ECE 422/522

Power System Operations & Planning/ Power Systems Analysis II

2 – Synchronous Machine Modeling

Spring 2014

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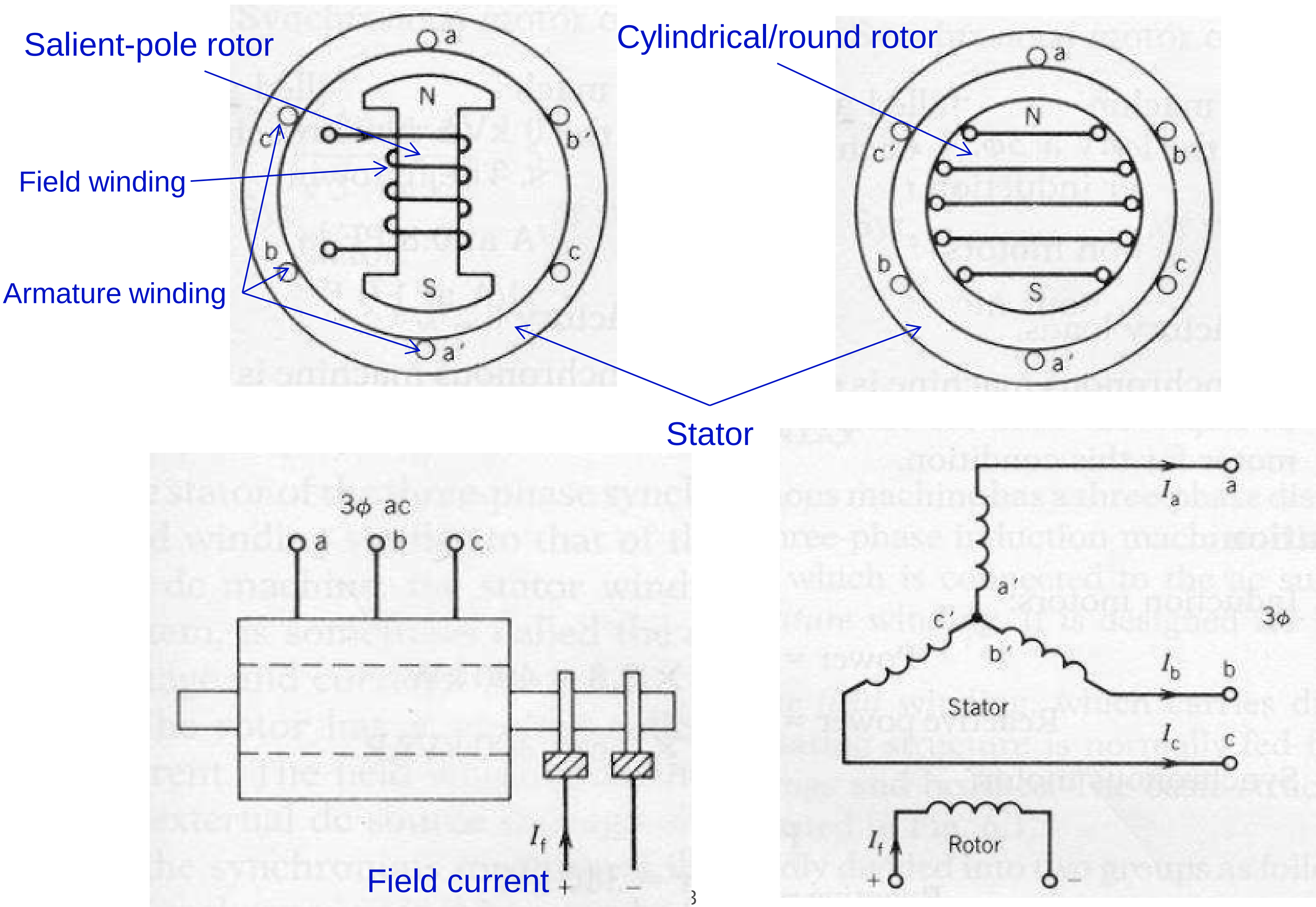
Outline

- Synchronous Machine Modeling
- Per Unit Representation
- Simplified Models for Stability Studies

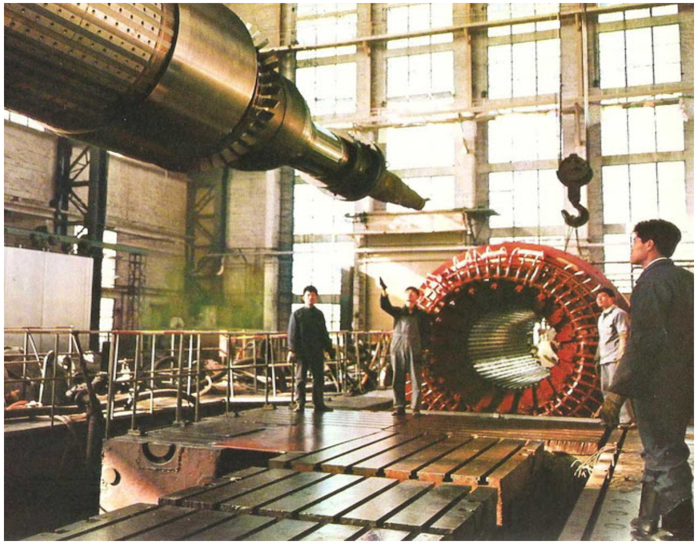
Synchronous Generators



Types of Rotors

- Salient pole rotors
  - Concentrated windings on poles and non-uniform air gap
  - Short axial length and large diameter
  - Hydraulic turbines operated at low speeds (large number of poles)
  - Have damper/amortisseur windings to help damp out speed oscillations
- Round rotors
  - 70% of large synchronous generators (150~1500MVA)
  - Distributed winding and uniform air gap
  - Large axial length and small diameter to limit the centrifugal forces
  - Steam and gas turbines, operated at high speeds, typically 3600 or 1800rpm (2 or 4-pole)
  - No special damper windings but eddy in the solid steal rotor gives damping effects

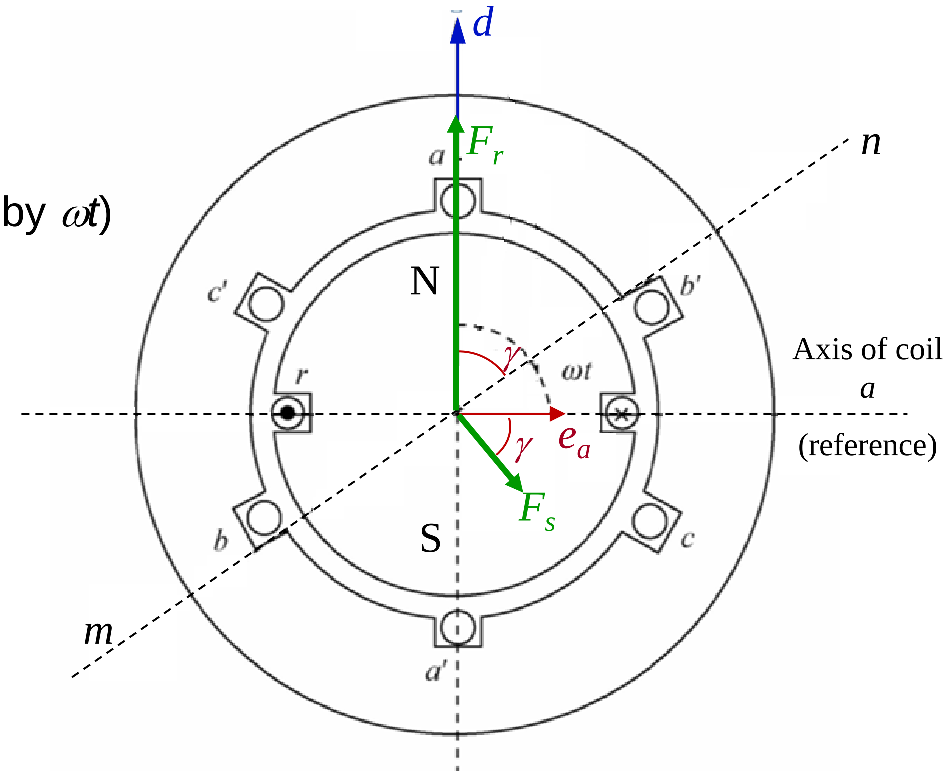
16 poles salient-pole rotor (12 MW)



Round rotor generator under construction

(Source: http://emadrlc.blogspot.com)

Generator Model



- Flux linkage with coil a (leading the axis of *a* by ωt)

$$\psi_a = N\phi \cos \omega t$$

- Induced voltage:

$$e_a = -\frac{d\psi_a}{dt} = \omega N\phi \sin \omega t = E_{\max} \sin \omega t$$

(reaches the maximum at the current position)

$$f = \frac{P}{2}\frac{n}{60}$$

(*n:* synchronous speed in rpm;
P: the number of poles)

- Assume: i_a is lagging e_a by γ (i_a reaches the maximum when *mn* aligns with *aa'*)

$$i_a = I_{\max} \sin(\omega t - \gamma) \qquad i_b = I_{\max} \sin(\omega t - \gamma - \frac{2}{3}\pi) \qquad i_c = I_{\max} \sin(\omega t - \gamma - \frac{4}{3}\pi)$$

- Magnetic motive forces (MMF's) of three phases:

$$F_a = Ki_a = F_m \sin(\omega t - \gamma)$$

$$F_b = Ki_b = F_m \sin(\omega t - \gamma - \frac{2}{3}\pi)$$

$$F_c = Ki_c = F_m \sin(\omega t - \gamma - \frac{4}{3}\pi)$$

$$F_s = \frac{3}{2} F_m$$

F_s is orthogonal to *mn and* revolving synchronously with MMF F_r due to the rotor

Under Steady-State Conditions

- $F_r + F_s$ gives MMF F_{sr} in air gap
- F_s induces EMF E_{ar}
- F_{sr} results air gap flux ϕ_{sr} to induce EMF

$$E_{sr}=E_a+E_{ar}$$

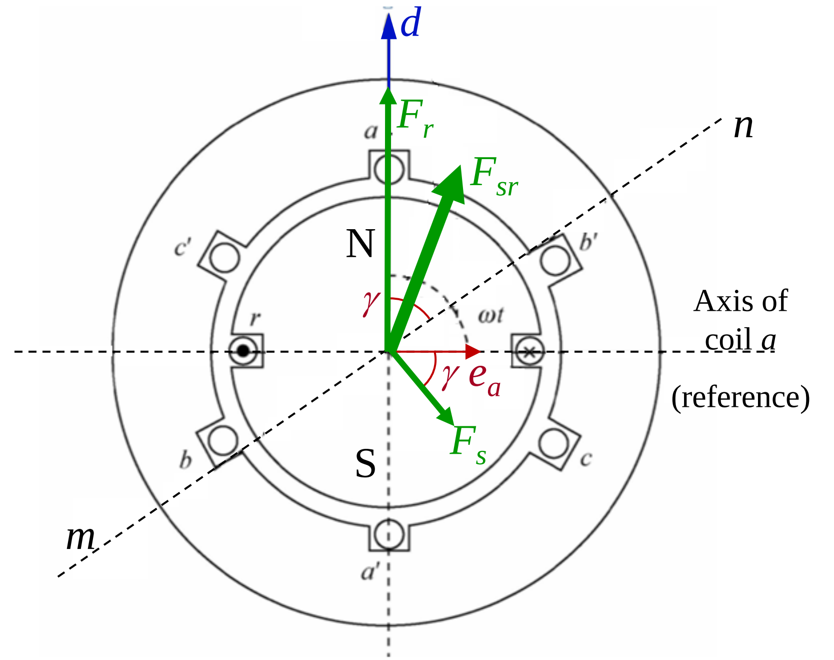
- For a round rotor, define the reactance of the armature reaction

