Digital Signal Processing
Lecture 2 - Discrete-Time Signals

Electrical Engineering and Computer Science
University of Tennessee, Knoxville

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Overview

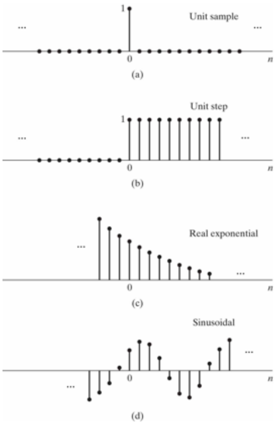
Recap

- Essential components of DSP
  - Frequency analysis
  - Sampling
  - Filter
- Different types of signals
  - continuous vs. discrete vs. digital
  - deterministic vs. random

Basic sequences

- Unit sample sequence (or impulse sequence):
  $$\delta[n] = \begin{cases} 0, & n \neq 0, \\ 1, & n = 0 \end{cases}$$
- Unit step sequence: $u[n] = \begin{cases} 1, & n \geq 0, \\ 0, & n < 0 \end{cases}$
- Exponential sequence: $x[n] = A\alpha^n$
  - real sequence
  - complex sequence
- Sinusoid sequence: $x[n] = A\cos(\omega_0 n + \phi)$

Basic sequences (cont')

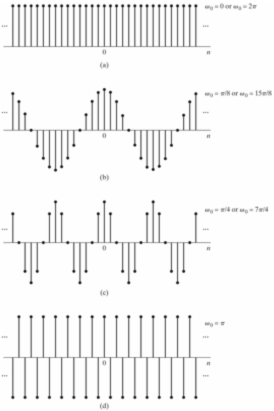


- What are their relationships?

On periodicity

- Continuous-time periodic signals: $x(t) = x(t+T)$
- Discrete-time periodic signals: $x[n] = x[n+N]$
- Exercises: What is the period of the following signals
  - $x[n] = \cos(\pi n/4)$
  - $x[n] = \cos(3\pi n/8)$
- Questions:
  - Is it always true that the higher the frequency, the lower the period?
  - Is it true that the sinusoidal sequence is always periodic?

On periodicity (cont')



Outline

Sinusoid

$$x_a(t) = A\cos(\Omega t + \theta), -\infty < t < \infty$$

or

$$x_a(t) = A\cos(2\pi F t + \theta), -\infty < t < \infty$$

where

- A: amplitude
- θ: phase (radians) or phase shift
- $\Omega = 2\pi F$: radian frequency (radians per second, rad/s)
- F: cyclic frequency (cycles per second, herz, Hz)
- $T_p = 1/F$: fundamental period (sec) such that $x_a(t + T_p) = x_a(t)$

More on frequency

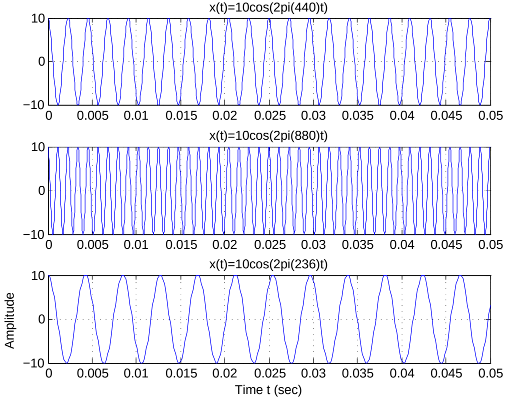


Figure: Sinusoids with different frequencies.

$\boxed{\text{What if } F = 0?}$

More on frequency - How does it sound?[1]

- A440
- A880
- C236
- A tuning fork demo

[1] The multimedia materials are from McClellan, Schafer and Yoder, *DSP FIRST: A Multimedia Approach.* Prentice Hall, Upper Saddle River, New Jersey, 1998. Copyright (c) 1998 Prentice Hall.

