# On Periodical Damping Ratio of a Controlled Dynamical System with Parametric Resonances

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***Abstract*— A dynamical system whose damping of oscillation or vibration is directly controlled may manifest a periodically varying damping ratio. In this paper, such a phenomenon is discovered to arise from a new type of parametric resonance, named the "zero-th order parametric resonance", which is different from the conventional principal parametric resonance. This letter proposes an approach using the method of multiple scales to find approximate, analytical solutions of the system and thus providing an interpretation on damping variations.**

***Index Terms*— Method of multiple scales, oscillation damping, parametric resonance, vibration damping.**

## I. Introduction

For a dynamical system that can be modeled as a network of oscillators, damping is important for attenuating its oscillation or vibration to eventually settle the system down to a stable equilibrium. Under disturbances, vibration or oscillation of the system is inevitable but is expected to be damped quickly to avoid detrimental effects; otherwise, open-loop or closed-loop damping control will be considered for a weakly-damped dynamical system to enhance its dynamic performance subject to a disturbance. For instance, weakly-damped and sustained oscillation of a power system can lead to rotor angle instability of generators, which can be avoided by properly-designed power system stabilizers or damping controllers [1], [2]. Similar cases that require damping control with a dynamical system can also be found in the fields of power electronics [3], robotics [4], etc.

When the damping ratio of a natural mode is controlled, an improperly tuned controller may introduce a new conjugated pair of complex eigenvalues that generate a new oscillatory mode. Such a phenomenon may induce a parametric resonance under certain conditions [5]. In literature, several types of parametric resonances are studied, e.g. fundamental parametric resonance [6] and principle parametric resonance [7]. For a dynamical system of multiple degrees of freedom, when the damping ratio of one dominant, natural oscillatory mode is concerned, the effect of a parametric resonance can be studied on its 1-DOF equivalent, i.e. a harmonic oscillator, regarding that mode by adding a controlled input $u$ to the differential equation of the oscillator. Most of existing studies consider $u$ to be an external force [6], [7] or a dimensionless scalar that scales the total system damping [8]. Very few papers have considered $u$ as an additional change on top of the original damping of the system, which is the focus of this letter.

Consider direct control of the damping with oscillation or vibration of a weakly-damped dynamical system. Its dynamics regarding a specific oscillation or vibration mode can be studied using a 1-degree-of-freedom (1-DOF) system as modeled by (1):

$$\ddot{x} + (-2\sigma + u)\dot{x} + (\sigma^2 + \omega^2)x = 0 \tag{1}$$

where $x$ and $\dot{x}$ are state variables respectively on the displacement and velocity. Parameter $u$ is a change on damping by control. An example is the real-time damping improvement of power systems by a feedback controller that measures the damping ratio and minimizes its error from a setpoint, and the controller changes the power output of the distributed energy resources in order to add a damping torque in phase with an oscillating generator's rotor speed deviation, which corresponds to the term $u\dot{x}$ [9]. If $u = 0$, the pair of eigenvalues $\lambda_{1,2}$ are calculated by (2), where $\zeta$ and $w_n$ are the damping ratio and natural angular frequency of the mode, respectively.

$$\lambda_{1,2} = \sigma \pm j\omega = -\zeta\omega_n \pm j\omega_n\sqrt{1-\zeta^2} \tag{2}$$

The controller for changing $u$ should be carefully designed to reach the expected performance; otherwise, the variation in $u$ could lead to unexpected dynamic response that complicates the damping control. For instance, the PID (proportional-integral-derivative) controller is a widely-used and easy-to-implement controller in practical applications, but an improperly-tuned PID controller could generate a (quasi-)periodical variation in $u$ that leads to a parametric resonance in the system response manifested by a periodically varying damping ratio.

In this letter, by considering the changes of $u$ over time due to continuous damping control, $u$ is represented by a periodic function $Kcos(\Omega t)$ with amplitude $K$ and angular frequency $\Omega$. Namely, eq. (1) is rewritten as (3). Then, by using the method of multiple scales (MMS) [10], this letter will show that, first, the principal parametric resonance can be excited when $\Omega \approx 2\omega$ and, second, when $\Omega \approx 0$, a different, new type of parametric resonance can arise, which is called a "zero-th order parametric resonance" in this paper. Note that here "$\Omega \approx 2\omega$" means $|\Omega - 2\omega| \ll \omega$, and similarly "$\Omega \approx 0$" means $\Omega \ll$

