

Optimal Distributed Control for Continuum Power Systems*

S. Sahyoun, S. M. Djouadi, K. Tomsovic¹, and S. Lenhart²

Abstract—Large electrical power networks viewed as continuum systems have been studied under constant voltage magnitude assumptions. The continuum system phase behavior was proved to follow the dynamics of a second order nonlinear wave equation. The latter represents electromechanical wave propagation in large electric power networks. In this paper, we generalize this work to time and space variant voltage magnitudes which is the case in real world applications. The resulting partial differential equations (PDEs) are also wave equations but include more nonlinearity terms. Optimal control theory is used to derive optimality conditions for two optimal control problems. The first problem is when the mechanical power is the control input where the constraint is a constant voltage PDE, while the second problem is when the variant voltage magnitude is the control input under a generalized variant voltage PDE as the optimization constraint. Numerical results are presented to illustrate the performance of the resulting closed loop control systems for large power networks. Due to page size limits we present the optimal control results for the variant voltage swing PDE in a different paper.

I. INTRODUCTION

Stability and control of power systems has been a concern since the early twenties and recently it become increasingly challenging due to several reasons: power systems are now being operated closer to their maximum operating points, environmental constrains limits the expansion needs, the number of long distance power transfers increased to serve far areas and the most recent challenge; the renewable energy integration [1].

Wind generation for instance is a growing renewable energy resource but the challenge is how to effectively integrate a significant amount of wind power into the power network [2].

The fundamental equation that describes the rotor dynamics in power systems is the swing equation [3]:

$$\frac{2H}{\omega} \ddot{\delta} + \omega D \dot{\delta} = P_m - P_e, \quad (1)$$

where δ is the rotor rotation angle, H is the inertia constant, ω is the electrical angular velocity, D is the

rotor damping constant, P_m and P_e are the mechanical and electrical power respectively, all expressed in per unit on the system base power. It represents the equation of motion of synchronous machines[5], [6], [9].

Sudden disturbances in power systems cause electromechanical oscillations, mainly of two types [3], local or inter area mode oscillations. Local mode oscillations are localized in one station or a small part of the power system and are associated with the swing of units at that station with respect to the rest of the power systems, while inter area mode oscillations are typically caused by two or more groups of closely coupled machines [4].

In [10], the power network consisting of generators and transmission lines was treated as a continuum system described by a nonlinear version of the standard second-order wave partial differential equation. However, authors assumed that the voltage is constant which is a limiting assumption in real applications, especially if voltage is used as a space/time variable control input to stabilize the system and quickly damp the oscillations that propagate after disturbances.

Optimal Control for PDEs is not as easy as ODEs since there is no complete generalization of the Pontryagin's Maximum Principle [12],[17]. However, basic ideas on how to deal with optimal control for PDEs were presented by Lions [14] and Li [15].

In this paper, we assume the voltage to vary in space and time and we develop the general continuum system PDE that describes the system. The stability analysis of a continuum system for which robust methods have been developed provides an alternative to overcome the extreme difficulties associated with the stability analysis of a large dimensional discrete system. Considering powerful mathematical tools for PDEs, our objective in this paper is to apply distributed control designs to our proposed continuum model and gain additional insight into mechanisms by which disturbance propagations in the power system can be mitigated.

This paper is organized as follows: In section II we start by summarizing the existent constant voltage Swing PDE, then in part B of this section we develop the variant voltage swing PDE. In section III we design the distributed optimal controller for the constant voltage swing PDE using the mechanical power as the control input while in section IV we solve the distributed optimal control problem using voltage as the control input, and then finally section V is the conclusion.

*This work was supported in part by the National Science Foundation under grant No CNS-1239366, and in part by the Engineering Research Center Program of the National Science Foundation and the Department of Energy under NSF Award Number EEC-1041877 and the CURENT Industry Partnership Program.

¹S. Sahyoun, S. Djouadi, and K. Tomsovic are with the Department of Electrical Engineering and Computer Science, University of Tennessee, Knoxville, Tennessee 37996-2250 USA; ssahyoun@utk.edu, djouadi@eecs.utk.edu, tomsovic@eecs.utk.edu

²S. Lenhart is with the Department of Mathematics, University of Tennessee, Knoxville, Tennessee 37996-1320 USA; lenhart@math.utk.edu

II. THE SWING PDE WITH A SPACE DEPENDENT VOLTAGE MAGNITUDE

A. Constant Voltage Magnitude Swing PDE

We will consider the distributed power system model shown in Fig. 1 [10]. Each node is a generator that supplies a variable current and voltage producing a variable power. The special case of a constant voltage was discussed in [10].