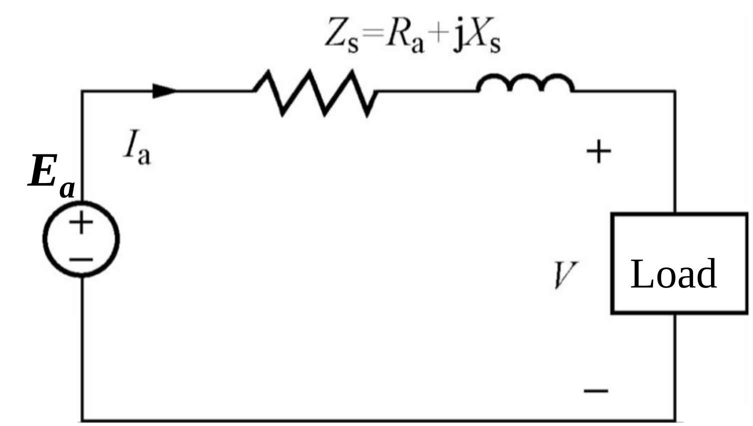
$$X_{ar}= -E_{ar}/(jI_a)$$

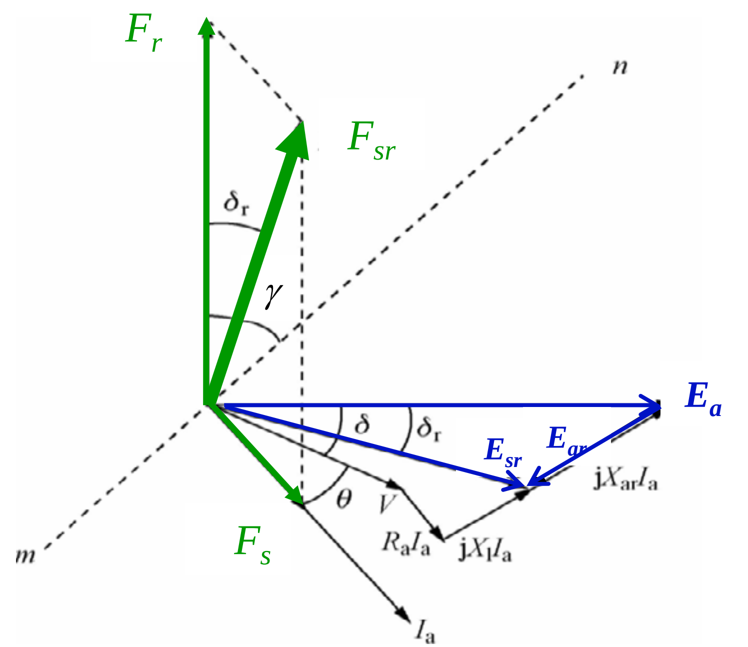
- Terminal voltage V, resistance R_a and leakage and reactance X_l satisfy

$$E_a = V + [R_a + j(X_\ell + X_{ar})]I_a = V + (R_a + jX_s)I_a$$

$X_s=X_l+X_{ar}$ is known as the synchronous reactance







Stator and Rotor Windings

Armature windings:

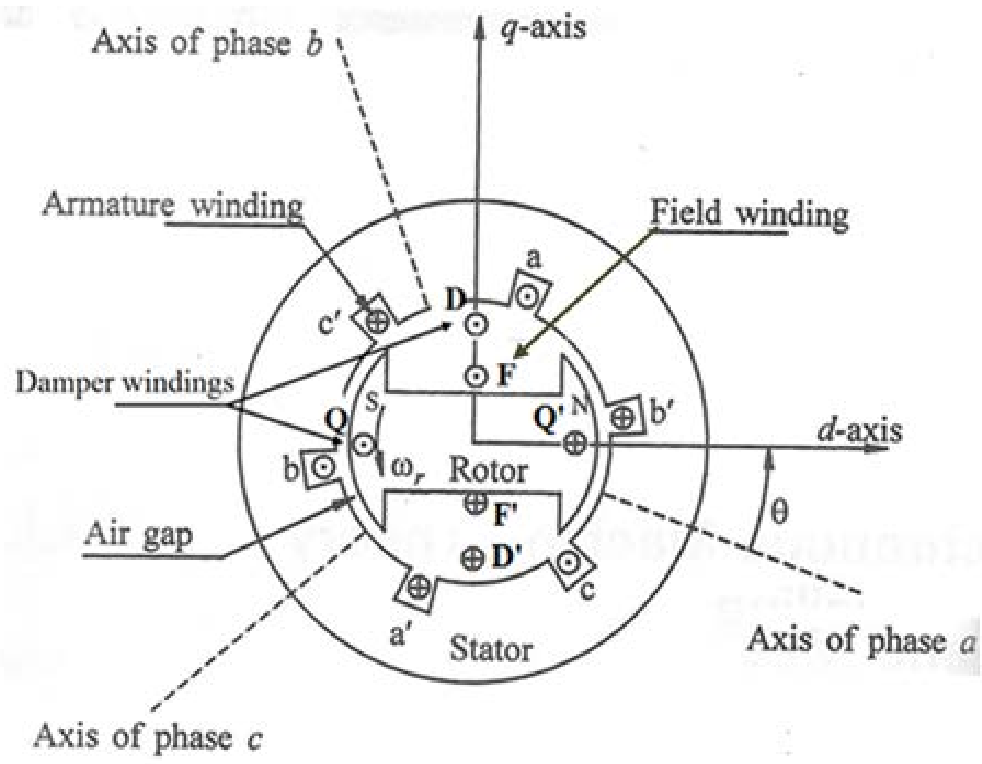
- a-a’, b-b’ and c-c’ windings

Rotor windings:

- Field windings
  - Field winding F-F’ produces a flux on the d-axis.

Damper windings

  - Two damper windings D-D’ and Q-Q’ respectively on d- and q-axes
  - For a round-rotor machine, consider a second damper winding G-G’ on the q-axis (two windings on each axis)



Total number of windings:

- Salient pole: 3+3 (discussed here)
- Round-rotor: 3+4

ANSI/IEEE standard 100-1977: the quadrature (q) axis is defined to lead the d-axis by 90^0

Winding Circuits

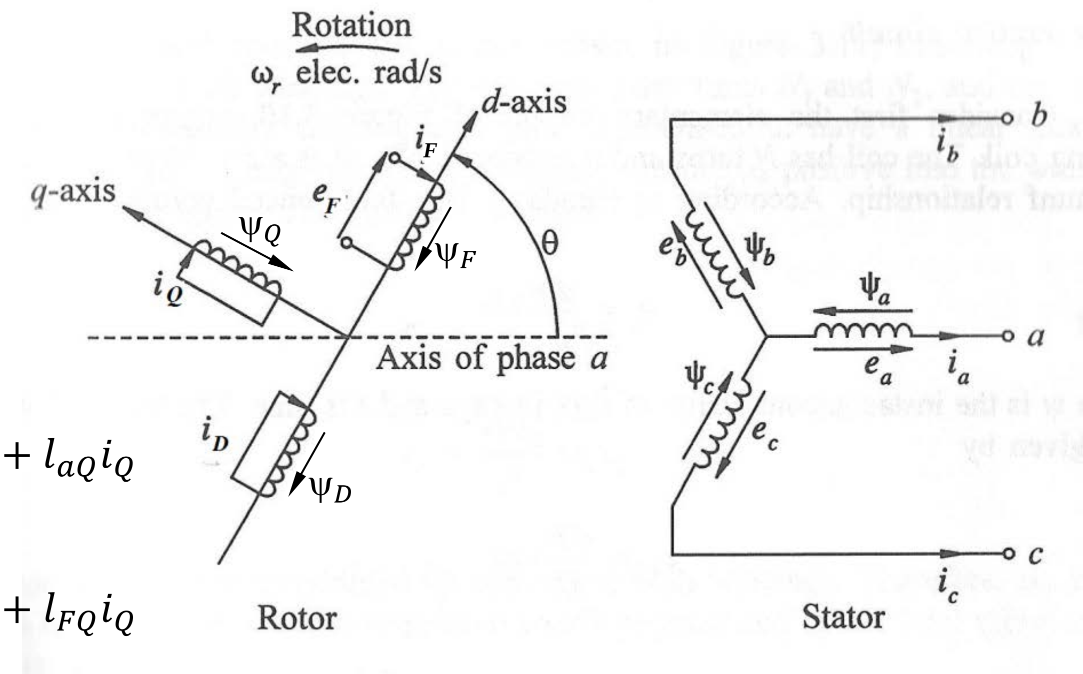
- Equations on the EMF (electromotive force) and flux of each winding

$e_a = d\psi_a/dt - R_a \times i_a$

$$\psi_a = -l_{aa}i_a - l_{ab}i_b - l_{ac}i_c + l_{aF}i_F + l_{aD}i_D + l_{aQ}i_Q$$

$e_F = d\psi_F/dt + R_F \times i_F$

$$\psi_F = -l_{Fa}i_a - l_{Fb}i_b - l_{Fc}i_c + l_{FF}i_F + l_{FD}i_D + l_{FQ}i_Q$$



$$\begin{bmatrix} \psi_a \\ \psi_b \\ \psi_c \\ - \\ \psi_F \\ \psi_D \\ \psi_Q \end{bmatrix} = \begin{bmatrix} l_{aa} & l_{ab} & l_{ac} & | & l_{aF} & l_{aD} & l_{aQ} \\ l_{ba} & l_{bb} & l_{bc} & | & l_{bF} & l_{bD} & l_{bQ} \\ l_{ca} & l_{cb} & l_{cc} & | & l_{cF} & l_{cD} & l_{cQ} \\ - & - & - & | & - & - & - \\ l_{Fa} & l_{Fb} & l_{Fc} & | & l_{FF} & l_{FD} & l_{FQ} \\ l_{Da} & l_{Db} & l_{Dc} & | & l_{DF} & l_{DD} & l_{DQ} \\ l_{Qa} & l_{Qb} & l_{Qc} & | & l_{QF} & l_{QD} & l_{QQ} \end{bmatrix} \begin{bmatrix} -i_a \\ -i_b \\ -i_c \\ \\ i_F \\ i_D \\ i_Q \end{bmatrix}$$

- Stator self-inductances (l_{aa}, l_{bb}, l_{cc})
- Stator mutual inductances (l_{ab}, l_{bc}, l_{ac})

- Stator-to-rotor mutual inductances (l_{aF}, l_{bD}, l_{aQ})

- Rotor self-inductances (l_{FF}, l_{DD}, l_{QQ})
- Rotor mutual inductances (l_{FD}, l_{DQ}, l_{FQ})

$$\begin{bmatrix} \psi_{abc} \\ \psi_{FDQ} \end{bmatrix} = \begin{bmatrix} \mathbf{L_{SS}} & \mathbf{L_{SR}} \\ \mathbf{L_{RS}} & \mathbf{L_{RR}} \end{bmatrix} \begin{bmatrix} -\mathbf{i}_{abc} \\ \mathbf{i}_{FDQ} \end{bmatrix}$$

Most of the efforts in synchronous machine modeling is to find constants and make the EMF and flux equations be simpler

$$\begin{bmatrix} \psi_a \\ \psi_b \\ \psi_c \\ - \\ \psi_F \\ \psi_D \\ \psi_Q \end{bmatrix} = \begin{bmatrix} l_{aa} & l_{ab} & l_{ac} & | & l_{aF} & l_{aD} & l_{aQ} \\ l_{ba} & l_{bb} & l_{bc} & | & l_{bF} & l_{bD} & l_{bQ} \\ l_{ca} & l_{cb} & l_{cc} & | & l_{bF} & l_{bD} & l_{bQ} \\ - & - & - & | & - & - & - \\ l_{Fa} & l_{Fb} & l_{Fc} & | & l_{FF} & l_{FD} & l_{FQ} \\ l_{Da} & l_{Db} & l_{Dc} & | & l_{DF} & l_{DD} & l_{DQ} \\ l_{Qa} & l_{Qb} & l_{Qc} & | & l_{QF} & l_{QD} & l_{QQ} \end{bmatrix} \begin{bmatrix} -i_a \\ -i_b \\ -i_c \\ - \\ i_F \\ i_D \\ i_Q \end{bmatrix} \text{Wb·Turns}$$

- The matrix is symmetric because the mutual inductance by definition is the flux linkage with one winding per unit current in the other winding, i.e.

$$l_{xy} \overset{\text{def}}{=} N_x \times \Phi_{my} / i_y = N_x \times N_y \times P_{xy} = l_{yx}$$

N_x ~ turns of winding x

Φ_{my} ~ mutual flux linking windings x and y due to current in winding y

P_{xy} ~ permeance of the mutual flux path

- A salient pole machine has significantly different permeances in d and q axes, such that the P_{xy} involving a stator winding (e.g. P_{ab} and P_{aF}) is a function of the rotor position α and reaches the maximum twice during 0º~360º

$$P_{xy} \approx P_0 + P_2 \cos 2\alpha$$

It is advisable to consider d- and q-axis components of P_{xy} individually

Stator self-inductances (l_{aa}, l_{bb}, l_{cc})

- l_{aa} is equal to the ratio of flux linking phase a winding to current i_a, with zero currents in all other circuits, and an be approximated as

$$l_{aa} = L_{aa0} + L_{aa2}\cos 2\theta$$

- Detailed calculation:
  - MMF$_a$ has a sinusoidal distribution in space with its peak centered on phase a axis. Resolve MMF$_a$ into two MMFs centered on d and q axes