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$\omega$. Those two conditions indicate that the parametric resonance is at a slower time scale than the oscillation in the state $x$, which motivates the use of MMS to identify the solution of $x$. Moreover, in practical applications, if the satisfaction of those two conditions is intentionally avoided in the design of control, the chances of principle and zero-th order parametric resonances can be eliminated or reduced.

$$\ddot{x} + (-2\sigma + Kcos(\Omega t))\dot{x} + (\sigma^2 + \omega^2)x = 0 \tag{3}$$

In the rest of the letter, principal and zero-th order parametric resonances of the dynamical system are respectively studied in sections II and III. The approximate solution under parametric resonance is derived using MMS, via which the mechanism of the periodically varying damping ratio is interpreted. Conclusions are drawn in section IV.

## II. Principal Parametric Resonance

If $\Omega \approx 2\omega$, the principal parametric resonance can be observed from the response of $x$. For instance, let $\zeta$ = 0.0098, $\omega_n$ = 3.8072 rad/s and $K$ = 0.5. Thus, $\sigma$ = 0.0373 and $\omega$ = 3.8070 rad/s from (2). If $\Omega$ = 6.9115 rad/s, which is close to $2\omega$ and the initial state has $x(t) = 1, \dot{x}(t) = 0$, the response of $x$ is shown in Fig. 1 with its envelop marked. The damping ratio is estimated using Prony's method [11] and is given in Fig. 2. The periodic variations in the damping ratio shows the parametric resonance due to $\Omega \approx 2\omega$.

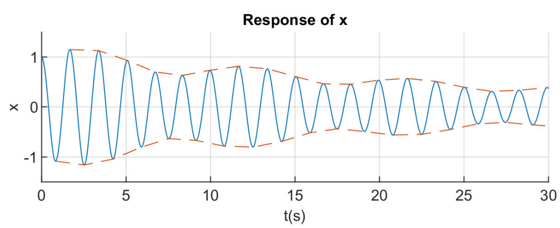


Fig. 1. Principal parametric resonance: response of $\boldsymbol{x}$.

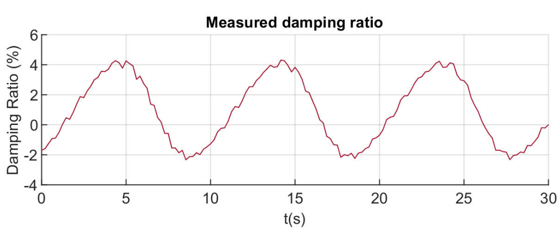


Fig. 2. Principal parametric resonance: Damping ratio.

The mechanism and properties of such a parametric resonance can be revealed from the solution of (3) obtained by the MMS. The basic idea of MMS is to find an asymptotic solution of a perturbed system considering different time scales. The use of MMS follows the approach in [6], [7]. First, the periodic parameter $Kcos(\Omega t)$ is treated as a perturbation by inserting a small dimensionless parameter $\varepsilon > 0$ as in (4), where $K_\varepsilon = K/\varepsilon$.

$$\ddot{x} + (-2\sigma + \varepsilon K_\varepsilon cos(\Omega t))\dot{x} + (\sigma^2 + \omega^2)x = 0 \tag{4}$$

Then, a first order uniform solution of (4) is:

$$\begin{gathered} x(\varepsilon, T_0, T_1) \approx x_0(T_0, T_1) + \varepsilon x_1(T_0, T_1) \\ T_0 = t, T_1 = \varepsilon t \end{gathered} \tag{5}$$

where $T_1$ is introduced as a slow-scale time variable, so that (5) describes how the solution evolves in a long time scale of order $\varepsilon^{-1}$, which is the key to identify the parametric resonance. Note that $x_0$ is exactly the solution of (4) when $\varepsilon$ = 0, and $x_1$ is the part caused by the perturbation. Substitute (5) into (4) and equate the coefficients of powers of $\varepsilon$:

$$D_0^2 x_0 - 2\sigma D_0 x_0 + (\sigma^2 + \omega^2)x_0 = 0 \tag{6}$$

$$\begin{aligned} &D_0^2 x_1 - 2\sigma D_0 x_1 + (\sigma^2 + \omega^2)x_1 = \\ &\quad - 2D_1 D_0 x_0 - K_\varepsilon cos(\Omega T_0) D_0 x_0 + 2\sigma D_1 x_0 \end{aligned} \tag{7}$$

where $D_n = \partial/\partial T_n$.

