)X(z)=\left(\frac{1}{1-\sum_{k=1}^{N} a_k z^{-k}}\right)X(z) \tag{6}$$

$$Y(z)=H_1(z)W(z)=\left(\sum_{k=0}^{M} b_k z^{-k}\right)W(z) \tag{7}$$

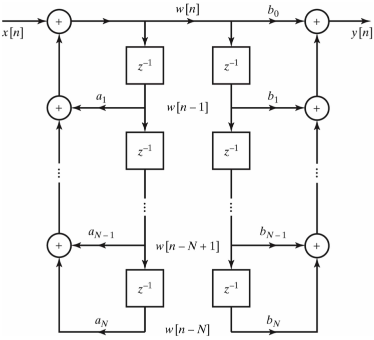
$$y[n]=\sum_{k=1}^{N} a_k y[n-k]+\sum_{k=0}^{M} b_k x[n-k] \tag{8}$$

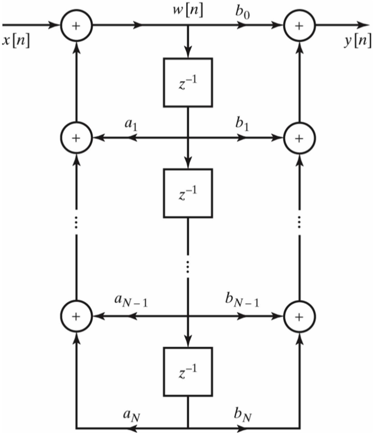
Comparison


x[n]
b0
v[n]
y[n]
z−1
x[n − 1]
b1
a1
y[n − 1]
x[n − 2]
bM−1
aN−1
y[n − 2]
bM
aN
x[n − M]
y[n − N]

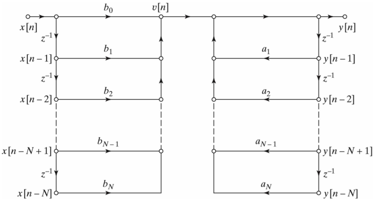

x[n]
w[n]
b0
y[n]
z−1
a1
w[n − 1]
b1
aN−1
w[n − N + 1]
bN−1
aN
w[n − N]
bN

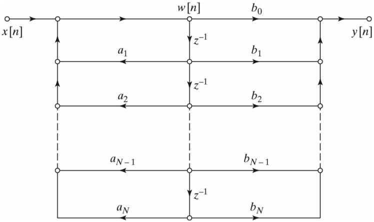
Comparison (cont')





Comparison (cont')

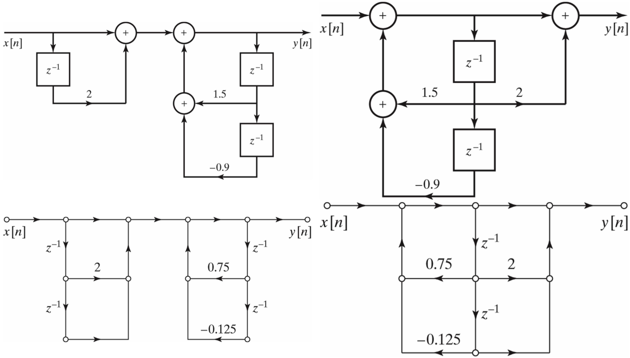




Exercises

- Ex1: $H(z) = \frac{1+2z^{-1}}{1-1.5z^{-1}+0.9z^{-2}}$
- Ex2: $H(z) = \frac{1+2z^{-1}+z^{-2}}{1-0.75z^{-1}+0.125z^{-2}}$

Solution

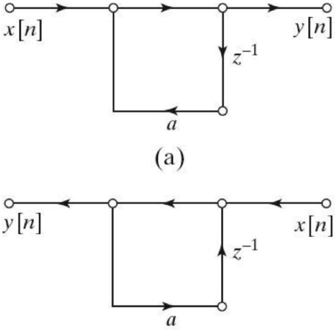


Canonic vs. Noncanonic structures

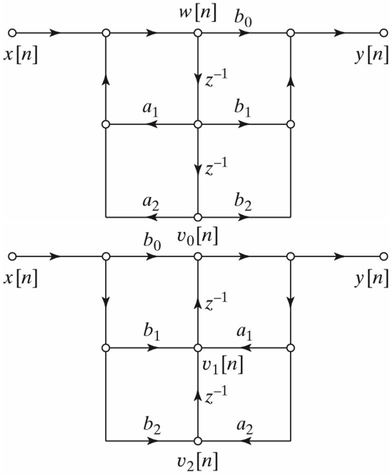
- A digital filter structure is said to be *canonic* if the number of delays is equal to the order of the difference equation. Otherwise, it is a *noncanonic* structure. That is, minimum number of delays required is $max(N, M)$.