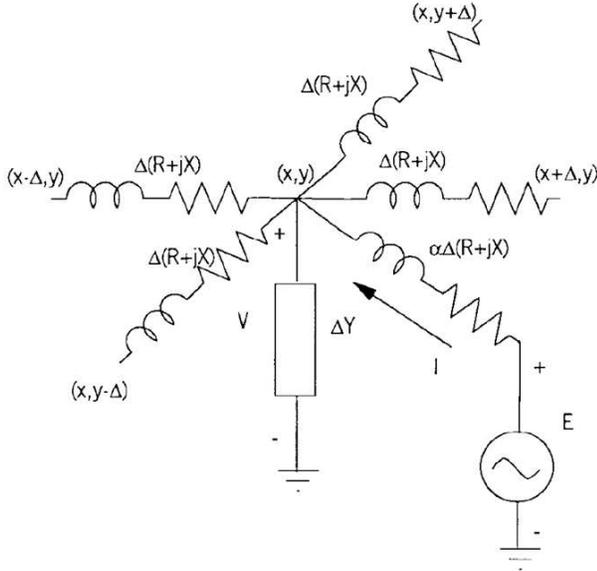


Fig. 1. Power System Model [10]

The continuum system for the case of a constant voltage is described by the following second order hyperbolic wave PDE [10]:

$$\frac{\partial^2 \delta}{\partial t^2} + v \frac{\partial \delta}{\partial t} - v^2 \nabla^2 \delta + u^2 (\nabla \delta)^2 = P, \quad (2)$$

and the PDE parameters are given by:

$$\begin{aligned} v^2 &= \frac{\omega V^2 \sin \theta}{2h|z|} \\ u^2 &= \frac{\omega V^2 \cos \theta}{2h|z|} \\ P &= \frac{\omega(p_m - GV^2)}{2h} \\ v &= \frac{\omega^2 d}{2h} \end{aligned}$$

where $E(x, y) = Ve^{j\delta(x, y)}$ and V is the constant voltage magnitude, $z = |z|(\cos \theta + j \sin \theta)$ is the transmission line impedance and G is the real part of the admittance and ∇ and ∇^2 are the first and second spacial derivatives respectively.

B. Space Dependent Voltage Magnitude

In this section, we derive the PDE that describes the electromechanical wave propagation for the space varying voltage magnitude, i.e. $E(x, y) = V(x, y)e^{j\delta(x, y)}$.

The space variant generator current is then given by:

$$I(x, y) = -\frac{\Delta^2}{z} \nabla^2 E(x, y) + \Delta Y E(x, y), \quad (3)$$

where Y is the shunt admittance, Δ is the separation between adjacent nodes and ∇ is the space derivative. So the first derivative of $E(x, y)$ is:

$$\nabla E = \nabla V e^{j\delta} + V j \nabla \delta e^{j\delta} \quad (4)$$

where the second derivative would be:

$$\begin{aligned} \nabla^2 E &= \nabla^2 V e^{j\delta} + \nabla V j \nabla \delta e^{j\delta} \\ &+ j[(\nabla V \nabla \delta + V \nabla^2 \delta) e^{j\delta} + V j (\nabla \delta)^2 e^{j\delta}] \\ &= [\nabla^2 V - V (\nabla \delta)^2 + j(2\nabla V \nabla \delta + V \nabla^2 \delta)] e^{j\delta} \end{aligned} \quad (5)$$

Using the current expression in (3), the electrical power P_e is given by:

$$\begin{aligned} P_e &= \text{Re}\{E I^*\} \\ &= \text{Re}\{V e^{j\delta} (-\frac{\Delta^2}{z} [\nabla^2 V - V (\nabla \delta)^2 - j(2\nabla V \nabla \delta + V \nabla^2 \delta)] e^{-j\delta} + \Delta Y^* V e^{-j\delta})\} \end{aligned}$$

The complex exponential terms cancel and the electrical power expression simplifies to:

$$\begin{aligned} P_e &= \text{Re}\{V (-\frac{\Delta^2}{z} [\nabla^2 V - V (\nabla \delta)^2 - j(2\nabla V \nabla \delta + V \nabla^2 \delta)] \\ &+ \Delta Y^* V)\} \end{aligned}$$

Using the complex form of the impedance $z = |z|(\cos \theta + j \sin \theta)$ we have:

$$\begin{aligned} P_e &= \text{Re}\{V (-\frac{\Delta^2}{|z|} [\nabla^2 V - V (\nabla \delta)^2 - j(2\nabla V \nabla \delta + V \nabla^2 \delta)] (\cos \theta \\ &- j \sin \theta) + \Delta Y V)\} \\ &= V (-\frac{\Delta^2}{|z|} [(\nabla^2 V - V (\nabla \delta)^2) \cos \theta - (2\nabla V \nabla \delta + V \nabla^2 \delta) \sin \theta] \\ &+ \Delta V^2 G), \end{aligned} \quad (6)$$

where $G = \text{Re}\{Y\}$.