The solution of (6) can be expressed in this complex form, where the bar denotes the complex conjugate:

$$x_0(T_0, T_1) = A(T_1)e^{(\sigma + j\omega)T_0} + \bar{A}(T_1)e^{(\sigma - j\omega)T_0} \tag{8}$$

Substitute (8) into (7) to obtain (9), where $C.C.$ for simplicity denotes the complex conjugates of the former two terms. By introducing a detuning parameter $\xi$ such that $\Omega = 2\omega + \varepsilon\xi$, (9) can be converted to (10). Before solving (10), it can be verified that the condition (11) should be met to avoid generating secular terms in the solution of $x$. The explanation of secular term can be found in [10]. In this problem it will be a term growing linearly with $t$ to result in an unbounded solution, which conflicts with the actually bounded true solution. With the condition (11), the solution of (10) is given in (12).

$x_1$ is usually small and negligible compared to $x_0$. Hence, assume $x(t) \approx x_0(t)$. $A(T_1)$ is determined by solving (11). For an undamped or weakly-damped system, $\sigma$ can be assumed to be zero and (11) changes to:

$$\frac{dA}{dT_1} = \frac{K_\varepsilon}{4} e^{j\xi T_1} \bar{A} \tag{13}$$

The analytical solutions of $A(T_1)$ and $x(t)$ are found for three cases:

*1)* $(\Omega - 2\omega)^2 - K^2/4 > 0$,
*2)* $(\Omega - 2\omega)^2 - K^2/4 < 0$,
*3)* $(\Omega - 2\omega)^2 - K^2/4 = 0$.

In the following, define $\omega_k$ and the $A_0$, the initial value of $A(T_1)$:

$$\omega_K = \sqrt{|\xi^2 - K_\varepsilon^2/4|} = \sqrt{|(\Omega - 2\omega)^2 - K^2/4|}/\varepsilon \tag{14}$$

$$A_0 = A_{re0} + jA_{im0} \tag{15}$$

$$
\begin{aligned}
D_0^2 x_1 - 2\sigma D_0 x_1 + (\sigma^2+\omega^2)x_1 = &-2(\sigma+j\omega)e^{(\sigma+j\omega)T_0}D_1A - 2\zeta\omega_n e^{(\sigma+j\omega)T_0}D_1A \\
&- \frac{K_\varepsilon}{2}\left[(\sigma+j\omega)e^{(\sigma+j(\Omega+\omega))T_0}A + (\sigma-j\omega)e^{(\sigma+j(\Omega-\omega))T_0}\bar{A}\right] + C.C.
\end{aligned} \tag{9}
$$

$$
\begin{aligned}
D_0^2 x_1 - 2\sigma D_0 x_1 + (\sigma^2+\omega^2)x_1 = &-\left[2(\sigma+j\omega)D_1A - 2\sigma D_1 A + \frac{K_\varepsilon}{2}(\sigma-j\omega)e^{j\varepsilon\xi T_0}\bar{A}\right]e^{(\sigma+j\omega)T_0} \\
&- \frac{K_\varepsilon}{2}(\sigma+j\omega)e^{(\sigma+j(\Omega+\omega))T_0}A + C.C.
\end{aligned} \tag{10}
$$

$$
2(\sigma+j\omega)D_1A - 2\sigma D_1A + \frac{K_\varepsilon}{2}(\sigma-j\omega)e^{j\varepsilon\xi T_0}\bar{A} = 0 \tag{11}
$$

$$
x_1(T_0,T_1) = \frac{K_\varepsilon(\sigma+j\omega)A}{2}\left(\frac{e^{(\sigma+j(\Omega+\omega))T_0}}{\Omega(\Omega+2\omega)} + \frac{e^{(\sigma-j\omega)T_0}}{2\omega(\Omega+2\omega)} - \frac{e^{(\sigma+j\omega)T_0}}{2\omega\Omega}\right) + C.C. \tag{12}
$$