More on frequency - The MATLAB code


```
% Lecture 1 - Sinusoid
% plot a sinusoidal signal and listen to it
% 440Hz is the frequency of A above middle C on a musical scale
% it is often used as the reference note for tuning purpose
%
clear buffer
clear all;
clf;

% specify parameters
F =440;
t = 0:1/F/30:1/F*5;
x = 10*cos(2*pi*F*t - 0.4*pi);

% plot the signal
plot(t,x);
title('Sinusoidal signal x(t)');
xlabel('Time t (sec)');
ylabel('Amplitude');
grid on;

% play the signal
sound(x)
```

More on phase - Phase shift vs. Time shift

- Phase shift θ determines the time location of the maxima and minima of a cosine wave
- $s(t)$ vs. $s(t-t_1)$ vs. $s(t+t_1)$ when t_1 is positive
  - Delayed in time vs.
  - Advanced in time
- The phase shift is negative when the time shift is positive (a delay)

$$x_a(t-t_1) = A\cos(\Omega(t-t_1)) = A\cos(\Omega t + \theta)$$

where $\theta = -\Omega t_1$, therefore, $t_1 = -\theta/\Omega$.

- Principal value of the phase shift: $-\pi$ and $+\pi$

$$|t_1| \leq T_p/2 \Longrightarrow -\pi < \theta \leq \pi$$

Outline

Complex exponential signals

- According to Euler's formula

$$\begin{aligned} x_a(t) &= A\cos(\Omega t+\theta) = \Re\{Ae^{j(\Omega t+\theta)}\} \\ &= \Re\{Ae^{j\theta}e^{j\Omega t}\} = \Re\{Xe^{j\Omega t}\} \end{aligned}$$

- The rotating phasor interpretation
  - Complex amplitude (or Phasor): $X = e^{j\theta}$
  - Rotating phasor: multiplying the fixed phasor X by $e^{j\Omega t}$ causes the phasor to rotate. If Ω is positive, the direction of rotation is counterclockwise; when Ω is negative, clockwise.
  - The phase shift θ defines where the phasor is pointing when $t = 0$
- A rotating phasor demo[2]

[2] The multimedia materials are from McClellan, Schafer and Yoder, *DSP FIRST: A Multimedia Approach.* Prentice Hall, Upper Saddle River, New Jersey, 1998. Copyright (c) 1998 Prentice Hall.

Outline

Spectrum and Time-frequency spectrum

- Spectrum: frequency domain representation of the signal that reveals the frequency content of the signal
- Two-sided spectrum: According to inverse Euler's formula

$$x_a(t) = A\cos(\Omega t + \theta) = \frac{A}{2}e^{j\theta}e^{j\Omega t} + \frac{A}{2}e^{-j\theta}e^{-j\Omega t}$$

  such that the sinusoid can be interpreted as made up of 2 complex phasors

$$\{(\frac{1}{2}X, F), (\frac{1}{2}X^*, -F)\}$$

- Spectrogram: frequency changes over time

Outline

Application 1: Phasor addition

- When adding several sinusoids having the same frequency but different amplitudes and phases, the resulting signal is a complex exponential signal with the same frequency

$$\sum_{k=1}^{N} A_k \cos(\Omega t + \theta_k) = A \cos(\Omega t + \theta)$$

- Proof
- Exercise:

$$1.7 \cos(2\pi(10)t + 70\pi/180) + 1.9 \cos(2\pi(10)t + 200\pi/180)$$

Application 2: Producing new signals from sinusoids

- Additive linear combination

$$\begin{aligned} x_a(t) &= A_0 + \sum_{k=1}^{N} A_k \cos(2\pi F_k t + \theta_k) \\ &= X_0 + \sum_{k=1}^{N} \Re\{X_k e^{j2\pi F_k t}\} \\ &= X_0 + \sum_{k=1}^{N} \{\frac{X_k}{2} e^{j2\pi F_k t} + \frac{X_k^*}{2} e^{-j2\pi F_k t}\} \end{aligned}$$

where $X_k = Ae^{j\theta_k}$.