    MMF$_{ad}$ has peak = $N_a i_a \cos\theta$

    MMF$_{aq}$ has peak = $-N_a i_a \sin\theta$
  - Air-gap fluxes

$$\Phi_{gad} = (N_a i_a \cos\theta) P_d$$

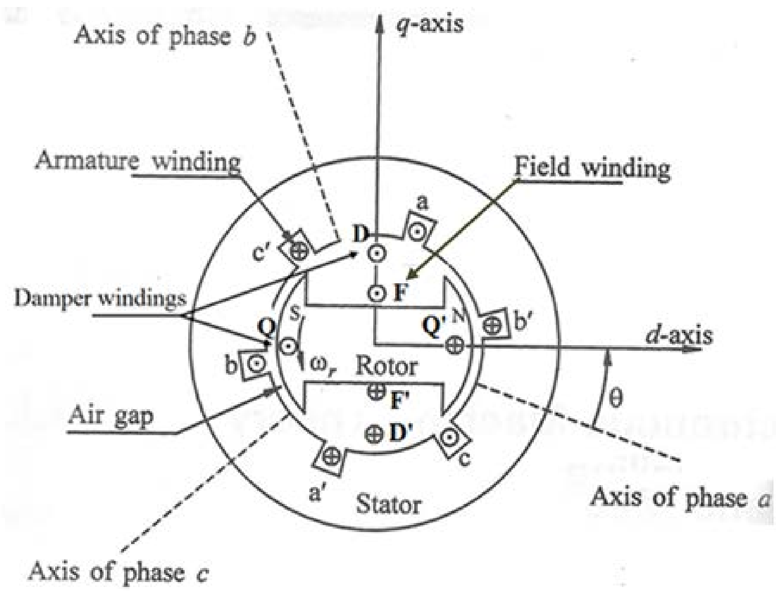
$$\Phi_{gaq} = (-N_a i_a \sin\theta) P_q$$

$$\Phi_{gaa} = \Phi_{gad}\cos\theta - \Phi_{gaq}\sin\theta = N_a i_a (P_d \cos^2\theta + P_q \sin^2\theta)$$

$$= N_a i_a \left(\frac{P_d + Pq}{2} + \frac{P_d - P_q}{2}\cos 2\theta\right)$$

$$l_{gaa} = N_a \Phi_{gad}/i_a = N_a^2 \left(\frac{P_d + Pq}{2} + \frac{P_d - P_q}{2}\cos 2\theta\right)$$

$$= L_{g0} + L_m \cos 2\theta$$



- Add the leakage inductance:

$$l_{aa} = l_{al} + l_{gaa}$$

$$= L_{al} + L_{g0} + L_m \cos 2\theta$$

$$= L_s + L_m \cos 2\theta$$

$L_s > L_m \geq 0$

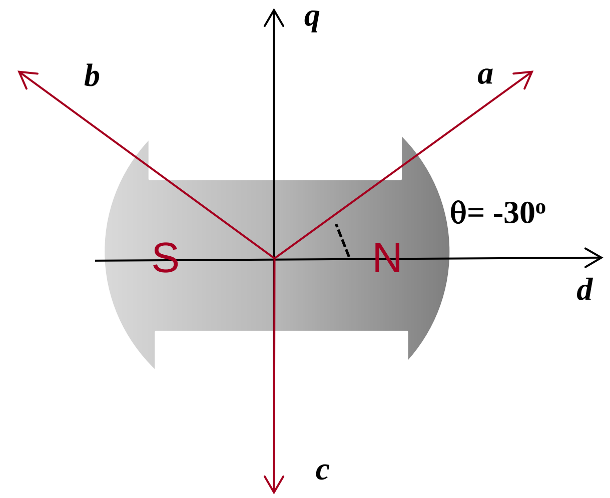
$$l_{aa} = L_s + L_m \cos 2\theta$$

$$l_{bb} = L_s + L_m \cos 2(\theta - 2\pi/3)$$

$$l_{cc} = L_s + L_m \cos 2(\theta + 2\pi/3)$$

Stator Mutual Inductances (l_{ab}, l_{bc}, l_{ac})

- l_{ab} <0 since windings a and b have120º (>90º) displacement
- Has the maximum absolute value when θ = -30⁰ or 150⁰.



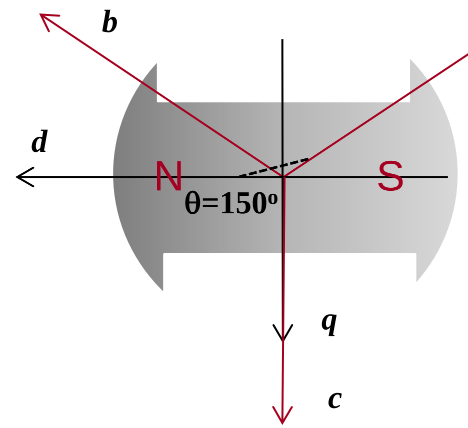
- Detailed calculation:

$$\Phi_{gba} = \Phi_{gad}\cos(\theta\text{-}2\pi/3) - \Phi_{gaq}\sin(\theta\text{-}2\pi/3)$$

$$=N_a i_a[P_d\cos\theta\cos(\theta\text{-}2\pi/3)+P_q\sin\theta\sin(\theta\text{-}2\pi/3)]$$

$$= N_a i_a[- \frac{P_d+Pq}{4} + \frac{P_d-P_q}{2}\cos(2\theta\text{-}2\pi/3)]$$

$$l_{gba} = N_a\Phi_{gba}/i_a = -L_{g0}/2 + L_m\cos(2\theta\text{-}2\pi/3)$$



  - Add leakage flux:

$$l_{ab} = l_{ba} = L_{al} - L_{g0}/2 + L_m\cos(2\theta\text{-}2\pi/3)$$

$$= -M_s + L_m\cos(2\theta\text{-}2\pi/3)$$

$$= -M_s - L_m\cos2(\theta+\pi/6)$$

$M_s \approx L_s/2$

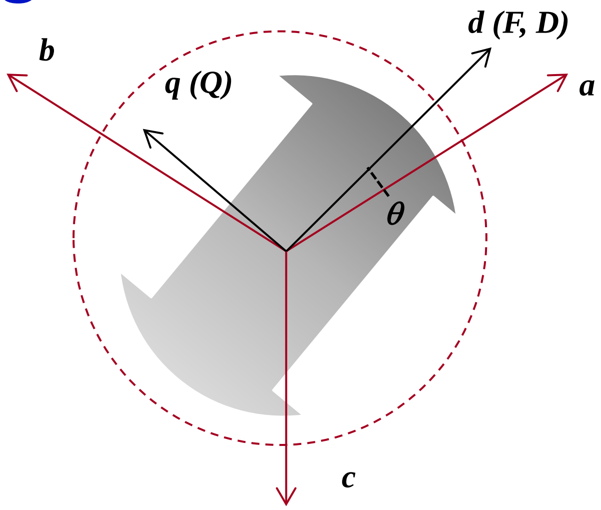
$$l_{ab} = -M_s - L_m\cos2(\theta+\pi/6)$$

$$l_{bc} = -M_s - L_m\cos2(\theta-\pi/2)$$

$$l_{ca} = -M_s - L_m\cos2(\theta+5\pi/6)$$

Stator to Rotor Mutual Inductances

(l_{aF}, l_{bF}, l_{cF}, l_{aD}, l_{bD}, l_{cD}, l_{aQ}, l_{bQ}, l_{cQ})



- The rotor sees a constant permeance if neglecting variations in the air gap due to stator slots
- When the flux linking a stator winding and a rotor winding reaches the maximum when they aligns with each other and is 0 when they are displaced by 90°
- d-axis

$$l_{aF} = l_{Fa} = M_F \cos\theta \qquad l_{aD} = l_{Da} = M_D \cos\theta$$

$$l_{bF} = l_{Fb} = M_F \cos(\theta-2\pi/3) \qquad l_{bD} = l_{Db} = M_D \cos(\theta-2\pi/3)$$

$$l_{cF} = l_{Fc} = M_F \cos(\theta+2\pi/3) \qquad l_{cD} = l_{Dc} = M_D \cos(\theta+2\pi/3)$$

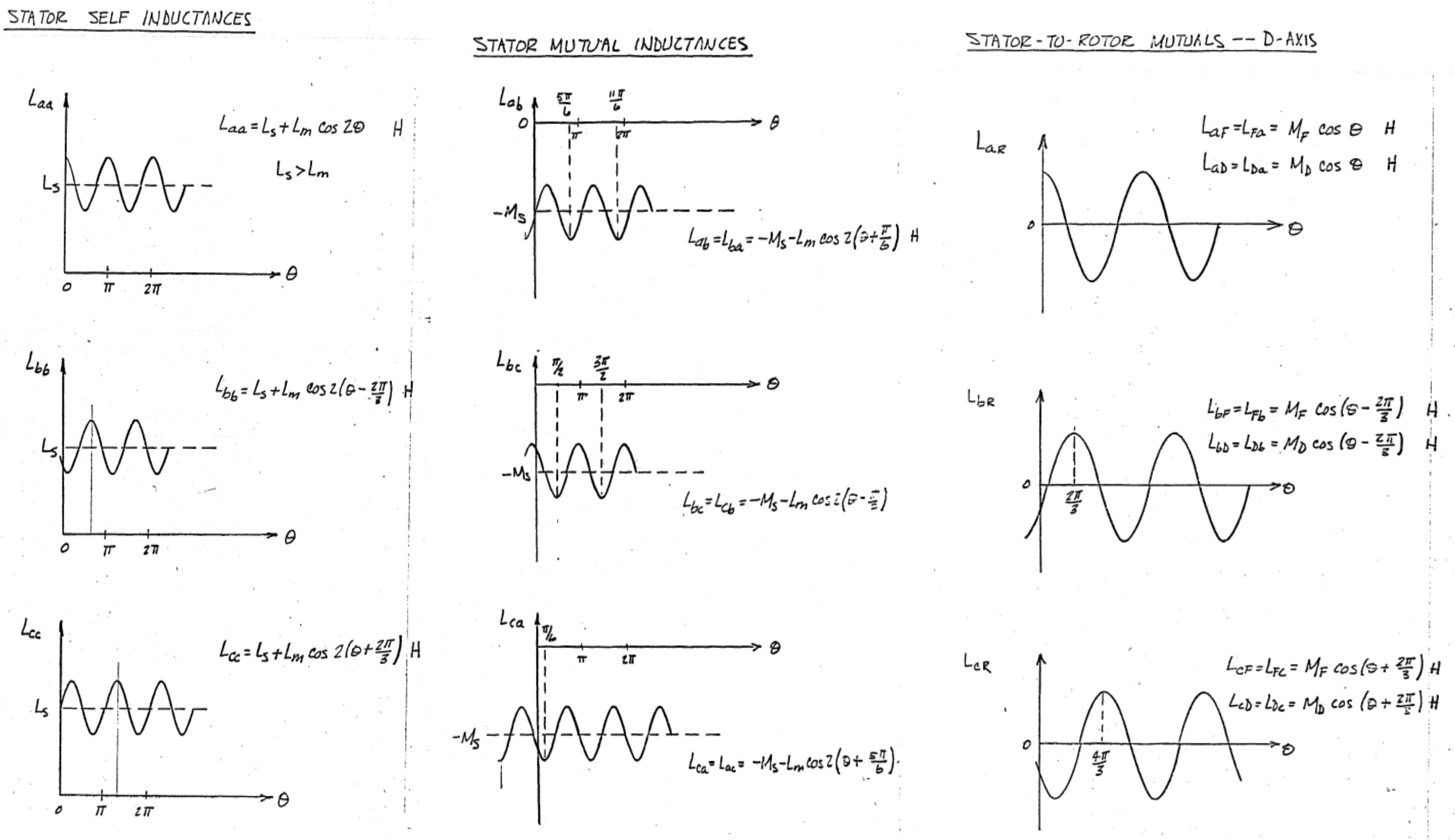
- q-axis

$$l_{aQ} = l_{Qa} = -M_Q \sin\theta$$

$$l_{bQ} = l_{Qb} = -M_Q \sin(\theta-2\pi/3)$$

$$l_{cQ} = l_{Qc} = -M_Q \sin(\theta+2\pi/3)$$

For Salient-pole Rotors



Which of the curves will be different for round rotors? *No 2[nd] harmonic*

Rotor Inductances (l_{FF}, l_{DD}, l_{QQ}, l_{FD}, l_{FQ}, l_{DQ})

- They are all constant
  - Rotor self inductances

$$l_{FF} \triangleq L_F$$
$$l_{DD} \triangleq L_D$$
$$l_{QQ} \triangleq L_Q$$

  - Rotor mutual inductances

$$l_{FD} = l_{DF} \triangleq M_R$$
$$l_{FQ} = l_{QF} = 0$$
$$l_{DQ} = l_{QD} = 0$$

Summary

$$\begin{bmatrix} \Psi_{abc} \\ \Psi_{FDQ} \end{bmatrix} = \begin{bmatrix} \boldsymbol{L_{SS}} & \boldsymbol{L_{SR}} \\ \boldsymbol{L_{RS}} & \boldsymbol{L_{RR}} \end{bmatrix} \begin{bmatrix} -\mathbf{i}_{abc} \\ \mathbf{i}_{FDQ} \end{bmatrix}$$

$$\boldsymbol{L_{SS}} = \begin{bmatrix} L_s + L_m\cos 2\theta & -M_s - L_m\cos 2\left(\theta + \frac{\pi}{6}\right) & -M_s - L_m\cos 2\left(\theta + \frac{5\pi}{6}\right) \\ -M_s - L_m\cos 2\left(\theta + \frac{\pi}{6}\right) & L_s + L_m\cos 2(\theta - \frac{2\pi}{3}) & -M_s - L_m\cos 2\left(\theta - \frac{\pi}{2}\right) \\ -M_s - L_m\cos 2\left(\theta + \frac{5\pi}{6}\right) & -M_s - L_m\cos 2\left(\theta - \frac{\pi}{2}\right) & L_s + L_m\cos 2(\theta + \frac{2\pi}{3}) \end{bmatrix}$$

$$\boldsymbol{L_{SR}} = \begin{bmatrix} M_F\cos\theta & M_D\cos\theta & -M_Q\sin\theta \\ M_F\cos\left(\theta - \frac{2\pi}{3}\right) & M_D\cos\left(\theta - \frac{2\pi}{3}\right) & -M_Q\sin\left(\theta - \frac{2\pi}{3}\right) \\ M_F\cos\left(\theta + \frac{2\pi}{3}\right) & M_D\cos\left(\theta + \frac{2\pi}{3}\right) & -M_Q\sin\left(\theta + \frac{2\pi}{3}\right) \end{bmatrix}$$

$$\boldsymbol{L_{RS}} = \boldsymbol{L^T}_{\boldsymbol{SR}}$$

$$\boldsymbol{L_{RR}} = \begin{bmatrix} L_F & M_R & 0 \\ M_R & L_D & 0 \\ 0 & 0 & L_Q \end{bmatrix}$$

What if we define q-axis lagging d-axis by 90°?