### A. Case 1: $(\Omega - 2\omega)^2 - K^2/4 > 0$

The solution of $A(T_1)$ is:

$$
A(T_1) = (C_1 e^{j(r_1-\omega_K T_1)} + C_2 e^{jr_2})e^{j\frac{\xi+\omega_K}{2}T_1} \tag{16}
$$

where:

$$
\begin{cases}
C_1 = \sqrt{\dfrac{\xi+\omega_K}{2\omega_K^2}}\sqrt{\xi(A_{re0}^2 + A_{im0}^2) + K_\varepsilon A_{re0}A_{im0}} \\
C_2 = \sqrt{\dfrac{\xi-\omega_K}{2\omega_K^2}}\sqrt{\xi(A_{re0}^2 + A_{im0}^2) + K_\varepsilon A_{re0}A_{im0}} \\
r_1 = arctan\left(\dfrac{2A_{im0}(\xi+\omega_K) + K_\varepsilon A_{re0}}{2A_{re0}(\xi+\omega_K) + K_\varepsilon A_{im0}}\right) \\
r_2 = arctan\left(\dfrac{2A_{im0}(\xi-\omega_K) + K_\varepsilon A_{re0}}{2A_{re0}(\xi-\omega_K) + K_\varepsilon A_{im0}}\right)
\end{cases} \tag{17}
$$

The solution of $x(t)$ is obtained by substituting (16) into (5) and ignoring $x_1$:

$$
\begin{aligned}
x(t) = &\, 2C_1 e^{\sigma t}cos(\omega_C t + r_1 - \varepsilon\omega_K t) \\
&+ 2C_2 e^{\sigma t}cos(\omega_C t + r_2)
\end{aligned} \tag{18}
$$

where $\omega_C = \omega + \frac{\xi+\omega_K}{2}\varepsilon$.

At the first glance, $x(t)$ seems to depend on $\varepsilon$ and $\xi$. Actually, such "dependence" does not exist after we substitute $\xi = (\Omega - 2\omega)/\varepsilon$, $\omega_K = \frac{\sqrt{|(\Omega-2\omega)^2 - K^2/4|}}{\varepsilon}$, $\omega_C = \omega + \frac{\xi+\omega_K}{2}\varepsilon$ and (17) into (18).

The resulting detailed expression of $x(t)$ is omitted for the sake of brevity, which consists of two components. The magnitude of the first component is $2C_1$ and the frequency is $\omega_C - \varepsilon\omega_K$. The magnitude of the second component is $2C_2$ and the frequency is $\omega_C$. The validity of the approximated solution can be visualized by the case when $\Omega = 6.9115$ rad/s. The comparison of the true response of $x(t)$ and the approximated $x(t)$ from (18) is shown in Fig. 3. The approximated $x(t)$ matches well the true response.

The principal parametric resonance in this case can be interpreted as follows. Without loss of generality, only consider the case when $C_2$ is larger than $C_1$. The second component can be more dominant than the first component. Since $\varepsilon\omega_K << \omega_C$, the term $\varepsilon\omega_K t$ can be viewed as a slow change in phase of the first component. Then, the change of damping of $x(t)$ can be interpreted as the periodic phase shift between the two components. When $r_1 - \varepsilon\omega_K t \approx r_2 + 2m\pi, m = 0, \pm1, \pm2, ...$, the two components are in-phase and the magnitude of $x(t)$ is amplified. When $r_1 - \varepsilon\omega_K t \approx r_2 + 2m\pi + \pi, m = 0, \pm1, \pm2, ...$, the two components are out-of-phase and the magnitude of $x(t)$ is reduced. Such changes have a frequency equal to $\varepsilon\omega_K = \sqrt{(\Omega - 2\omega)^2 - K^2/4}$, resulting in periodically varying damping of the response of $x(t)$.

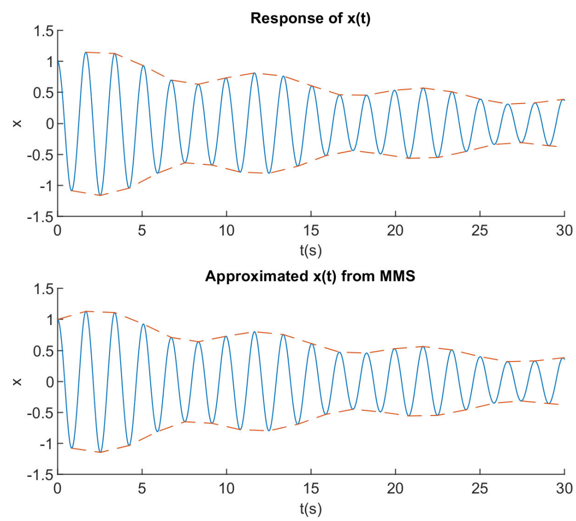


Fig. 3. Comparison of true response and approximated solution: $\boldsymbol{(\Omega - 2\omega)^2 - K^2/4 > 0}$.

