Real-Time Digital Signal Processing Lecture 8 - Infinite Impulse Response Filter - IIR

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Overview

Recap

- Week 1: Background
- Week 2: DSK and Lab
- Week 3: I/O - Sampling and Reconstruction
- Week 4: The z-transform and Design Structure
- Week 5-6: The FIR filter with linear phase
- Week 7-8: FIR filter design techniques

Review - FIR filter with linear phase

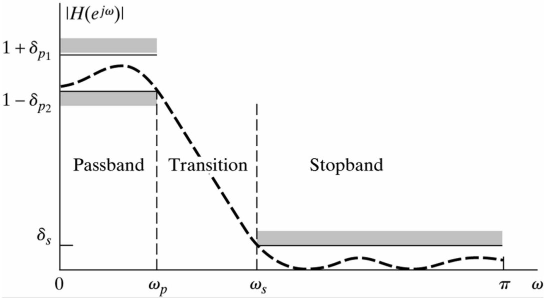
- Linear-phase FIR filter
  - Generalized linear phase, $\sphericalangle H(e^{j\omega}) = \beta - \omega\alpha, 0 < \omega < \pi$
  - Group delay, α
  - Phase shift, β
- Four types of causal FIR filters with generalized linear phase, $h[n] = \pm h[M-n], n = 0, \cdots, M$
  - Definition and derivation (the type of symmetry, with or w/o phase shift (β), integer or non-integer group delay (α), even or odd number of filter coefficients)
  - Design structure (halved # of multiplication)
  - Zero patterns (the set of four reciprocal zeros)
  - $z = \pm 1$ being zero?
- When to use what?
  - Constraints on the zeros
  - Integer/non-integer group delay
  - Phase shift

Review - FIR filter design techniques

- Windows - Kaiser
  - Why using window?
  - The two essential parameters: width of the main lobe and height of the side lobe
  - Pros and cons of windowing techniques
- Optimal methods - Park-McClellan equiripple
  - How does PM overcome issues with Kaiser window?

Review - Practical frequency-selective filters

- Approximate ideal filters by a rational function or LCDE



- Factors that affect the filter performance
  - the maximum tolerable passband ripple, $20 \log_{10} \delta_p$
  - the maximum tolerable stopband ripple, $20 \log_{10} \delta_s$
  - the passband edge frequency ω_p
  - the stopband edge frequency ω_s
  - M and N: order of the LCDE

Design techniques for IIR filters

- Analytical — closed-form solution of transfer function
- Continuous-time $\rightarrow$ Discrete-time
- Algorithmic

General guidelines for CT->DT

$$\begin{array}{ccc} \text{continuous} & \rightarrow & \text{discrete} \\ H_a(s) & \rightarrow & H(z) \\ h_a(t) & \rightarrow & h[n] \end{array}$$

- $j\Omega$-axis (s-plane) $\rightarrow$ unit circle (z-plane)
- if $H_a(s)$ is stable $\rightarrow$ $H(z)$ is stable

Different CT->DT approaches

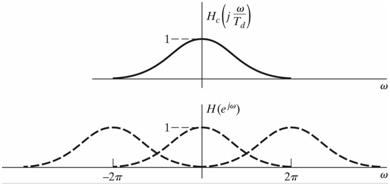
- Mapping differentials to differences
  - $z = 1 + sT$
  - the $j\Omega$-axis is NOT mapped to the unit circle
  - stable poles might not be mapped to inside the unit circle
- Impulse invariance
- Bilinear transformation

Impulse invariance

$$h[n] = T_d h_c(nT_d) \tag{1}$$

$$H(e^{j\omega}) = \sum_{k=-\infty}^{\infty} H_c[\frac{j\omega}{T_d} + \frac{j2\pi k}{T_d}] \tag{2}$$

- Preserve good time-domain characteristics
- Linear scaling of frequency axis, $\omega = \Omega T$
- Existence of aliasing
- Impulse invariance doesn't imply step invariance



Impulse invariance (cont')

$$H_c(s) = \sum_{k=1}^{N} \frac{A_k}{s - s_k} \to H(z) = \sum_{k=1}^{N} \frac{T_d A_k}{1 - e^{s_k T_d} z^{-1}}$$

- Mapping poles

$$s = s_k \to z = e^{s_k T_d}$$

- Preserve residues
- $s = j\Omega \to z = e^{j\Omega T_d} = e^{j\omega}$, the unit circle
- if s_k is stable, i.e., region of s_k is less than 0, $\to |z_k| < 1 \to$ digital filter is stable

Impulse invariance - An example

- Find the system function of the digital filter mapped from the analog filter with a system function $H_c(s) = \frac{s+a}{(s+a)^2+b^2}$. Compare magnitude of the frequency response and pole-zero distributions in the s- and z-plane
- Sol: $H(z) = \frac{1-(e^{-aT}\cos bT)z^{-1}}{(1-e^{-(a+jb)T}z^{-1})(1-e^{-(a-jb)T}z^{-1})}$
- Note that zeros are not mapped. Also note that $|H_s(j\Omega)|$ is not periodic but $|H(e^{j\omega})|$ is.