- $2N+1$ complex phasors

$$\{(X_0, 0), (\frac{1}{2}X_1, F_1), (\frac{1}{2}X_1^*, -F_1), (\frac{1}{2}X_2, F_2), (\frac{1}{2}X_2^*, -F_2), \cdots\}$$

- Exercise

$$x_a(t) = 10 + 14\cos(200\pi t - \pi/3) + 8\cos(500\pi t + \pi/2)$$

Application 3: Adding two sinusoids with nearly identical frequencies - Beat notes

- Adding two sinusoids with frequencies, F_1 and F_2, very close to each other

$$x_a(t) = \cos(2\pi F_1 t) + \cos(2\pi F_2 t)$$

  where
  - $F_1 = F_c - F_\Delta$ and $F_2 = F_c + F_\Delta$.
  - $F_c = \frac{1}{2}(F_1 + F_2)$ is the *center frequency*
  - $F_\Delta = \frac{1}{2}(F_2 - F_1)$ is the *deviation frequency*
  - In general, $F_\Delta << F_c$
- Two-sided spectrum representation,

$$\{(\frac{1}{2}, F_1), (\frac{1}{2}, -F_1), (\frac{1}{2}, F_2), (\frac{1}{2}, -F_2)\}$$

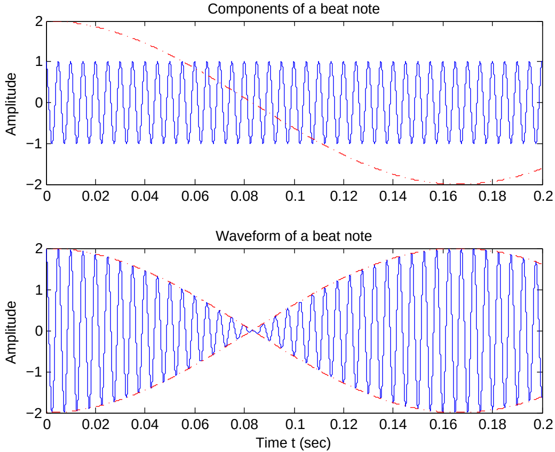
Adding two sinusoids with nearly identical frequencies - Beat notes (cont')

- Rewrite $x_a(t)$ as a product of two cosines

$$
\begin{aligned}
x_a(t) &= \Re\{e^{j2\pi F_1 t}\} + \Re\{e^{j2\pi F_2 t}\} \\
&= \Re\{e^{j2\pi (F_c - F_\Delta) t} + e^{j2\pi (F_c + F_\Delta) t}\} \\
&= \Re\{e^{j2\pi F_c t}(e^{-j2\pi F_\Delta t} + e^{j2\pi F_\Delta t})\} \\
&= \Re\{e^{j2\pi F_c t}(2\cos(2\pi F_\Delta t))\} \\
&= 2\cos(2\pi F_\Delta t)\cos(2\pi F_c t)
\end{aligned}
$$

- Adding two sinusoids with nearly identical frequencies = Multiplying two sinusoids with frequencies far apart
- What is the effect of multiplying a higher-frequency sinusoid (e.g., 2000 Hz) by a lower-frequency sinusoid (e.g., 20 Hz)? The “beating” phenomenon.

Adding two sinusoids with nearly identical frequencies - Beat notes (cont')



- A demo

Adding two sinusoids with nearly identical frequencies: Beat notes (cont')

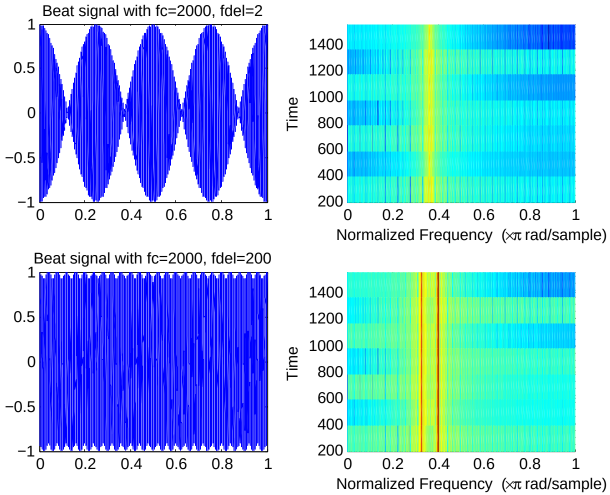


Figure: Beat notes and the spectrogram.