### B. Case 2: $(\Omega - 2\omega)^2 - K^2/4 < 0$

In this case, the solution of $A(T_1)$ is:

$$
A(T_1) = (C_3 e^{jr_3} + C_4 e^{j(r_4-\omega_K T_1)})e^{j\frac{\xi+\omega_K}{2}T_1} \tag{19}
$$

where:

$$\begin{cases} C_3 = \sqrt{\dfrac{K_\varepsilon(2K_\varepsilon+\omega_K)}{\omega_K^2}}\left|A_{re0}+A_{im0}\dfrac{2\xi}{K_\varepsilon+2\omega_K}\right| \\ C_4 = \sqrt{\dfrac{K_\varepsilon(2K_\varepsilon-\omega_K)}{\omega_K^2}}\left|A_{re0}+A_{im0}\dfrac{2\xi}{K_\varepsilon-2\omega_K}\right| \\ r_3 = arctan\left(\dfrac{-A_{im0}(K_\varepsilon-2\omega_K)-2\xi A_{re0}}{A_{re0}(K_\varepsilon+2\omega_K)+2\xi A_{im0}}\right) \\ r_4 = arctan\left(\dfrac{A_{im0}(K_\varepsilon+2\omega_K)+2\xi A_{re0}}{-A_{re0}(K_\varepsilon-2\omega_K)-2\xi A_{im0}}\right) \end{cases} \tag{20}$$

Then, the solution of $x(t)$ is obtained by substituting (19) into (5) and ignoring $x_1$. Similarly to (18), it does not depend on $\varepsilon$ and $\xi$, either.

$$\begin{aligned} x(t) = {} & 2C_3 e^{(\sigma+\varepsilon\frac{\omega_K}{2})t} cos\left((\omega+\frac{\Omega}{2})t+r_3\right) \\ & +2C_4 e^{(\sigma-\varepsilon\frac{\omega_K}{2})t} cos\left((\omega+\frac{\Omega}{2})t+r_4\right) \end{aligned} \tag{21}$$

The resulting $x(t)$ consists of two components. The magnitude of the first component is $2C_3$ and the magnitude of the second component is $2C_4$. The frequency of both components is $\omega+\Omega/2$. The two components of $x(t)$ have damping different from $\sigma$. The validity of the approximated solution can be verified by the case when $\Omega$ is changed to 7.5524 rad/s. A comparison of the true response of $x(t)$ and the approximate response from (21) is shown in Fig. 4, which match well. Note that the response has a growing amplitude since the damping of the first component becomes negative. In this case, the response of $x(t)$ does not exhibit periodical damping although damping changes with time. If assume $C_4 \gg C_3$, the second component will become dominant at the early stage and the response of $x(t)$ is damped more quickly. Thereafter, the first component will become dominant after the second component is damped out.

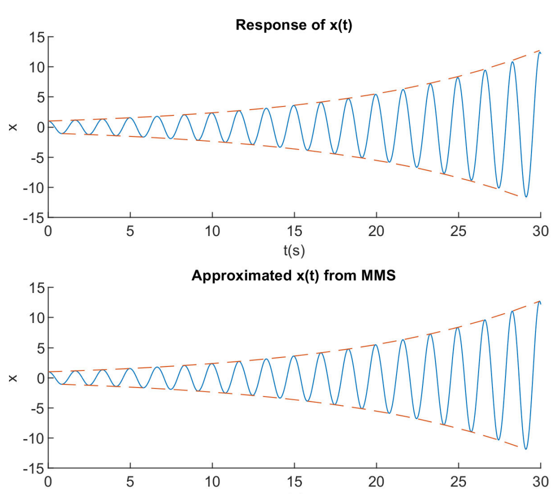


Fig. 4. Comparison of true response and approximated solution: $(\boldsymbol{\Omega}-\boldsymbol{2\omega})^{\mathbf{2}}-\boldsymbol{K^2/4}<\mathbf{0}$.