The discrete swing equation parameters H, D, Z , and P_m in (1) translate for the continuum system into the distributed parameters $\Delta h(x, y), \Delta d(x, y), \Delta z(x, y)$, and $\Delta p_m(x, y)$ respectively. Substituting the electrical power expression (6) into the discrete swing equation (1) and taking the continuum limits yields:

$$\begin{aligned} \frac{2h}{\omega} \frac{\partial^2 \delta}{\partial t^2} + \omega d \frac{\partial \delta}{\partial t} &= p_m + V (-\frac{1}{|z|} [(\nabla^2 V - V (\nabla \delta)^2) \cos \theta \\ &- (2\nabla V \nabla \delta + V \nabla^2 \delta) \sin \theta] - GV^2) \end{aligned} \quad (7)$$

where the dependence on Δ cancels. The PDE (7) is also a hyperbolic second order wave equation but includes nonlinearities that didn't show up in the constant voltage swing PDE (2). For a particular but also practical choice of $\theta = \frac{\pi}{2}$, (7) simplifies to:

$$\ddot{\delta} + v\dot{\delta} = -\alpha(2V\nabla V\nabla\delta + V\nabla^2\delta) + \beta(p_m - GV^2) \quad (8)$$

where $v = \frac{\omega^2 d}{2h}$, $\alpha = \frac{\omega}{2h|z|}$ and $\beta = \frac{\omega}{2h}$

Figure (2) shows a numerical simulation for the electromechanical wave propagation for the angle δ in a continuous 2D system. Initial disturbance is a Gaussian function of space where its peak is at the center. The voltage magnitude V is allowed to have a random (uncontrolled) space variation from 0.9 to 1.1 pu. Control design for systems governed

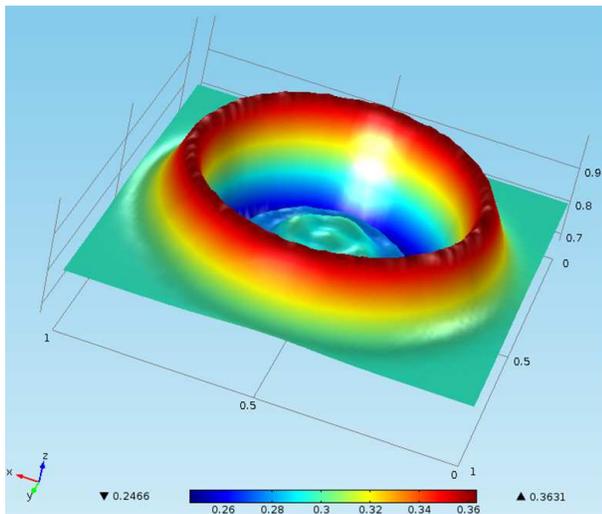


Fig. 2. Electromechanical wave propagation in the continuous 2D system

by these types of nonlinear PDEs is not an easy task due to the existence of the nonlinear terms in the right hand side of (8). Optimal Control design for ODE systems is not difficult. Space discretization can be implemented on the PDE (8) to obtain a state space system of ODEs for which control techniques are well studied in the literature. But before discretization, since it is a second order type PDE, we define states x_1 and x_2 as follows:

$$\begin{aligned} x_1 &= \delta, & \dot{x}_1 &= \dot{\delta} \\ x_2 &= \dot{\delta}, & \dot{x}_2 &= \ddot{\delta} \end{aligned}$$

Then (8) can be written in the form:

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -vx_2 - \alpha(2VA_1VA_1 + VA_2)x_1 + \beta(p_m - G(V).(V)) \end{aligned} \quad (9)$$

where $\nabla = A_1$ and $\nabla^2 = A_2$ are the discretization matrices and

$$(V).(V) = \begin{pmatrix} V_1 & 0 & \dots \\ 0 & \ddots & 0 \\ 0 & \dots & V_N \end{pmatrix} \begin{pmatrix} V_1 \\ \vdots \\ V_N \end{pmatrix}$$

If the control input to be implemented is the voltage magnitude V then the system (9) is highly nonlinear with cross terms state-control nonlinearities. Linearization is always the easiest option but not the best one in this case as shown in Figures (3) and (4) that show the angle time trajectory at two different locations of the power grid using the nonlinear system (9) and a linearized version of it.

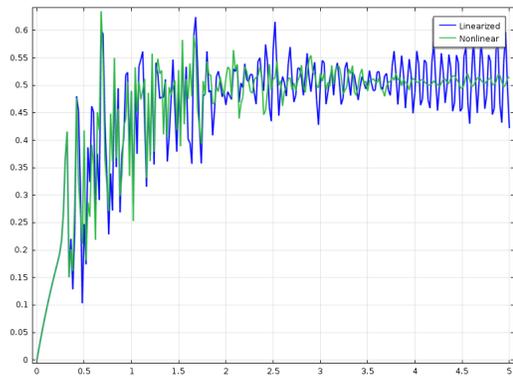


Fig. 3. Time trajectory comparison between linearized and nonlinear systems at a chosen space location in the power Grid. Not a satisfactory agreement.