Application 4: Multiplying sinusoids - Amplitude modulation

- Modulation for communication systems: multiplying a low-frequency signal by a high-frequency sinusoid

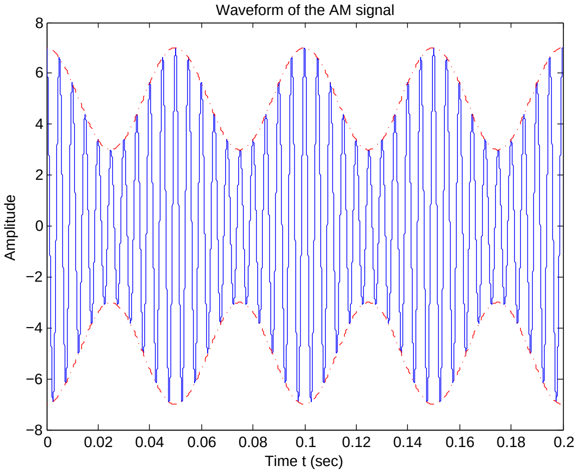
$$x_a(t) = v_a(t)\cos(2\pi F_c t)$$

  - $v_a(t)$: the modulation signal to be transmitted, must be a sum of sinusoids
  - $\cos(2\pi F_c t)$: the carrier signal
  - F_c: the carrier frequency
  - F_c should be much higher than any frequencies contained in the spectrum of $v_a(t)$.
- Exercise:

$$v_a(t) = 5 + 2\cos(40\pi t), F_c = 200 \text{ Hz}$$

- Difference between a beat note and an AM signal?

Multiplying sinusoids - Amplitude modulation (cont')



- A demo

Application 5: Adding cosine waves with harmonically related frequencies - Periodic waveforms

- Fourier Series Theorem: Any periodic signal can be approximated with a sum of harmonically related sinusoids, although the sum may need an infinite number of terms.

$$\begin{aligned} x_a(t) &= A_0 + \sum_{k=1}^{N} A_k \cos(2\pi k F_0 t + \theta_k) \\ &= X_0 + \Re\{\sum_{k=1}^{N} X_k e^{j2\pi k F_0 t}\} \end{aligned}$$

  - $F_k = kF_0$: the harmonic of F_0
  - F_0: the fundamental frequency
- Estimate interesting waveforms by clever choice of $X_k = A_k e^{j\theta_k}$

Adding cosine waves with harmonically related frequencies - Periodic waveforms (cont')

- Fourier analysis: starting from $x_a(t)$ and calculate X_k. X_k can be calculated using the Fourier integral

$$X_k = \frac{2}{T_0}\int_0^{T_0} x_a(t)e^{-j2\pi kt/T_0}dt, X_0 = \frac{1}{T_0}\int_0^{T_0} x_a(t)dt$$

  - T_0: the fundamental period of $x_a(t)$
  - X_0: the DC component
- Fourier synthesis: starting from X_k and calculate $x_a(t)$
- Demo: synthetic vowel ('ah'), $F_0 = 100$ Hz

$$\begin{aligned} x_a(t) &= \Re\{X_2 e^{j2\pi 2F_0 t} + X_4 e^{j2\pi 4F_0 t} + X_5 e^{j2\pi 5F_0 t} + \\ & X_{16} e^{j2\pi 16F_0 t} + X_{17} e^{j2\pi 17F_0 t}\} \end{aligned}$$

- Exercise: How to approximate a square wave?