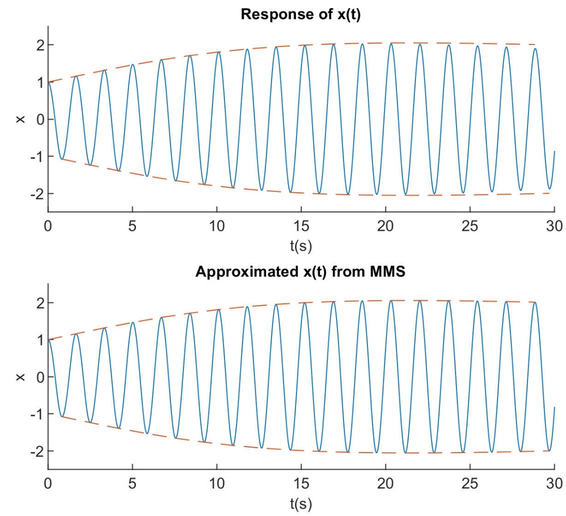


Fig. 5. Comparison of true response and approximated solution: $(\boldsymbol{\Omega}-\boldsymbol{2\omega})^{\mathbf{2}}-\boldsymbol{K^2/4}=\mathbf{0}$.

### C. Case 3: $(\Omega-2\omega)^2-K^2/4<0$

In this case, the solution of $A(T_1)$ is:

$$\begin{cases} A(T_1) = A_0 e^{j\frac{K_\varepsilon}{4}T_1}+\dfrac{K_\varepsilon}{4}(\bar{A}_0-jA_0)T_1 e^{j\frac{K_\varepsilon}{4}T_1}, \\ \qquad if\ \Omega-2\omega=\dfrac{K}{2} \\ A(T_1) = A_0 e^{-j\frac{K_\varepsilon}{4}T_1}-\dfrac{K_\varepsilon}{4}(\bar{A}_0-jA_0)T_1 e^{-j\frac{K_\varepsilon}{4}T_1}, \\ \qquad if\ \Omega-2\omega=-\dfrac{K}{2} \end{cases} \tag{22}$$

By substituting (22) into (5) and ignoring $x_1$, the solution of $x(t)$ is obtained as (23).

Each of the resulting $x(t)$ consists of two components at the frequency $\Omega/2$. Note that part of the result depends on $t$, indicating varying damping with time.

This validity of the approximate solution can be verified by the case when $\Omega$ is changed to 7.3639 rad/s. The comparison of the true response of $x(t)$ and the approximated $x(t)$ from (23) is shown in Fig. 5, which match well.

Since the condition $(\Omega-2\omega)^2-K^2/4=0$ can hardly be met, this case is rare in realistic systems.

## III. Zero-th Order Parametric Resonance

When $\Omega \approx 0$, the parametric resonance can also be observed in the response of $x$, which is named as the "zero-th order parametric resonance" in this paper.

For instance, let $\zeta$ = 0.0098, $\omega_n$ = 3.8072 rad/s, $K$ = 0.5, and consequently, $\sigma$ = 0.0373 and $\omega$ = 3.8070 rad/s. Then, if $\Omega$ = 0.6283 rad/s and the initial state has $x(0)=1, \dot{x}(0)=0$, the response is shown in Fig. 6 with the envelop marked. The damping ratio estimated using the Prony's method is given in Fig. 7. The periodic variations of the damping ratio manifest the parametric resonance due to $\Omega \approx 0$.

Through MMS, some properties of such a zero-th order parametric resonance are revealed. Again, consider a small dimensionless parameter $\varepsilon$ as in (4), and take the same derivation

$$\begin{cases} x(t) = 2e^{\sigma t}\left[\left(A_{re0} + \frac{K}{4}(A_{re0}+A_{im0})t\right) cos\left(\frac{\Omega}{2}t\right) - \left(A_{im0} - \frac{K}{4}(A_{re0}+A_{im0})t\right) sin\left(\frac{\Omega}{2}t\right)\right], & if \quad \Omega - 2\omega = \frac{K}{2} \\ x(t) = 2e^{\sigma t}\left[\left(A_{re0} - \frac{K}{4}(A_{re0}+A_{im0})t\right) cos\left(\frac{\Omega}{2}t\right) - \left(A_{im0} + \frac{K}{4}(A_{re0}+A_{im0})t\right) sin\left(\frac{\Omega}{2}t\right)\right], & if \quad \Omega - 2\omega = -\frac{K}{2} \end{cases} \tag{23}$$

$$\begin{aligned} D_0^2 x_1 - 2\sigma D_0 x_1 + (\sigma^2+\omega^2)x_1 = &- \left[2(\sigma+j\omega)D_1 A - 2\sigma D_1 A + \frac{K_\varepsilon}{2}(\sigma+j\omega)e^{j\varepsilon\xi T_0}A\right] e^{(\sigma+j\omega)T_0} \\ &- \frac{K_\varepsilon}{2}(\sigma-j\omega)e^{(\sigma+j(\varepsilon\xi+\omega))T_0}\bar{A} + C.C. \end{aligned} \tag{24}$$