- Only $\boldsymbol{L_{RR}}$ is constant
- $\boldsymbol{L_{SS}}$ and $\boldsymbol{L_{SR}}$ are θ or time dependent
- How to simplify $\boldsymbol{L_{SS}}$ and $\boldsymbol{L_{SR}}$?
  - Diagonalize $\boldsymbol{L_{SS}}$
  - Remove time dependency

Observations

$$\begin{bmatrix} \Psi_{abc} \\ \Psi_{FDQ} \end{bmatrix} = \begin{bmatrix} L_{SS} & L_{SR} \\ L_{RS} & L_{RR} \end{bmatrix} \begin{bmatrix} -i_{abc} \\ i_{FDQ} \end{bmatrix}$$

- All harmonic terms in $\mathbf{L_{SS}}$ (1st harmonic) and $\mathbf{L_{SR}}$ (2nd harmonic) are due to the rotor rotating relative to a, b and c to cause variations in permeance
- Constant $\mathbf{L_{RR}}$ doesn’t have harmonic terms because it is in a reference frame rotating with the rotor

$$\psi_{FDQ} = -\mathbf{L}_{SR}\,\mathbf{i}_{abc} + \mathbf{L}_{RR}\,\mathbf{i}_{FDQ}$$

$$\mathbf{L}_{SR}\,\mathbf{i}_{abc} = -\psi_{FDQ} + \mathbf{L}_{RR}\,\mathbf{i}_{FDQ}$$

– $\mathbf{L}_{SR}\,\mathbf{i}_{abc}$ may be represented by functions independent of θ

– Represent stator currents and flux linkages also in a reference frame rotating with the rotor.

$$\boldsymbol{\psi_{FDQ}} = \boxed{\mathbf{-\, L_{SR}\, i_{abc}}} + \mathbf{L_{RR}\, i_{FDQ}}$$

$$\psi_F = -l_{aF}i_a - l_{bF}i_b - l_{cF}i_c \quad + l_{FF}i_F + l_{FD}i_D + l_{FQ}i_Q$$

$$= -M_F\cos\theta\, i_a - M_F\cos(\theta-2\pi/3)\, i_b - M_F\cos(\theta+2\pi/3) i_c \;\; + L_F i_F + M_R i_D + 0$$

$$= -M_F\, \mathbf{[}\boldsymbol{i_a}\, \mathbf{\cos\theta} + \boldsymbol{i_b}\, \mathbf{\cos(\theta-2\pi/3)} + \boldsymbol{i_c}\, \mathbf{\cos(\theta+2\pi/3)]} \quad + L_F i_F + M_R i_D$$

$$= K_1 \times i_d$$

$$\psi_D = -l_{aD}i_a - l_{bD}i_b - l_{cD}i_c \quad + l_{DF}i_F + l_{DD}i_D + l_{DQ}i_Q$$

$$= -M_D\cos\theta\, i_a - M_D\cos(\theta-2\pi/3)\, i_b - M_D\cos(\theta+2\pi/3) i_c \;\; + M_R i_F + L_D i_D + 0$$

$$= -M_D\, \mathbf{[}\boldsymbol{i_a}\, \mathbf{\cos\theta} + \boldsymbol{i_b}\, \mathbf{\cos(\theta-2\pi/3)} + \boldsymbol{i_c}\, \mathbf{\cos(\theta+2\pi/3)]} \quad + M_R i_F + L_D i_D$$

$$= K_2 \times i_d$$

$$\psi_Q = -l_{aQ}i_a - l_{bQ}i_b - l_{cQ}i_c \quad + l_{QF}i_F + l_{QD}i_D + l_{QQ}i_Q$$

$$= M_Q\sin\theta\, i_a + M_Q\sin(\theta-2\pi/3)\, i_b + M_Q\sin(\theta+2\pi/3)\, i_c \quad + 0 + 0 + L_Q i_Q$$

$$= M_Q\, \mathbf{[}\boldsymbol{i_a}\, \mathbf{\sin\theta} + \boldsymbol{i_b}\, \mathbf{\sin(\theta-2\pi/3)} + \boldsymbol{i_c}\, \mathbf{\sin(\theta+2\pi/3)\ ]} \quad + L_Q i_Q$$

$$= K_3 \times i_q$$

Park's (dq0) Transformation

- Define

$i_d = \; k_d\,[i_a \cos\theta + i_b \cos(\theta-2\pi/3) + i_c \cos(\theta+2\pi/3)]$

$i_q = -\, k_g\,[i_a \sin\theta + i_b \sin(\theta-2\pi/3) + i_c \sin(\theta+2\pi/3)\,]$

For balanced steady-state conditions:

$i_a = I_m \sin\omega_s t \quad i_b = I_m \sin(\omega_s t - 2\pi/3)$

$i_c = I_m \sin(\omega_s t + 2\pi/3)$

⇒

$i_d = k_d\, I_m \sin(\omega_s t-\theta)\times 3/2$

$i_q = -k_q\, I_m \cos(\omega_s t-\theta)\times 3/2$

$\theta=\omega_r t+\theta_0,\ \omega_r\approx\omega_s$

⇒

$i_d = -k_d\, I_m \sin\theta_0 \times 3/2$

$i_q = -k_q\, I_m \cos\theta_0 \times 3/2$

Constant

- Define

$i_0 = k_0(i_a + i_b + i_c)$

$$\begin{bmatrix} i_d \\ i_q \\ i_0 \end{bmatrix} = \begin{bmatrix} k_d\cos\theta & k_d\cos\left(\theta-\frac{2\pi}{3}\right) & k_d\cos\left(\theta+\frac{2\pi}{3}\right) \\ -k_q\sin\theta & -k_q\sin\left(\theta-\frac{2\pi}{3}\right) & -k_q\sin\left(\theta+\frac{2\pi}{3}\right) \\ k_0 & k_0 & k_0 \end{bmatrix} \begin{bmatrix} i_a \\ i_b \\ i_c \end{bmatrix}$$

What if we define q-axis lagging d-axis by 90°?

Park's Transformation Matrix - P

$$\begin{bmatrix}\Psi_{abc}\\ \Psi_{FDQ}\end{bmatrix}=\begin{bmatrix}L_{SS} & L_{SR}\\ L_{RS} & L_{RR}\end{bmatrix}\begin{bmatrix}-i_{abc}\\ i_{FDQ}\end{bmatrix}$$

$$\boldsymbol{P}=\begin{bmatrix}k_d cos\theta & k_d cos\left(\theta-\frac{2\pi}{3}\right) & k_d cos\left(\theta+\frac{2\pi}{3}\right)\\ -k_q sin\theta & -k_q sin\left(\theta-\frac{2\pi}{3}\right) & -k_q sin\left(\theta+\frac{2\pi}{3}\right)\\ k_0 & k_0 & k_0\end{bmatrix}$$

$\Psi_{dq0}= P\,\Psi_{abc}$ $\qquad$ $i_{dq0}= P\,i_{abc}$

$$\begin{bmatrix}\Psi_{dq0}\\ \Psi_{FDQ}\end{bmatrix}=\begin{bmatrix}P & 0\\ 0 & I_3\end{bmatrix}\begin{bmatrix}\Psi_{abc}\\ \Psi_{FDQ}\end{bmatrix}=\begin{bmatrix}P & 0\\ 0 & I_3\end{bmatrix}\begin{bmatrix}L_{SS} & L_{SR}\\ L_{RS} & L_{RR}\end{bmatrix}\begin{bmatrix}P^{-1} & 0\\ 0 & I_3\end{bmatrix}\begin{bmatrix}P & 0\\ 0 & I_3\end{bmatrix}\begin{bmatrix}-i_{abc}\\ i_{FDQ}\end{bmatrix}=\begin{bmatrix}PL_{SS}P^{-1} & PL_{SR}\\ L_{RS}P^{-1} & L_{RR}\end{bmatrix}\begin{bmatrix}-i_{dq0}\\ i_{FDQ}\end{bmatrix}$$

$$\begin{bmatrix}\Psi_{dq0}\\ \Psi_{FDQ}\end{bmatrix}\triangleq\begin{bmatrix}L'_{SS} & L'_{SR}\\ L'_{RS} & L_{RR}\end{bmatrix}\begin{bmatrix}-i_{dq0}\\ i_{FDQ}\end{bmatrix}$$

We hope $L'_{RS}=(L'_{SR})^T$ like $L_{RS}=(L_{SR})^T$

$$L'_{RS}=L_{RS}P^{-1}=L^T_{SR}P^{-1}=(P^{-T}L_{SR})^T$$

$$(L'_{SR})^T=(P\,L_{SR})^T$$

$P^{-T}=P$ or $P^TP=I$ (**P** is a unitary matrix)

$k_d=k_q=\sqrt{\frac{2}{3}}$ and $k_0=\frac{1}{\sqrt{3}}$

$$\mathbf{P}=\sqrt{\frac{2}{3}}\begin{bmatrix}\cos\theta & \cos\left(\theta-\frac{2\pi}{3}\right) & \cos\left(\theta+\frac{2\pi}{3}\right)\\ -\sin\theta & -\sin\left(\theta-\frac{2\pi}{3}\right) & -\sin\left(\theta+\frac{2\pi}{3}\right)\\ 1/\sqrt{2} & 1/\sqrt{2} & 1/\sqrt{2}\end{bmatrix}$$

$$\mathbf{P}^{-1}=\sqrt{\frac{2}{3}}\begin{bmatrix}\cos\theta & -\sin\theta & 1/\sqrt{2}\\ \cos\left(\theta-\frac{2\pi}{3}\right) & -\sin\left(\theta-\frac{2\pi}{3}\right) & 1/\sqrt{2}\\ \cos\left(\theta+\frac{2\pi}{3}\right) & -\sin\left(\theta+\frac{2\pi}{3}\right) & 1/\sqrt{2}\end{bmatrix}$$

Flux Equations after Park's Transformation

$$\begin{bmatrix}\psi_d\\ \psi_q\\ \psi_0\\ -\\ \psi_F\\ \psi_D\\ \psi_Q\end{bmatrix} = \begin{bmatrix} L_d & 0 & 0 & | & kM_F & kM_D & 0\\ 0 & L_q & 0 & | & 0 & 0 & kM_Q\\ 0 & 0 & L_0 & | & 0 & 0 & 0\\ - & - & - & | & - & - & -\\ kM_F & 0 & 0 & | & L_F & M_R & 0\\ kM_D & 0 & 0 & | & M_R & L_D & 0\\ 0 & kM_Q & 0 & | & 0 & 0 & L_Q \end{bmatrix} \begin{bmatrix}-i_d\\ -i_q\\ -i_0\\ -\\ i_F\\ i_D\\ i_Q\end{bmatrix}$$

$$k = \sqrt{\frac{3}{2}}$$

$$L_d = L_s + M_s + 3L_m/2$$

$$L_q = L_s + M_s - 3L_m/2$$

$$L_0 = L_s - 2M_s$$

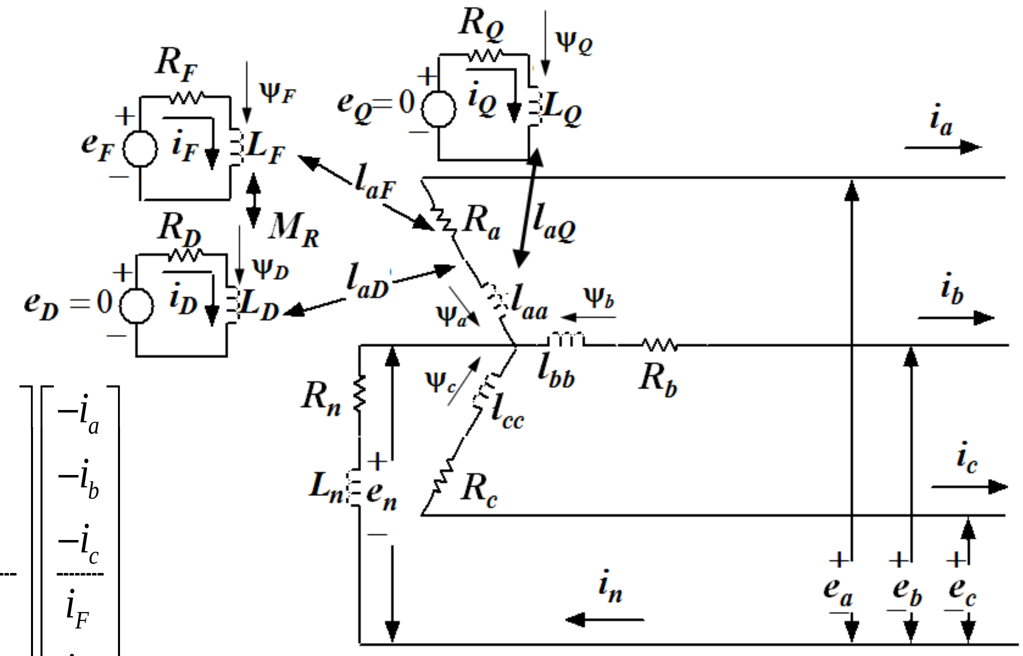
$$\begin{bmatrix}\psi_0\\ \psi_d\\ \psi_F\\ \psi_D\\ \psi_q\\ \psi_Q\end{bmatrix} = \begin{bmatrix} L_0 & & & & & \\ & L_d & kM_F & kM_D & & \\ & kM_F & L_F & M_R & & \\ & kM_D & M_R & L_D & & \\ & & & & L_q & kM_Q\\ & & & & kM_Q & L_Q \end{bmatrix} \begin{bmatrix}-i_0\\ -i_d\\ i_F\\ i_D\\ -i_q\\ i_Q\end{bmatrix}$$