Application 6: Frequency modulation - the Chirp signal

- A "chirp" signal is a swept-frequency signal whose frequency changes linearly from some low value to a high one.
- How to generate it?
  - concatenate a large number of short constant-frequency sinusoids, whose frequencies step from low to high
  - time-varying phase $\psi(t)$ as a function of time

$$x_a(t) = \Re\{Ae^{j\psi(t)}\} = A\cos(\psi(t))$$

  - instantaneous frequency: the derivative (slope) of the phase

$$\Omega(t) = \frac{d}{dt}\psi(t), F(t) = \Omega(t)/(2\pi)$$

- Frequency modulation: frequency variation produced by the time-varying phase. Signals of this class are called FM signals

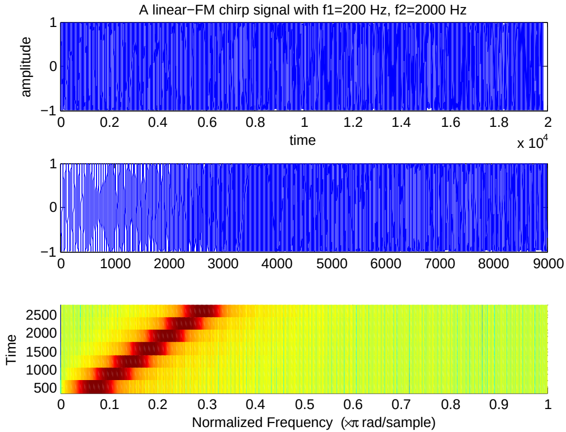
Frequency modulation - the Chirp signal (cont')

- Linear FM signal: chirp signal
- Exercise: quadratic phase

$$\psi(t) = 2\pi\mu t^2 + 2\pi F_0 t + \theta, F(t) = 2\mu t + F_0$$

- Reverse process: If a certain linear frequency sweep is desired, the actual phase can be obtained from the integral of $\Omega(t)$.
- Exercise: synthesize a frequency sweep from $F_1 = 220$ Hz to $F_2 = 2320$ Hz over the time interval $t = 0$ to $t = T_2 = 3$ sec.

Frequency modulation - the Chirp signal



- A demo

Euler's formula and Inverse Euler's formula

- Euler's formula

$$e^{j\theta} = \cos\theta + j\sin\theta$$

- Inverse Euler's formula

$$\cos\theta = \frac{e^{j\theta} + e^{-j\theta}}{2}$$

$$\sin\theta = \frac{e^{j\theta} - e^{-j\theta}}{2j}$$

Basic trignometric identities

$$\sin^2\theta + \cos^2\theta = 1$$

$$\cos 2\theta = \cos^2\theta - \sin^2\theta$$

$$\sin 2\theta = 2\sin\theta\cos\theta$$

$$\sin(\alpha \pm \beta) = \sin\alpha\cos\beta \pm \cos\alpha\sin\beta$$

$$\cos(\alpha \pm \beta) = \cos\alpha\cos\beta \mp \sin\alpha\sin\beta$$

Basic properties of the sine and cosine functions

- Equivalence

$$\sin\theta = \cos(\theta - \pi/2) \text{ or } \cos\theta = \sin(\theta + \pi/2)$$

- Periodicity

$$\cos(\theta + 2k\pi) = \cos\theta, \text{ when k is an integer}$$

- Evenness of cosine

$$\cos(-\theta) = \cos\theta$$

- Oddness of sine

$$\sin(-\theta) = -\sin\theta$$

Basic properties of the sine and cosine functions (cont')

- Zeros of sine

$$\sin(\pi k) = 0, \text{ when k is an integer}$$

- Ones of cosine

$$\cos(2\pi k) = 1, \text{ when k is an integer}$$

- Minus ones of cosine

$$\cos[2\pi(k + \frac{1}{2})] = -1, \text{ when k is an integer}$$

- Derivatives

$$\frac{d \sin\theta}{d\theta} = \cos\theta, \frac{d \cos\theta}{d\theta} = -\sin\theta$$