$$2(\sigma+j\omega)D_1 A - 2\sigma D_1 A + \frac{K_\varepsilon}{2}(\sigma+j\omega)e^{j\varepsilon\xi T_0}A = 0 \tag{25}$$

$$x_1(T_0,T_1) = \frac{K_\varepsilon(\sigma-j\omega)\bar{A}}{2}\left(\frac{e^{(\sigma+j(\Omega-\omega))T_0}}{\Omega(\Omega-2\omega)} + \frac{e^{(\sigma-j\omega)T_0}}{2\omega\Omega} - \frac{e^{(\sigma+j\omega)T_0}}{2\omega(2\omega-\Omega)}\right) + C.C. \tag{26}$$

$$\frac{dA}{dT_1} = \frac{K_\varepsilon(-\omega+j\sigma)}{4\omega}e^{j\xi T_1}A \tag{27}$$

$$A(T_1) = e^{\frac{K_\varepsilon(\sigma+j\omega)}{4\xi\omega}e^{j\xi T_1}} + A_0 - e^{\frac{K_\varepsilon(\sigma+j\omega)}{4\xi\omega}} \tag{28}$$

$$\begin{aligned} x(t) = 2e^{\frac{K\sqrt{\sigma^2+\omega^2}}{4\Omega\omega}cos(\Omega t+\theta)+\sigma t} cos\left(\frac{K\sqrt{\sigma^2+\omega^2}}{4\Omega\omega}sin(\Omega t+\theta)+\omega t\right) \\ + 2e^{\sigma t}\left(A_{re0}cos(\omega t) - A_{im0}sin(\omega t) - e^{\frac{K\sigma}{4\omega\Omega}}cos(\omega t + \frac{K}{4\Omega})\right) \end{aligned} \tag{29}$$

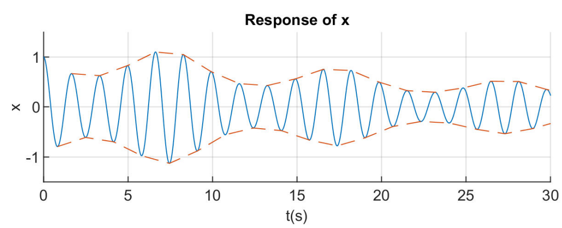


Fig. 6. Zero-th order parametric resonance: response of $\boldsymbol{x}$.

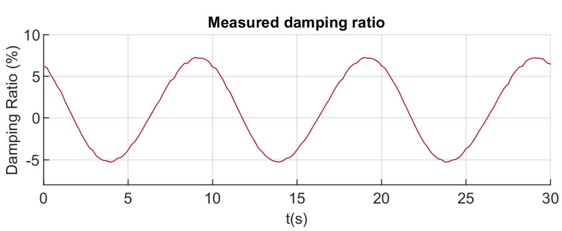


Fig. 7. Zero-th order parametric resonance: measured damping ratio.

as from (4) to (9). By introducing a detuning parameter $\xi$ such that $\Omega = \varepsilon\xi$, (9) can be converted to (24).

It can be verified that the condition (25) should be met to avoid generating secular terms in the solution of $x(t)$. Hence, the solution of (24) is given in (26).

$x_1$ could be ignored compared to $x_0$, since it is small. Hence, assume $x(t) \approx x_0(t)$. $A(T_1)$ can be determined by solving (25). First convert (25) to (27). Then, the analytical solution of $A(T_1)$ is shown in (28), where $A_0 = A_{re0} + jA_{im0}$ is the initial value of $A(T_1)$.

Substitute (28) into (5) and ignore $x_1$, and then (29) is obtained, where $\theta = arctan\left(\frac{\omega}{\sigma}\right)$.

The solution includes two components. Note that the first component has $cos(\Omega t+\theta)$ added to the damping $\sigma t$. This leads to a periodically varying damping in the response of $x(t)$. The $sin(\Omega t+\theta)$ term in the first component results in a periodic phase shift relative to the second component. The solution $x(t)$ exhibits a periodically varying damping ratio at a frequency close to $\Omega$.

The validity of the approximated solution can be verified by the case when $\Omega$ is changed to 0.6283 rad/s. The comparison of the true response of $x(t)$ and the approximated $x(t)$ from (29) is shown in Fig. 8. The approximated $x(t)$ is almost the same as the true response.