Voltage Equations

$$\mathbf{e} = d\psi/dt \pm \mathbf{R}\times\mathbf{i}$$

Stator: ψ and i are in opposite directions

Rotor: ψ and i are the same directions



(A neutral line is added compared to slide #8)

$$\begin{bmatrix} e_a \\ e_b \\ e_c \\ e_F \\ e_D = 0 \\ e_Q = 0 \end{bmatrix} - \begin{bmatrix} e_n \\ e_n \\ e_n \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} \dot{\psi}_a \\ \dot{\psi}_b \\ \dot{\psi}_c \\ \dot{\psi}_F \\ \dot{\psi}_D \\ \dot{\psi}_Q \end{bmatrix} + \begin{bmatrix} R_a & & & & & \\ & R_b & & & 0 & \\ & & R_c & & & \\ & & & R_F & & \\ & 0 & & & R_D & \\ & & & & & R_Q \end{bmatrix} \begin{bmatrix} -i_a \\ -i_b \\ -i_c \\ i_F \\ i_D \\ i_Q \end{bmatrix}$$

$$\begin{bmatrix} \mathbf{e_{abc}} \\ \mathbf{e_{FDQ}} \end{bmatrix} = \begin{bmatrix} \mathbf{R_{abc}} & \mathbf{0} \\ \mathbf{0} & \mathbf{R_{FDQ}} \end{bmatrix} \begin{bmatrix} -\mathbf{i_{abc}} \\ \mathbf{i_{FDQ}} \end{bmatrix} + \begin{bmatrix} \dot{\boldsymbol{\psi}}_{\mathbf{abc}} \\ \dot{\boldsymbol{\psi}}_{\mathbf{FDQ}} \end{bmatrix} + \begin{bmatrix} \mathbf{e_n} \\ \mathbf{0} \end{bmatrix}$$

Neutral line:

$$\mathbf{e_n} \stackrel{\Delta}{=} \begin{bmatrix} e_n \\ e_n \\ e_n \end{bmatrix} = -(R_n i_n + L_n \frac{di_n}{dt}) \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = -\begin{bmatrix} R_n & R_n & R_n \\ R_n & R_n & R_n \\ R_n & R_n & R_n \end{bmatrix} \begin{bmatrix} i_a \\ i_b \\ i_c \end{bmatrix} - \begin{bmatrix} L_n & L_n & L_n \\ L_n & L_n & L_n \\ L_n & L_n & L_n \end{bmatrix} \frac{d}{dt} \begin{bmatrix} i_a \\ i_b \\ i_c \end{bmatrix}$$

$$= -\mathbf{R_n} \mathbf{i_{abc}} - \mathbf{L_n} \frac{d}{dt} \mathbf{i_{abc}}$$

$$\begin{bmatrix} \mathbf{e}_{abc} \\ \mathbf{e}_{FDQ} \end{bmatrix} = \begin{bmatrix} \mathbf{R}_{abc} & \mathbf{0} \\ \mathbf{0} & \mathbf{R}_{FDQ} \end{bmatrix} \begin{bmatrix} -\mathbf{i}_{abc} \\ \mathbf{i}_{FDQ} \end{bmatrix} + \begin{bmatrix} \dot{\psi}_{abc} \\ \dot{\psi}_{FDQ} \end{bmatrix} + \begin{bmatrix} \mathbf{e}_{n} \\ \mathbf{0} \end{bmatrix}$$

Assume $R_a = R_b = R_c$

$$\mathbf{R} \triangleq \begin{bmatrix} R_a & & \\ & R_a & \\ & & R_a \end{bmatrix}, \quad \mathbf{R_R} \triangleq \begin{bmatrix} R_F & & \\ & R_D & \\ & & R_Q \end{bmatrix}$$

$$\mathbf{e}_{dq0} \triangleq \mathbf{P} \cdot \mathbf{e}_{abc} \qquad \psi_{dq0} = \mathbf{P}\, \psi_{abc} \qquad \mathbf{i}_{dq0} = \mathbf{P}\, \mathbf{i}_{abc}$$

$$\begin{bmatrix} \mathbf{e}_{dq0} \\ \mathbf{e}_{FDQ} \end{bmatrix} = \begin{bmatrix} \mathbf{P} & \mathbf{0} \\ \mathbf{0} & \mathbf{I}_3 \end{bmatrix} \begin{bmatrix} \mathbf{e}_{abc} \\ \mathbf{e}_{FDQ} \end{bmatrix}$$

$$= \begin{bmatrix} \mathbf{P} & \mathbf{0} \\ \mathbf{0} & \mathbf{I}_3 \end{bmatrix} \begin{bmatrix} \mathbf{R} & \mathbf{0} \\ \mathbf{0} & \mathbf{R_R} \end{bmatrix} \begin{bmatrix} -\mathbf{i}_{abc} \\ \mathbf{i}_{FDQ} \end{bmatrix} + \begin{bmatrix} \mathbf{P} & \mathbf{0} \\ \mathbf{0} & \mathbf{I}_3 \end{bmatrix} \begin{vmatrix} \dot{\Psi}_{abc} \\ \dot{\Psi}_{FDO} \end{vmatrix} + \begin{bmatrix} \mathbf{P} & \mathbf{0} \\ \mathbf{0} & \mathbf{I}_3 \end{bmatrix} \begin{bmatrix} \mathbf{e}_{n} \\ \mathbf{0} \end{bmatrix}$$

$$= \begin{bmatrix} \mathbf{P} & \mathbf{0} \\ \mathbf{0} & \mathbf{I}_3 \end{bmatrix} \begin{bmatrix} \mathbf{R} & \mathbf{0} \\ \mathbf{0} & \mathbf{R_R} \end{bmatrix} \begin{bmatrix} \mathbf{P}^{-1} & \mathbf{0} \\ \mathbf{0} & \mathbf{I}_3 \end{bmatrix} \begin{bmatrix} \mathbf{P} & \mathbf{0} \\ \mathbf{0} & \mathbf{I}_3 \end{bmatrix} \begin{bmatrix} -\mathbf{i}_{abc} \\ \mathbf{i}_{FDQ} \end{bmatrix} + \begin{bmatrix} \mathbf{P} & \mathbf{0} \\ \mathbf{0} & \mathbf{I}_3 \end{bmatrix} \begin{bmatrix} \dot{\Psi}_{abc} \\ \dot{\Psi}_{FDQ} \end{bmatrix} + \begin{bmatrix} \mathbf{P} & \mathbf{0} \\ \mathbf{0} & \mathbf{I}_3 \end{bmatrix} \begin{bmatrix} \mathbf{e}_{n} \\ \mathbf{0} \end{bmatrix}$$

$$= \begin{bmatrix} \mathbf{R} & \mathbf{0} \\ \mathbf{0} & \mathbf{R_R} \end{bmatrix} \begin{bmatrix} -\mathbf{i}_{dq0} \\ \mathbf{i}_{FDQ} \end{bmatrix} + \begin{bmatrix} \mathbf{P} \cdot \dot{\Psi}_{abc} \\ \dot{\Psi}_{FDQ} \end{bmatrix} + \begin{bmatrix} \mathbf{P} \cdot \mathbf{e}_{n} \\ \mathbf{0} \end{bmatrix}$$

$\mathbf{P \cdot \dot{\psi}_{abc}}$

$$\begin{bmatrix} e_{dq0} \\ e_{FDQ} \end{bmatrix} = \begin{bmatrix} R & 0 \\ 0 & R_R \end{bmatrix} \begin{bmatrix} -i_{dq0} \\ i_{FDQ} \end{bmatrix} + \begin{bmatrix} P \cdot \dot{\psi}_{abc} \\ \dot{\psi}_{FDQ} \end{bmatrix} + \begin{bmatrix} P \cdot e_n \\ 0 \end{bmatrix}$$

$$P \cdot \dot{\psi}_{abc} \neq \dot{\psi}_{dq0}$$

$$\dot{\psi}_{dq0} = \frac{d(P\psi_{abc})}{dt} = \dot{P}\psi_{abc} + P\dot{\psi}_{abc} = \dot{P}P^{-1}\psi_{dq0} + P\dot{\psi}_{abc}$$

$$P = \sqrt{\frac{2}{3}} \begin{bmatrix} \cos\theta & \cos\left(\theta - \frac{2\pi}{3}\right) & \cos\left(\theta + \frac{2\pi}{3}\right) \\ -\sin\theta & -\sin\left(\theta - \frac{2\pi}{3}\right) & -\sin\left(\theta + \frac{2\pi}{3}\right) \\ 1/\sqrt{2} & 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix} \quad P^{-1} = \sqrt{\frac{2}{3}} \begin{bmatrix} \cos\theta & -\sin\theta & 1/\sqrt{2} \\ \cos\left(\theta - \frac{2\pi}{3}\right) & -\sin\left(\theta - \frac{2\pi}{3}\right) & 1/\sqrt{2} \\ \cos\left(\theta + \frac{2\pi}{3}\right) & -\sin\left(\theta + \frac{2\pi}{3}\right) & 1/\sqrt{2} \end{bmatrix}$$

$$\theta = \omega_r t + \theta_0$$

$$\dot{P} = \sqrt{\frac{2}{3}} \begin{bmatrix} -\sin\theta \cdot \omega_r & -\sin\left(\theta - \frac{2\pi}{3}\right) \cdot \omega_r & -\sin\left(\theta + \frac{2\pi}{3}\right) \cdot \omega_r \\ -\cos\theta \cdot \omega_r & -\cos\left(\theta - \frac{2\pi}{3}\right) \cdot \omega_r & -\cos\left(\theta + \frac{2\pi}{3}\right) \cdot \omega_r \\ 0 & 0 & 0 \end{bmatrix} \qquad \dot{P}P^{-1} = \begin{bmatrix} 0 & \omega_r & 0 \\ -\omega_r & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\mathbf{P}\cdot\dot{\psi}_{\mathbf{abc}} = \dot{\psi}_{\mathbf{dq0}} - \dot{\mathbf{P}}\mathbf{P}^{-1}\psi_{\mathbf{dq0}}$$

$$= \dot{\psi}_{\mathbf{dq0}} - \begin{bmatrix} 0 & \omega_r & 0 \\ -\omega_r & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \psi_{\mathbf{dq0}} = \dot{\psi}_{\mathbf{dq0}} + \begin{bmatrix} -\omega_r \psi_q \\ \omega_r \psi_d \\ 0 \end{bmatrix}$$

Transformer voltages due to flux change in time (=0 under steady-state conditions)

Speed voltages due to flux change in space

$$\mathbf{S} \triangleq \begin{bmatrix} -\omega_r \psi_q \\ \omega_r \psi_d \\ 0 \end{bmatrix} = \begin{bmatrix} -\omega_r(-L_q i_q + k M_Q i_Q) \\ \omega_r(-L_d i_d + k M_F i_F + k M_D i_D) \\ 0 \end{bmatrix} = \begin{bmatrix} -\omega_r L_q(-i_q) - \omega_r k M_Q i_Q \\ \omega_r L_d(-i_d) + \omega_r k M_F i_F + \omega_r k M_D i_D \\ 0 \end{bmatrix}$$