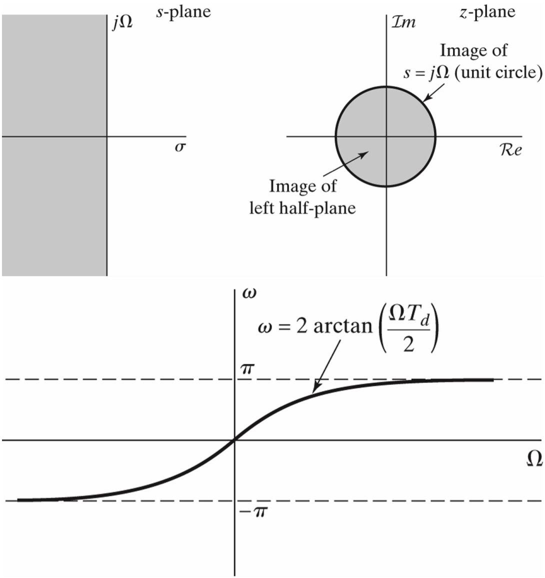
Bilinear transformation

- Mapping from s-plane to z-plane by relating s and z according to a bilinear transformation. $H_c(s) \to H(z)$

$$s = \frac{2}{T}\left(\frac{1 - z^{-1}}{1 + z^{-1}}\right), \text{ or } z = \frac{1 + \frac{sT}{2}}{1 - \frac{sT}{2}}$$

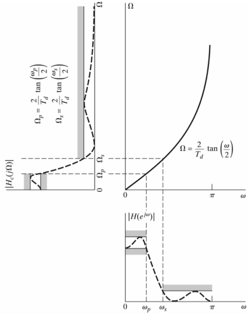
- Two guidelines
  - Preserves the frequency characteristics? I.e., maps the $j\Omega$-axis to the unit circle?
  - Stable analog filter mapped to stable digital filter?
- Important properties of bilinear transformation
  - Left-side of the s-plane $\to$ interior of the unit circle; Right-side of the s-plane $\to$ exterior of the unit circle. Therefore, stable analog filters $\to$ stable digital filters.
  - The $j\Omega$-axis gets mapped exactly once around the unit circle.
    - No aliasing
    - The $j\Omega$-axis is infinitely long but the unit circle isn't $\to$ nonlinear distortion of the frequency axis

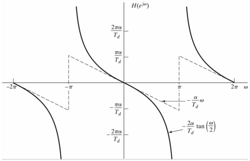
Blinear transformation - Mappings



Bilinear transformation - How to tolerate distortions?

- Prewarp the digital cutoff frequency to an analog cutoff frequency through $\Omega = \frac{2}{T}\tan\frac{\omega}{2}$
- Better used to approximate piecewise constant filters which will be mapped as constant as well
- Can't be used to obtain digital lowpass filter with linear-phase





- Avoid aliasing at the price of distortion of the frequency axis

The class of analog filters

- Butterworth filter
  - $|H_c(j\Omega)|^2 = \frac{1}{1+(\frac{j\Omega}{j\Omega_c})^{2N}}$
  - Note about the butterworth circle with radius Ω_c
  - Ω_c is also called the 3dB-cutoff frequency when $-10log_{10}|H_c(j\Omega)|^2|_{\Omega=\Omega_c} \approx 3$
  - Monotonic function in both passband and stopband
  - Matlab functions: `buttord, butter`
- Chebyshev filter
  - Type I Chebyshev has an equiripple freq response in the passband and varies monotonicaly in the stopband, $|H_c(j\Omega)|^2 = \frac{1}{1+\epsilon^2 T_N^2(\Omega/\Omega_p)}$
  - Type II Chebyshev is monotonic in the passband and equiripple in the stopband, $|H_c(j\Omega)|^2 = \frac{1}{1+\epsilon^2[\frac{T_N(\Omega_s/\Omega_p)}{T_N(\Omega_s/\Omega)}]^2}$
  - Matlab functions: `cheb1ord, cheby1, cheb2ord, cheby2`

The class of analog filters (cont'd)

- Elliptic filter
  - $|H_c(j\Omega)^2 = \frac{1}{1+\epsilon^2 R_N^2(\Omega/\Omega_p)}$ where $R_N(\Omega)$ is a rational function of order N satisfying the perperty $R_N(1/\Omega) = 1/R_N(\Omega)$ with the roots of its numerator lying within the interval $0 < \Omega < 1$ and the roots of its denominator lying in the interval $1 < \Omega < \infty$
  - Equiripple in both the passband and the stopband
  - Matlab functions: `ellipord`, `ellip`

Example

- Specs of the discrete-time filter: passband gain between 0dB and -1dB, and stopband attenuation of at least -15dB.

$$1-\delta_p \geq -1dB, \delta_s \leq -15dB$$

$$20\log_{10}|H(e^{j0.2\pi})| \geq -1 \rightarrow |H(e^{j0.2\pi})| \geq 10^{-0.05} = 0.891 \quad (3)$$

$$20\log_{10}|H(e^{j0.3\pi})| \leq -15 \rightarrow |H(e^{j0.3\pi})| \leq 10^{-0.75} = 0.177 \quad (4)$$

Example (cont'd)

- Impulse invariance
  - Round up to the next integer of N
  - Due to aliasing problem, meet the passband exactly with exceeded stopband

$$1+\left(\frac{j\frac{0.2\pi}{T}}{j\Omega_c}\right)^{2N}=10^{0.1} \tag{5}$$

$$1+\left(\frac{j\frac{0.3\pi}{T}}{j\Omega_c}\right)^{2N}=10^{1.5} \tag{6}$$

- Bilinear transformation
  - Round up to the next integer of N
  - By convention, choose to meet the stopband exactly with exceeded passband

$$1+\left(\frac{j2\tan(0.1\pi)}{j\Omega_c}\right)^{2N}=10^{0.1} \tag{7}$$

$$1+\left(\frac{j2\tan(0.15\pi)}{j\Omega_c}\right)^{2N}=10^{1.5} \tag{8}$$

Example - Comparison