## IV. Conclusion

The response of a weakly-damped 1-DOF system with direct control of its damping ratio can exhibit a principal parametric resonance or the new type of zero-th order parametric resonance. It is shown that the principal parametric resonance can be classified into three cases depending on $(\Omega-2\omega)^2 - K^2/4 > 0, < 0$, or $= 0$. Specifically, when $(\Omega-2\omega)^2 - K^2/4 > 0$, the magnitude of $x$ periodically variation in time at a frequency close to $\sqrt{(\Omega-2\omega)^2 - K^2/4}$, which manifests periodical changes of its damping ratio. When a zero-th order parametric resonance is excited, the magnitude of $x(t)$ can periodically vary in time at a frequency close to $\Omega$. Hence, when designing damping controller for a dynamical system, the principal and zero-th order parametric resonance need to be taken account of to avoid unexpected dynamic behaviors with control.

## References


[1] B. C. Pal et al., "Robust damping controller design in power systems with superconducting magnetic energy storage devices," *IEEE Trans. Power Syst.*, vol. 15, no. 1, pp. 320325, Feb. 2000.

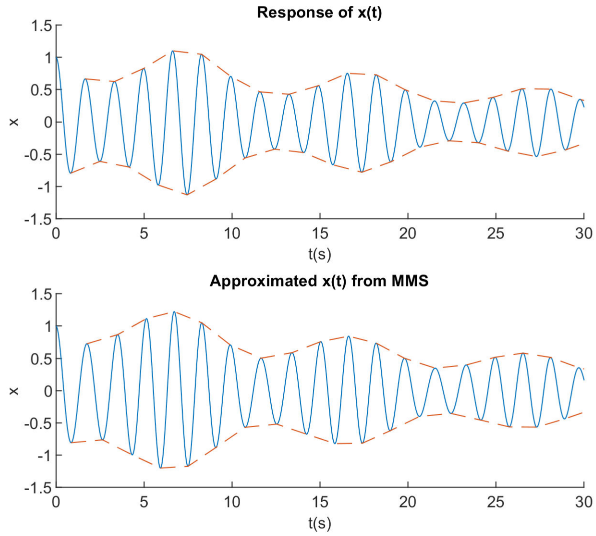


Fig. 8. Comparison of true response and approximated solution: zero-th order parametric resonance.

[2] Y. Zhu, C. Liu, K. Sun, D. Shi, and Z. Wang, "Optimization of battery energy storage to improve power system oscillation damping," *IEEE Trans. Sustain. Energy*, vol. 10, no. 3, pp. 10151024, Jul. 2019.

[3] S. G. Parker, B. P. McGrath, D. G. Holmes,"Regions of Active Damping Control for LCL Filters," *IEEE Trans. Ind. Appl.*, vol. 50, no. 1, pp. 424-432, Jan/Feb. 2014.

[4] F. Petit, A. Albu-Schffer, "State feedback damping control for a multi DOF variable stiffness robot arm," *2011 IEEE International Conference on Robotics and Automation*, pp. 5561-5567, 2011.

[5] T. I. Fossen and H. Nijmeijer, *Parametric Resonance in Dynamical Systems*, New York: Springer-Verlag; 2012.

[6] L. D. Zavodney, A. H. Nayfeh,"The response of a single-degree-of-freedom system with quadratic and cubic non-linearities to a fundamental parametric resonance," *Journal of Sound and Vibration*, vol. 120, no. 1, pp. 63-93, 1988.

[7] L. D. Zavodney, A. H. Nayfeh, N. E. Sanchez,"The response of a single-degree-of-freedom system with quadratic and cubic non-linearities to a principal parametric resonance," *Journal of Sound and Vibration*, vol. 129, no. 3, pp. 417-442, 1989.

[8] R. S. Johnson, *Singular perturbation theory: mathematical and analytical techniques with applications to engineering*, Springer Science & Business Media; 2005.

[9] X. Xu, K. Sun,"Direct Damping Feedback Control Using Power Electronics-Interfaced Resources," *IEEE Trans. Power Syst.*, in press.

[10] J. K. Kevorkian, J. D. Cole, *Multiple scale and singular perturbation methods*, Springer Science & Business Media; 2012.

[11] J. Hauer et al., "Initial results in Prony analysis of power system response signals," *IEEE Trans. Power Syst.*, vol. 5, no. 1, pp. 8089, Feb. 1990.