$\mathbf{P}\cdot\mathbf{e}_n$

$$\mathbf{P}\,\mathbf{e}_n = \mathbf{P}\,(-\mathbf{R}_n\,\mathbf{i}_{abc} - \mathbf{L}_n\, d\mathbf{i}_{abc}/dt)$$

$$= -\mathbf{P}\,\mathbf{R}_n\,\mathbf{P}^{-1}\,\mathbf{P}\,\mathbf{i}_{abc} - \mathbf{P}\,\mathbf{L}_n\,\mathbf{P}^{-1}\,\mathbf{P}\, d\mathbf{i}_{abc}/dt$$

$$= -\mathbf{P}\,\mathbf{R}_n\,\mathbf{P}^{-1}\,\mathbf{i}_{dq0} - \mathbf{P}\,\mathbf{L}_n\,\mathbf{P}^{-1}\,(d\mathbf{i}_{dq0}/dt - d\mathbf{P}/dt \times \mathbf{i}_{abc})$$

$$\mathbf{P} = \sqrt{\frac{2}{3}}\begin{bmatrix} \cos\theta & \cos\left(\theta-\frac{2\pi}{3}\right) & \cos\left(\theta+\frac{2\pi}{3}\right) \\ -\sin\theta & -\sin\left(\theta-\frac{2\pi}{3}\right) & -\sin\left(\theta+\frac{2\pi}{3}\right) \\ 1/\sqrt{2} & 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}$$

$$\mathbf{P}^{-1} = \sqrt{\frac{2}{3}}\begin{bmatrix} \cos\theta & -\sin\theta & 1/\sqrt{2} \\ \cos\left(\theta-\frac{2\pi}{3}\right) & -\sin\left(\theta-\frac{2\pi}{3}\right) & 1/\sqrt{2} \\ \cos\left(\theta+\frac{2\pi}{3}\right) & -\sin\left(\theta+\frac{2\pi}{3}\right) & 1/\sqrt{2} \end{bmatrix}$$

$$\mathbf{P}\mathbf{R}_n\mathbf{P}^{-1} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 3R_n \end{bmatrix} \qquad \mathbf{P}\mathbf{L}_n\mathbf{P}^{-1} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 3L_n \end{bmatrix}$$

$$\dot{\mathbf{P}} = \sqrt{\frac{2}{3}}\begin{bmatrix} -\sin\theta\cdot\omega_r & -\sin\left(\theta-\frac{2\pi}{3}\right)\cdot\omega_r & -\sin\left(\theta+\frac{2\pi}{3}\right)\cdot\omega_r \\ -\cos\theta\cdot\omega_r & -\cos\left(\theta-\frac{2\pi}{3}\right)\cdot\omega_r & -\cos\left(\theta+\frac{2\pi}{3}\right)\cdot\omega_r \\ 0 & 0 & 0 \end{bmatrix}$$

Note: $\mathbf{P}\,\mathbf{L}_n\,\mathbf{P}^{-1} \times d\mathbf{P}/dt \times \mathbf{i}_{abc} = \mathbf{0}$

$$\mathbf{R}_n = \begin{bmatrix} R_n & R_n & R_n \\ R_n & R_n & R_n \\ R_n & R_n & R_n \end{bmatrix}$$

$$\mathbf{P}\mathbf{e}_n = -\begin{bmatrix} 0 \\ 0 \\ 3R_n i_0 \end{bmatrix} - \begin{bmatrix} 0 \\ 0 \\ 3L_n \frac{d}{d} i_0 \end{bmatrix} \triangleq \mathbf{n}_{dq0}$$

$$\mathbf{L}_n = \begin{bmatrix} L_n & L_n & L_n \\ L_n & L_n & L_n \\ L_n & L_n & L_n \end{bmatrix}$$

$$\begin{bmatrix} \mathbf{e_{dq0}} \\ \mathbf{e_{FDQ}} \end{bmatrix} = \begin{bmatrix} \mathbf{R} & \mathbf{0} \\ \mathbf{0} & \mathbf{R_R} \end{bmatrix} \begin{bmatrix} -\mathbf{i_{dq0}} \\ \mathbf{i_{FDQ}} \end{bmatrix} + \begin{bmatrix} \mathbf{P} \cdot \dot{\boldsymbol{\psi}}_{\mathbf{abc}} \\ \dot{\boldsymbol{\psi}}_{\mathbf{FDQ}} \end{bmatrix} + \begin{bmatrix} \mathbf{P} \cdot \mathbf{e_n} \\ \mathbf{0} \end{bmatrix}$$

$$\begin{bmatrix} \mathbf{e_{dq0}} \\ \mathbf{e_{FDQ}} \end{bmatrix} = \begin{bmatrix} \mathbf{R} & \mathbf{0} \\ \mathbf{0} & \mathbf{R_R} \end{bmatrix} \begin{bmatrix} -\mathbf{i_{dq0}} \\ \mathbf{i_{FDQ}} \end{bmatrix} + \begin{bmatrix} \dot{\boldsymbol{\psi}}_{\mathbf{dq0}} \\ \dot{\boldsymbol{\psi}}_{\mathbf{FDQ}} \end{bmatrix} + \begin{bmatrix} \mathbf{S} \\ \mathbf{0} \end{bmatrix} + \begin{bmatrix} \mathbf{n_{dq0}} \\ \mathbf{0} \end{bmatrix}$$

$$\begin{bmatrix} e_0 \\ e_d \\ e_F \\ e_D \\ e_q \\ e_Q \end{bmatrix} = \begin{bmatrix} R_a + 3R_n & & & & & \\ & R_a & & & & \\ & & R_F & & & \\ & & & R_D & & \\ & & & & R_a & \\ & & & & & R_Q \end{bmatrix} \begin{bmatrix} -i_0 \\ -i_d \\ i_F \\ i_D \\ -i_q \\ i_Q \end{bmatrix} + \begin{bmatrix} L_0 + 3L_n & & & & & \\ & L_d & kM_F & kM_D & & \\ & kM_F & L_F & M_R & & \\ & kM_D & M_R & L_D & & \\ & & & & L_q & kM_Q \\ & & & & kM_Q & L_Q \end{bmatrix} \times d \begin{bmatrix} -i_0 \\ -i_d \\ i_F \\ i_D \\ -i_q \\ i_Q \end{bmatrix} / dt + \begin{bmatrix} 0 \\ -\omega_r \psi_q \\ 0 \\ 0 \\ \omega_r \psi_d \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} e_0 \\ e_d \\ e_F \\ e_D \\ e_q \\ e_Q \end{bmatrix} = \begin{bmatrix} R_a + 3R_n & & & & & \\ & R_a & & & & \\ & & R_F & & & \\ & & & R_D & & \\ & & & & R_a & \\ & & & & & R_Q \end{bmatrix} \begin{bmatrix} -i_0 \\ -i_d \\ i_F \\ i_D \\ -i_q \\ i_Q \end{bmatrix} + \begin{bmatrix} L_0 + 3L_n & & & & & \\ & L_d & kM_F & kM_D & & \\ & kM_F & L_F & M_R & & \\ & kM_D & M_R & L_D & & \\ & & & & L_q & kM_Q \\ & & & & kM_Q & L_Q \end{bmatrix} \times d \begin{bmatrix} -i_0 \\ -i_d \\ i_F \\ i_D \\ -i_q \\ i_Q \end{bmatrix} / dt + \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -\omega_r L_q & -\omega_r kM_Q \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \omega_r L_d & \omega_r kM_F & \omega_r kM_D & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} -i_0 \\ -i_d \\ i_F \\ i_D \\ -i_q \\ i_Q \end{bmatrix}$$

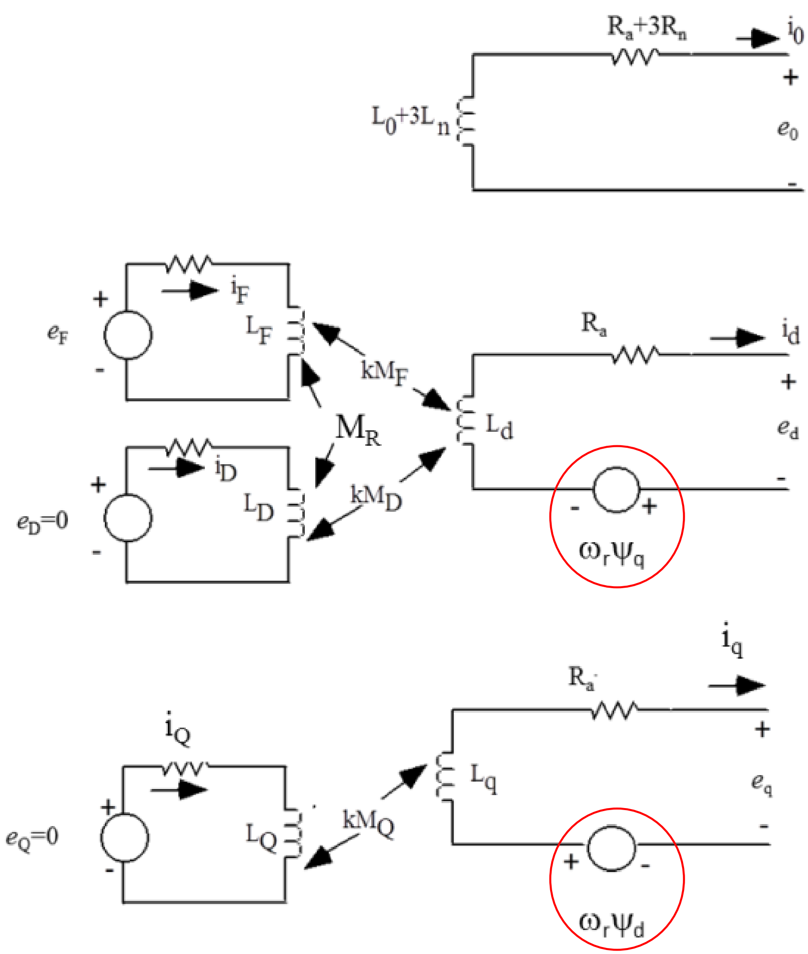
Voltage Equations after Park's Transformation

$$\begin{bmatrix} e_0 \\ e_d \\ e_F \\ e_D \\ e_q \\ e_Q \end{bmatrix} = \begin{bmatrix} \mathrm{R_a} + 3\mathrm{R_n} & 0 & 0 & 0 & 0 & 0 \\ 0 & \mathrm{R_a} & 0 & 0 & -\omega_r \mathrm{L_q} & -\omega_r k\mathrm{M_Q} \\ 0 & 0 & \mathrm{R_F} & 0 & 0 & 0 \\ 0 & 0 & 0 & \mathrm{R_D} & 0 & 0 \\ 0 & \omega_r \mathrm{L_d} & \omega_r k\mathrm{M_F} & \omega_r k\mathrm{M_D} & \mathrm{R_a} & 0 \\ 0 & 0 & 0 & 0 & 0 & \mathrm{R_Q} \end{bmatrix} \begin{bmatrix} -i_0 \\ -i_d \\ i_F \\ i_D \\ -i_q \\ i_Q \end{bmatrix} +$$

$$\begin{bmatrix} \mathrm{L_0} + 3\mathrm{L_n} & & & & & \\ & \mathrm{L_d} & k\mathrm{M_F} & k\mathrm{M_D} & & \\ & k\mathrm{M_F} & \mathrm{L_F} & \mathrm{M_R} & & \\ & k\mathrm{M_D} & \mathrm{M_R} & \mathrm{L_D} & & \\ & & & & \mathrm{L_q} & k\mathrm{M_Q} \\ & & & & k\mathrm{M_Q} & \mathrm{L_Q} \end{bmatrix} \times \mathrm{d} \begin{bmatrix} -i_0 \\ -i_d \\ i_F \\ i_D \\ -i_q \\ i_Q \end{bmatrix} / \mathrm{dt}$$

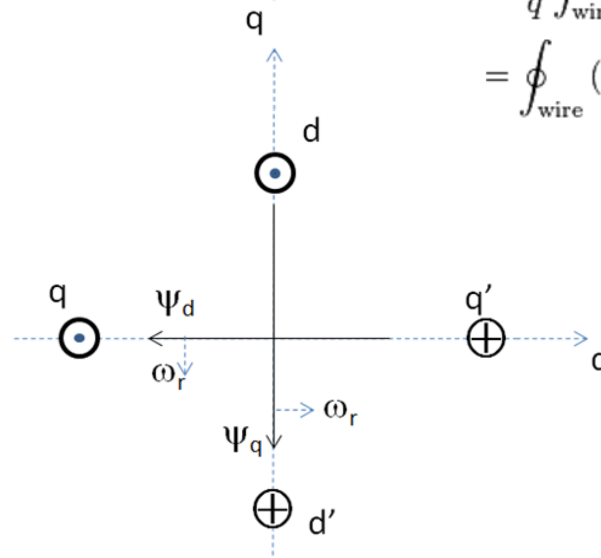
Winding Circuits after Park's Transformation

$$\begin{bmatrix} e_0 \\ e_d \\ e_F \\ e_D \\ e_q \\ e_Q \end{bmatrix} = \begin{bmatrix} R_a+3R_n & & & & & \\ & R_a & & & & \\ & & R_F & & & \\ & & & R_D & & \\ & & & & R_a & \\ & & & & & R_Q \end{bmatrix} \begin{bmatrix} -i_0 \\ -i_d \\ i_F \\ i_D \\ -i_q \\ i_Q \end{bmatrix} + \begin{bmatrix} L_0+3L_n & & & & & \\ & L_d & kM_F & kM_D & & \\ & kM_F & L_F & M_R & & \\ & kM_D & M_R & L_D & & \\ & & & & L_q & kM_Q \\ & & & & kM_Q & L_Q \end{bmatrix} \times d \begin{bmatrix} -i_0 \\ -i_d \\ i_F \\ i_D \\ -i_q \\ i_Q \end{bmatrix} /dt + \begin{bmatrix} 0 \\ -\omega_r \psi_q \\ 0 \\ 0 \\ \omega_r \psi_d \\ 0 \end{bmatrix}$$