Transposed form

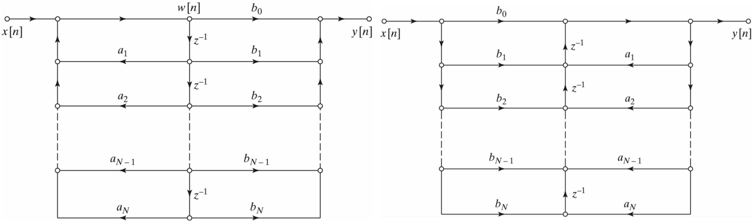
- The transposition theorem: For single-input, single-output systems, the resulting flow graph has the same system function as the original graph if the input and output nodes are interchanged.
  - reverse direction of all branches
  - interchange input and output
- Implement zeros first, then poles as compared to the direct II form



Example



Comparison - Direct form II vs. Transposed direct form II

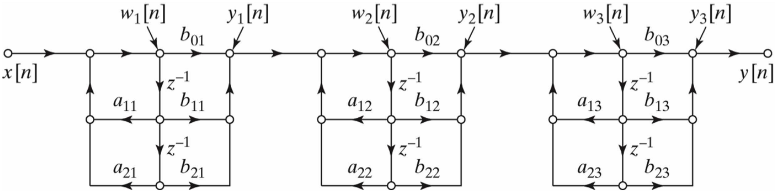


Cascade form

$$H(z) = \frac{\sum_{k=0}^{M} b_k z^{-k}}{1 - \sum_{k=1}^{N} a_k z^{-k}}$$

$$H(z) = \prod_{k=1}^{N_s} \frac{b_{0k} + b_{1k} z^{-1} + b_{2k} z^{-2}}{1 - a_{1k} z^{-1} - a_{2k} z^{-2}}$$

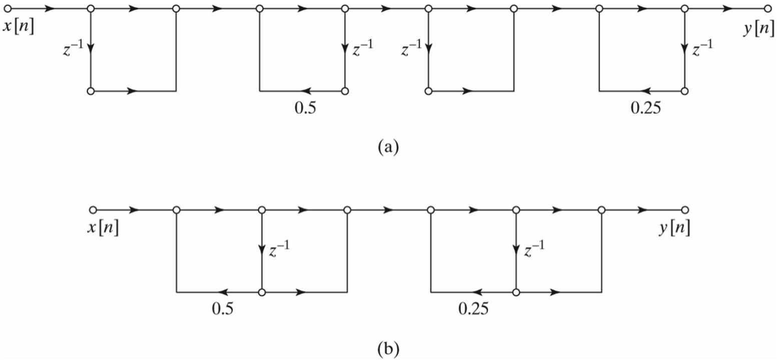
$$N_s = \lfloor (N+1)/2 \rfloor$$



Exercises

- Ex: $H(z) = \frac{1+2z^{-1}+z^{-2}}{1-0.75z^{-1}+0.125z^{-2}} = \frac{(1+z^{-1})(1+z^{-1})}{(1-0.5z^{-1})(1-0.25z^{-1})}$

Solution



Why cascading?

- Use of computation resource
  - Direct form II structure: $2N + 1$ constant multipliers

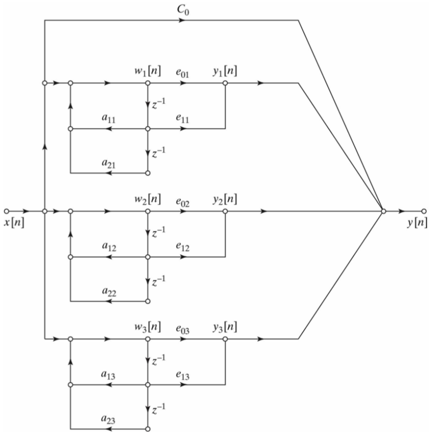
$$H(z) = \frac{\sum_{k=0}^{M} b_k z^{-k}}{1 - \sum_{k=1}^{N} a_k z^{-k}}$$

  - Cascade form structure: $5N/2$ constant multipliers (assume $M = N$ and N is even)

$$H(z) = \prod_{k=1}^{N_s} \frac{b_{0k} + b_{1k} z^{-1} + b_{2k} z^{-2}}{1 - a_{1k} z^{-1} - a_{2k} z^{-2}}$$

- Precision

Parallel form



$$H(z) = \sum_{k=0}^{N_p} C_k z^{-k} + \sum_{k=1}^{N_s} \frac{e_{0k} + e_{1k} z^{-1}}{1 - a_{1k} z^{-1} - a_{2k} z^{-2}}$$

Exercises

- Ex: $H(z) = \frac{1+2z^{-1}+z^{-2}}{1-0.75z^{-1}+0.125z^{-2}} = 8 + \frac{-7+8z^{-1}}{1-0.75z^{-1}+0.125z^{-2}}$

Solution