- d−axis flux causes a speed voltage $\omega_r \psi_d$ in the q−axis winding
- q−axis flux causes a speed voltage $-\omega_r \psi_q$ in the d−axis winding

$$\mathcal{E} = \frac{1}{q} \oint_{\text{wire}} \mathbf{F} \cdot d\boldsymbol{\ell}$$
$$= \oint_{\text{wire}} (\mathbf{E} + \mathbf{v} \times \mathbf{B}) \cdot d\boldsymbol{\ell}$$



Alternative Park's Transformation

$$i_a = I_m \sin\omega_s t \qquad\qquad i_d = k_d\, I_m \sin(\omega_s t - \theta) \times 3/2$$

$$i_b = I_m \sin(\omega_s t - 2\pi/3) \quad \xleftrightarrow{\quad P \quad} \quad i_q = -k_q\, I_m \cos(\omega_s t - \theta) \times 3/2$$

$$i_c = I_m \sin(\omega_s t + 2\pi/3) \qquad\qquad i_0 = k_0(i_a + i_b + i_c)$$

- If $k_d = k_q = 2/3$ and $k_0 = 1/3$, a unit-to-unit relationship holds between *abc* and *dq0* variables.

$$\mathbf{P} = \frac{2}{3}\begin{bmatrix} \cos\theta & \cos\left(\theta - \frac{2\pi}{3}\right) & \cos\left(\theta + \frac{2\pi}{3}\right) \\ -\sin\theta & -\sin\left(\theta - \frac{2\pi}{3}\right) & -\sin\left(\theta + \frac{2\pi}{3}\right) \\ 1/2 & 1/2 & 1/2 \end{bmatrix} \qquad \mathbf{P}^{-1} = \begin{bmatrix} \cos\theta & -\sin\theta & 1 \\ \cos\left(\theta - \frac{2\pi}{3}\right) & -\sin\left(\theta - \frac{2\pi}{3}\right) & 1 \\ \cos\left(\theta + \frac{2\pi}{3}\right) & -\sin\left(\theta + \frac{2\pi}{3}\right) & 1 \end{bmatrix}$$

$$\begin{bmatrix} \psi_d \\ \psi_q \\ \psi_0 \\ \psi_F \\ \psi_D \\ \psi_Q \end{bmatrix} = \begin{bmatrix} L_d & 0 & 0 & M_F & M_D & 0 \\ 0 & L_q & 0 & 0 & 0 & M_Q \\ 0 & 0 & L_0 & 0 & 0 & 0 \\ \frac{3}{2}M_F & 0 & 0 & L_F & M_R & 0 \\ \frac{3}{2}M_D & 0 & 0 & M_R & L_D & 0 \\ 0 & \frac{3}{2}M_Q & 0 & 0 & 0 & L_Q \end{bmatrix} \begin{bmatrix} -i_d \\ -i_q \\ -i_0 \\ i_F \\ i_D \\ i_Q \end{bmatrix}$$

- By defining proper base inductances, the matrix may become symmetric in per unit

Per Unit Representation

Table 4.1. Electrical Quantities, Units, and Dimensions

Quantity	Symbol	Units	v-i-t Dimensions	Relationship
Voltage	v	volts (V)	$[v]$	
Current	i	amperes (A)	$[i]$	
Power or voltamperes	p or S	watts (W) voltamperes (VA)	$[vi]$	$p = vi$
Flux linkage	λ	weber turns (Wb turns)	$[vt]$	$v = \dot{\lambda}$
Resistance	r	ohm (Ω)	$[v/i]$	$v = ri$
Inductance	L or M	henry (H)	$[vt/i]$	$v = L\dot{i}$
Time	t	second (s)	$[t]$	
Angular velocity	ω	radians per second (rad/s)	$[1/t]$	
Angle	θ or δ	radian (rad)	dimensionless	

Quantity in p.u. = Actual quantity / Base quantity

$$\bar{x} = \frac{x}{x_{base}} \quad \text{p.u.}$$

Base Quantities for Synchronous Machines

$S_{base} \sim i_{base}\, e_{base}$ $\quad Z_{base} \sim e_{base}/i_{base}$ $\quad L_{base} \sim Z_{base}/\omega_{base}$

$\psi_{base} \sim L_{base} \times i_{base}$ $\quad T_{base} \sim S_{base} / \omega_{base}$

- For steady-state conditions, only two base quantities for each voltage level should be provided, e.g. e_{base} and i_{base}, or S_{base} and e_{base}

- Considering dynamics, 3 base quantities are needed, e.g.
  - f_{base} , e_{base}, i_{base} $\rightarrow$ S_{base}, Z_{base}, L_{base}, ψ_{base}, T_{base}
  - f_{base} , e_{base}, S_{base} $\rightarrow$ i_{base}, Z_{base}, L_{base}, ψ_{base}, T_{base}

Base	d	q	0	F	D	Q
1	f_{base}	f_{base}	f_{base}	f_{base}	f_{base}	f_{base}
2						
3						

Stator Base Quantities

- Using the machine ratings as the base values
  - $e_{s\,base}$ (V) — peak value of rated line-to-neutral voltage
  - $i_{s\,base}$ (A) — peak value of rated line current
  - f_{base} (Hz) — rated frequency
- Accordingly:
  - $S_{3\phi\,base}$ (VA) $= 3E_{RMS\,base} \times I_{RMS\,base} = 3(e_{s\,base}/\sqrt{2}) \times (i_{s\,base}/\sqrt{2}) = \frac{3}{2} e_{s\,base} \times i_{s\,base}$
  - $Z_{s\,base}$ (Ω) $= e_{s\,base}/i_{s\,base}$
  - $L_{s\,base}$ (H) $= Z_{s\,base}/\omega_{base}$
  - ω_{base} (elec. rad/s) $= 2\pi f_{base}$
  - ω_{mbase} (mech. rad/s) $= \omega_{base} \times (2/p_f)$
  - t_{base} (s) $= 1/\omega_{base} = 1/(2\pi f_{base})$
  - $\psi_{s\,base}$ (Wb·turns) $= L_{s\,base} \times i_{s\,base} = e_{s\,base}/\omega_{base}$
  - T_{base} (N·m) $= S_{3\phi\,base}/\omega_{mbase} = \frac{3}{2}\left(\frac{p_f}{2}\right) \psi_{s\,base} \times i_{s\,base}$

Base	d	q	0	F	D	Q
1	f_{base}	f_{base}	f_{base}	f_{base}	f_{base}	f_{base}
2	$e_{s\,base}$	$e_{s\,base}$	$e_{s\,base}$	$S_{3\phi\,base}$	$S_{3\phi\,base}$	$S_{3\phi\,base}$
3	$i_{s\,base}$	$i_{s\,base}$	$i_{s\,base}$			

How to select rotor base quantities?

Base	d	q	0	F	D	Q
1	f_{base}	f_{base}	f_{base}	f_{base}	f_{base}	f_{base}
2	$e_{s\ base}$	$e_{s\ base}$	$e_{s\ base}$	$S_{3\phi\ base}$	$S_{3\phi\ base}$	$S_{3\phi\ base}$
3	$i_{s\ base}$	$i_{s\ base}$	$i_{s\ base}$	$i_{F\ base}$	$i_{D\ base}$	$i_{Q\ base}$

$$\begin{bmatrix} \psi_d \\ \psi_q \\ \psi_0 \\ \psi_F \\ \psi_D \\ \psi_Q \end{bmatrix} = \begin{bmatrix} L_d & 0 & 0 & M_F & M_D & 0 \\ 0 & L_q & 0 & 0 & 0 & M_Q \\ 0 & 0 & L_0 & 0 & 0 & 0 \\ \frac{3}{2}M_F & 0 & 0 & L_F & M_R & 0 \\ \frac{3}{2}M_D & 0 & 0 & M_R & L_D & 0 \\ 0 & \frac{3}{2}M_Q & 0 & 0 & 0 & L_Q \end{bmatrix} \begin{bmatrix} -i_d \\ -i_q \\ -i_0 \\ i_F \\ i_D \\ i_Q \end{bmatrix}$$

i_{Fbase}, i_{Dbase} and i_{Qbase} should enable a symmetric per-unit inductance matrix

$$\begin{bmatrix}\psi_d\\ \psi_q\\ \psi_0\\ \psi_F\\ \psi_D\\ \psi_Q\end{bmatrix}=\begin{bmatrix}L_d & 0 & 0 & M_F & M_D & 0\\ 0 & L_q & 0 & 0 & 0 & M_Q\\ 0 & 0 & L_0 & 0 & 0 & 0\\ \frac{3}{2}M_F & 0 & 0 & L_F & M_R & 0\\ \frac{3}{2}M_D & 0 & 0 & M_R & L_D & 0\\ 0 & \frac{3}{2}M_Q & 0 & 0 & 0 & L_Q\end{bmatrix}\begin{bmatrix}-i_d\\ -i_q\\ -i_0\\ i_F\\ i_D\\ i_Q\end{bmatrix}$$

$$\Rightarrow$$

$$\begin{bmatrix}\bar\psi_d\\ \bar\psi_q\\ \bar\psi_0\\ \bar\psi_F\\ \bar\psi_D\\ \bar\psi_Q\end{bmatrix}=\begin{bmatrix}\bar L_d & 0 & 0 & \bar M_F & \bar M_D & 0\\ 0 & \bar L_q & 0 & 0 & 0 & \bar M_Q\\ 0 & 0 & \bar L_0 & 0 & 0 & 0\\ \bar M'_F & 0 & 0 & \bar L_F & \bar M_R & 0\\ \bar M'_D & 0 & 0 & \bar M_R & \bar L_D & 0\\ 0 & \bar M'_Q & 0 & 0 & 0 & \bar L_Q\end{bmatrix}\begin{bmatrix}-\bar i_d\\ -\bar i_q\\ -\bar i_0\\ \bar i_F\\ \bar i_D\\ \bar i_Q\end{bmatrix}$$

$$\frac{\psi_d}{\psi_{s\ base}}=-\frac{L_d\times i_d}{L_{s\ base}\times i_{s\ base}}+\frac{M_F\times i_F}{L_{s\ base}\times i_{s\ base}}+\frac{M_D\times i_D}{L_{s\ base}\times i_{s\ base}}$$

$$=-\frac{L_d\times i_d}{L_{s\ base}\times i_{s\ base}}+\frac{M_F\times i_{F\ base}}{L_{s\ base}\times i_{s\ base}}\times\frac{i_F}{i_{F\ base}}+\frac{M_D\times i_{D\ base}}{L_{s\ base}\times i_{s\ base}}\times\frac{i_D}{i_{D\ base}}$$

$$\bar\psi_d=-\bar L_d\times\bar i_d+\bar M_F\times\bar i_F+\bar M_D\times\bar i_D$$

$$\frac{\psi_F}{\psi_{F\ base}}=-\frac{3}{2}\frac{M_F\times i_d}{L_{F\ base}\times i_{F\ base}}+\frac{L_F\times i_F}{L_{F\ base}\times i_{F\ base}}+\frac{M_R\times i_D}{L_{F\ base}\times i_{F\ base}}$$

$$=-\frac{3}{2}\frac{M_F\times i_{S\ base}}{L_{F\ base}\times i_{F\ base}}\times\frac{i_d}{i_{s\ base}}+\frac{L_F\times i_F}{L_{F\ base}\times i_{F\ base}}+\frac{M_R\times i_{D\ base}}{L_{F\ base}\times i_{F\ base}}\times\frac{i_D}{i_{D\ base}}$$

$$\bar\psi_F=-\bar M'_F\times\bar i_d+\bar L_F\times\bar i_F+\bar M_R\times\bar i_D$$

$$\bar M_F=\bar M'_F$$

$$\frac{M_F\times i_{F\ base}}{L_{s\ base}\times i_{s\ base}}=\frac{3}{2}\frac{M_F\times i_{S\ base}}{L_{F\ base}\times i_{F\ base}}$$

$$L_{F\ base}\,i^2_{F\ base}=\frac{3}{2}L_{S\ base}\,i^2_{S\ base}$$

$$\omega_{base}L_{F\ base}\,i^2_{F\ base}=\frac{3}{2}\omega_{base}L_{S\ base}\,i^2_{S\ base}$$

$$e_{F\ base}\,i_{F\ base}=\frac{3}{2}e_{S\ base}\,i_{S\ base}$$

$$=S_{3\phi\ base}$$

$$=e_{D\ base}\,i_{D\ base}$$

$$=e_{Q\ base}\,i_{Q\ base}$$

Rotor Base Quantities

$$\begin{bmatrix} \bar{\psi}_d \\ \bar{\psi}_q \\ \bar{\psi}_0 \\ \bar{\psi}_F \\ \bar{\psi}_D \\ \bar{\psi}_Q \end{bmatrix} = \begin{bmatrix} \bar{L}_d & 0 & 0 & \bar{M}_F & \bar{M}_D & 0 \\ 0 & \bar{L}_q & 0 & 0 & 0 & \bar{M}_Q \\ 0 & 0 & \bar{L}_0 & 0 & 0 & 0 \\ \bar{M}_F & 0 & 0 & \bar{L}_F & \bar{M}_R & 0 \\ \bar{M}_D & 0 & 0 & \bar{M}_R & \bar{L}_D & 0 \\ 0 & \bar{M}_Q & 0 & 0 & 0 & \bar{L}_Q \end{bmatrix} \begin{bmatrix} -\bar{i}_d \\ -\bar{i}_q \\ -\bar{i}_0 \\ \bar{i}_F \\ \bar{i}_D \\ \bar{i}_Q \end{bmatrix}$$

- Stator self-inductance $\overline{L}_d$ or $\overline{L}_q$ can be split into two parts:
  - Leakage inductance due to flux that does not link any rotor circuit
  - Mutual inductance due to flux that links the rotor circuits
- Stator leakage inductances in d and q axes are nearly equal. Then

$$\overline{L}_d = \overline{L}_l + \overline{L}_{ad} \qquad \overline{L}_q = \overline{L}_l + \overline{L}_{aq}$$

- Assume that all the per unit mutual inductances between the stator and rotor circuits in each axis are equal

$$\overline{M}_F = \overline{M}_D = \overline{L}_{ad} \qquad \overline{M}_Q = \overline{L}_{aq}$$

- Some references suggest rotor mutual inductance $\overline{M}_R = \overline{L}_{ad}$ to further simplify equivalent circuits

$$\bar{L}_{ad} = \bar{M}_F = \bar{M}_D$$

$$\frac{L_{ad}}{L_{s\ base}} = \frac{M_F \times i_{F\ base}}{L_{s\ base} \times i_{s\ base}} = \frac{M_D \times i_{D\ base}}{L_{s\ base} \times i_{s\ base}}$$

$$\bar{L}_{aq} = \bar{M}_Q$$

$$\frac{L_{aq}}{L_{s\ base}} = \frac{M_Q \times i_{Q\ base}}{L_{s\ base} \times i_{s\ base}}$$

$i_{F\ base} \triangleq \frac{L_{ad}}{M_F} \times i_{S\ base}$, A

$e_{F\ base} \triangleq S_{3\phi\ base} / i_{F\ base}$, V

$Z_{F\ base} \triangleq e_{F\ base} / i_{F\ base} = S_{3\phi\ base} / i^2_{F\ base}$, Ω

$L_{F\ base} \triangleq Z_{F\ base} / \omega_{base}$, H

$\psi_{F\ base} \triangleq L_{F\ base} \times i_{F\ base}$, Wb·turns

$M_{F\ base} = L_{S\ base} \times i_{S\ base} / i_{F\ base}$

$i_{Q\ base} \triangleq \frac{L_{aq}}{M_Q} \times i_{S\ base}$, A

$e_{Q\ base} \triangleq S_{3\phi\ base} / i_{Q\ base}$, V

$Z_{Q\ base} = e_{Q\ base} / i_{Q\ base} \triangleq S_{3\phi\ base} / i^2_{Q\ base}$, Ω

$L_{Q\ base} \triangleq Z_{Q\ base} / \omega_{base}$, H

$\psi_{Q\ base} \triangleq L_{Q\ base} \times i_{Q\ base}$, Wb·turns

$M_{Q\ base} = L_{S\ base} \times i_{S\ base} / i_{Q\ base}$

$i_{D\ base} \triangleq \frac{L_{ad}}{M_D} \times i_{S\ base}$, A

$e_{D\ base} \triangleq S_{3\phi\ base} / i_{D\ base}$, V

$Z_{D\ base} = e_{D\ base} / i_{D\ base} \triangleq S_{3\phi\ base} / i^2_{D\ base}$, Ω

$L_{D\ base} \triangleq Z_{D\ base} / \omega_{base}$, H

$\psi_{D\ base} \triangleq L_{D\ base} \times i_{D\ base}$, Wb·turns

$M_{D\ base} = L_{S\ base} \times i_{S\ base} / i_{D\ base}$

$$\begin{bmatrix} \bar{\psi}_d \\ \bar{\psi}_q \\ \bar{\psi}_0 \\ - \\ \bar{\psi}_F \\ \bar{\psi}_D \\ \bar{\psi}_Q \end{bmatrix} = \begin{bmatrix} \bar{L}_l + \bar{L}_{ad} & 0 & 0 & | & \bar{L}_{ad} & \bar{L}_{ad} & 0 \\ 0 & \bar{L}_l + \bar{L}_{aq} & 0 & | & 0 & 0 & \bar{L}_{aq} \\ 0 & 0 & \bar{L}_0 & | & 0 & 0 & 0 \\ - & - & - & | & - & - & - \\ \bar{L}_{ad} & 0 & 0 & | & \bar{L}_F & \bar{M}_R & 0 \\ \bar{L}_{ad} & 0 & 0 & | & \bar{M}_R & \bar{L}_D & 0 \\ 0 & \bar{L}_{aq} & 0 & | & 0 & 0 & \bar{L}_Q \end{bmatrix} \times \begin{bmatrix} -\bar{i}_d \\ -\bar{i}_q \\ -\bar{i}_0 \\ - \\ \bar{i}_F \\ \bar{i}_D \\ \bar{i}_Q \end{bmatrix}$$

L_{ad}-L_{aq} based per unit system

Per Unit Voltage Equations

$$\mathbf{e_{dq0}} = -\mathbf{Ri_{dq0}} + \dot{\psi}_{\mathbf{dq0}} + \mathbf{S} + \mathbf{n_{dq0}}$$

$$\mathbf{e_{FDQ}} = \mathbf{R_R i_{FDQ}} + \dot{\psi}_{\mathbf{FDQ}}$$

$$\begin{bmatrix} e_d \\ e_q \\ e_0 \end{bmatrix} = -\begin{bmatrix} R_a & & \\ & R_a & \\ & & R_a \end{bmatrix}\begin{bmatrix} i_d \\ i_q \\ i_0 \end{bmatrix} + \begin{bmatrix} p\psi_\mathrm{d} \\ p\psi_\mathrm{q} \\ p\psi_0 \end{bmatrix} + \begin{bmatrix} -\omega_r\psi_\mathrm{q} \\ \omega_r\psi_\mathrm{d} \\ 0 \end{bmatrix} - \begin{bmatrix} 0 \\ 0 \\ 3R_n i_0 \end{bmatrix} - \begin{bmatrix} 0 \\ 0 \\ 3L_n p i_0 \end{bmatrix}$$

$$\begin{bmatrix} e_F \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} R_F & & \\ & R_D & \\ & & R_Q \end{bmatrix}\begin{bmatrix} i_F \\ i_D \\ i_Q \end{bmatrix} + \begin{bmatrix} p\psi_\mathrm{F} \\ p\psi_\mathrm{D} \\ p\psi_\mathrm{Q} \end{bmatrix}$$

$p = d/dt$ differential operator

- Divide both sides of each equation by one of the following:

$$e_{S\ base} = \omega_{base} \times \psi_{S\ base} = \omega_{base} \times L_{S\ base} \times i_{S\ base} = Z_{S\ base} \times i_{S\ base}$$

$$e_{F\ base} = \omega_{base} \times \psi_{F\ base} = \omega_{base} \times L_{F\ base} \times i_{F\ base} = Z_{F\ base} \times i_{F\ base}$$

$$e_{D\ base} = \omega_{base} \times \psi_{D\ base} = \omega_{base} \times L_{D\ base} \times i_{D\ base} = Z_{D\ base} \times i_{D\ base}$$

$$e_{Q\ base} = \omega_{base} \times \psi_{Q\ base} = \omega_{base} \times L_{Q\ base} \times i_{Q\ base} = Z_{Q\ base} \times i_{Q\ base}$$

- For example:

$$\frac{e_d}{e_{s\,base}} = -\frac{R_a \times i_d}{Z_{s\,base} \times i_{s\,base}} + \frac{p\psi_d}{\omega_{base} \times \psi_{s\,base}} - \frac{\omega_r \psi_q}{\omega_{base} \times \psi_{s\,base}}$$

- Note: $\frac{p}{\omega_{base}} = \frac{d}{\omega_{base} dt} = \frac{t_{base} d}{dt} = \frac{d}{d\bar{t}} \triangleq \bar{p}$

Per unit differential operator

$$\bar{e}_d = -\bar{R}_a \times \bar{i}_d + \overline{p\psi}_d - \bar{\omega}_r \bar{\psi}_q$$

$$\frac{e_F}{e_{F\,base}} = \frac{R_F \times i_F}{Z_{F\,base} \times i_{F\,base}} + \frac{p\psi_F}{\omega_{base} \times \psi_{F\,base}}$$

$$\bar{e}_F = \bar{R}_F \times \bar{i}_F + \overline{p\psi}_F$$

$$\begin{bmatrix} \bar{e}_d \\ \bar{e}_q \\ \bar{e}_0 \end{bmatrix} = -\begin{bmatrix} \bar{R}_a & & \\ & \bar{R}_a & \\ & & \bar{R}_a \end{bmatrix} \begin{bmatrix} \bar{i}_d \\ \bar{i}_q \\ \bar{i}_0 \end{bmatrix} + \begin{bmatrix} \overline{p\psi}_{\mathrm{d}} \\ \overline{p\psi}_{\mathrm{q}} \\ \overline{p\psi}_0 \end{bmatrix} + \begin{bmatrix} -\bar{\omega}_r \bar{\psi}_{\mathrm{q}} \\ \bar{\omega}_r \bar{\psi}_{\mathrm{d}} \\ 0 \end{bmatrix} - \begin{bmatrix} 0 \\ 0 \\ 3\bar{R}_n \bar{i}_0 \end{bmatrix} - \begin{bmatrix} 0 \\ 0 \\ 3\bar{L}_n \bar{p} \bar{i}_0 \end{bmatrix}$$

$$\begin{bmatrix} \bar{e}_F \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} \bar{R}_F & & \\ & \bar{R}_D & \\ & & \bar{R}_Q \end{bmatrix} \begin{bmatrix} \bar{i}_F \\ \bar{i}_D \\ \bar{i}_Q \end{bmatrix} + \begin{bmatrix} \overline{p\psi}_{\mathrm{F}} \\ \overline{p\psi}_{\mathrm{D}} \\ \overline{p\psi}_{\mathrm{Q}} \end{bmatrix}$$

Per Unit Power and Torque

$$\begin{bmatrix} e_d \\ e_q \\ e_0 \end{bmatrix} = -\begin{bmatrix} R_a & & \\ & R_a & \\ & & R_a \end{bmatrix}\begin{bmatrix} i_d \\ i_q \\ i_0 \end{bmatrix} + \begin{bmatrix} p\psi_d \\ p\psi_q \\ p\psi_0 \end{bmatrix} + \begin{bmatrix} -\omega_r\psi_q \\ \omega_r\psi_d \\ 0 \end{bmatrix} - \begin{bmatrix} 0 \\ 0 \\ 3R_n i_0 \end{bmatrix} - \begin{bmatrix} 0 \\ 0 \\ 3L_n p i_0 \end{bmatrix}$$

- Instantaneous power at the machine terminal:

$$P_t = e_a i_a + e_b i_b + e_c i_c = [e_a \ \ e_b \ \ e_c]\times[i_a \ \ i_b \ \ i_c]^T = [e_d \ \ e_q \ \ e_0]\times \mathbf{P^{-T}P^{-1}} \times [i_d \ \ i_q \ \ i_0]^T$$

$$= [e_d \ \ e_q \ \ e_0]\times(\mathbf{P^T P})^{-1}\times[i_d \ \ i_q \ \ i_0]^T$$

$$\mathbf{P^T P} = \frac{1}{3}\begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \ (\mathbf{P^T P})^{-1} = \frac{3}{2}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

$$P_t = \frac{3}{2}(e_d i_d + e_q i_q + 2e_0 i_0)$$

$$= \frac{3}{2}(e_d i_d + e_q i_q)$$ (under balanced conditions)

Divided by $S_{3\phi\, base} = \frac{3}{2} e_{s\, base} \times i_{s\, base}$

$$\bar{P}_t = \bar{e}_d \bar{i}_d + \bar{e}_q \bar{i}_q$$

$$P_t = \frac{3}{2}[\,(i_d p\psi_d + i_q p\psi_q) + (\psi_d i_q - \psi_q i_d)\,\omega_r - (i_d^2 + i_q^2)R_a]$$

Power transferred across the air-gap

- The air-gap torque (i.e. electrical torque):

$$T_e = \frac{3}{2}(\psi_d i_q - \psi_q i_d)\,\omega_r/\omega_{mech} = \frac{3}{2}(\psi_d i_q - \psi_q i_d)\,p_f/2$$

Divided by $T_{base} = \frac{3}{2}(\frac{p_f}{2})\,\psi_{s\, base} \times i_{s\, base}$

$$\bar{T}_e = \bar{\psi}_d \bar{i}_q - \bar{\psi}_q \bar{i}_d$$

Per Unit Reactance

$$X = 2\pi f L$$

$$\frac{X}{Z_{base}} = \frac{2\pi f}{2\pi f_{base}} \times \frac{L}{L_{base}}$$

- If $f = f_{base}$

$$\overline{X} = \overline{L}$$

- The per unit reactance of a winding is numerically equal to the per unit inductance.