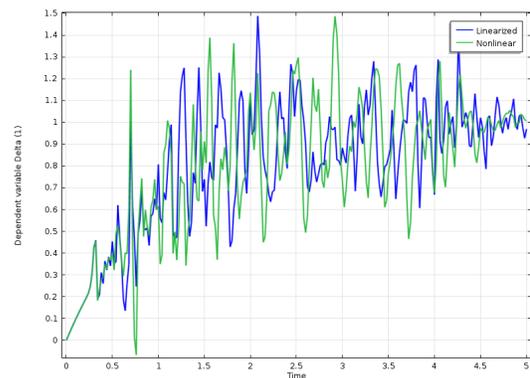


Fig. 4. Time trajectory comparison between linearized and nonlinear systems at a different space location in the power Grid. Not a satisfactory agreement.

For the constant voltage swing PDE (2) and for the practical choice of $\theta = \frac{\pi}{2}$ the only nonlinear term in the

equation $u^2(\Delta\delta)^2$ vanishes and the equation becomes linear in the form:

$$\frac{\partial^2 \delta}{\partial t^2} + v \frac{\partial \delta}{\partial t} - v^2 \nabla^2 \delta = P$$

This form of the wave PDE can be analytically solved using the transformation:

$$\delta = \varphi e^{-\frac{vt}{2}}$$

Then,

$$v \frac{\partial \delta}{\partial t} = v \frac{\partial \varphi}{\partial t} e^{-\frac{vt}{2}} - \frac{v^2}{2} e^{-\frac{vt}{2}} \varphi$$

And

$$\frac{\partial^2 \delta}{\partial t^2} = \frac{\partial^2 \varphi}{\partial t^2} e^{-\frac{vt}{2}} - \frac{\partial \varphi}{\partial t} v e^{-\frac{vt}{2}} + \frac{v^2}{4} e^{-\frac{vt}{2}} \varphi - \frac{\partial \varphi}{\partial t} v e^{-\frac{vt}{2}}$$

So the PDE becomes:

$$\frac{\partial^2 \varphi}{\partial t^2} - \frac{v}{4} \varphi - v^2 \nabla^2 \varphi = P e^{\frac{vt}{2}}$$

The damping term doesn't exist, this equation is called the Klein Gordon Equation for which an analytical solution exists.

III. DISTRIBUTED OPTIMAL CONTROL OF THE CONSTANT VOLTAGE SWING PDE USING POWER AS THE CONTROL INPUT

In this section we use the mechanical power P as the control input that drives the angle $\delta(x, y)$ to track a reference value $\delta_r(x, y)$ for the constant voltage swing PDE. So our goal is to minimize the cost function:

$$J(P) = \frac{1}{2} \int_0^T \int_{\Omega} [|\delta(x, t) - \delta_r(x, t)|^2 + P(x, t)^2] dxdt \quad (10)$$

Subject to:

$$\begin{aligned} \frac{\partial^2 \delta}{\partial t^2} + v \frac{\partial \delta}{\partial t} - v^2 \nabla^2 \delta &= P \\ \delta(x, t) &= 0 \quad \text{for } \delta \in \partial\Omega \times [0, T] \\ \delta(x, 0) &= \delta_i \end{aligned}$$

To derive necessary conditions for optimality, we need to differentiate the cost function with respect to P , i.e. we need to differentiate the map $P \mapsto J(P)$. However, δ contributes to $J(P)$ so we must also differentiate the map $P \mapsto \delta(P)$. Let

$$\psi = \lim_{\varepsilon \rightarrow 0} \frac{\delta(P + \varepsilon l) - \delta(P)}{\varepsilon} \quad (11)$$

be the sensitivity of the state with respect to the control where l is a variation function and $\varepsilon > 0$.

Then the PDE that corresponds to the control $P + \varepsilon l$ is:

$$\delta_{tt}^\varepsilon + v \delta_t^\varepsilon - v^2 \delta_{xx}^\varepsilon = P + \varepsilon l \quad (12)$$

Subtracting the constraint in (10) from (12) and dividing both sides by ε yields:

$$\begin{aligned} \left(\frac{\delta^\varepsilon - \delta}{\varepsilon} \right)_{tt} + v \left(\frac{\delta^\varepsilon - \delta}{\varepsilon} \right)_t - v^2 \left(\frac{\delta^\varepsilon - \delta}{\varepsilon} \right)_{xx} &= l \\ \psi_{tt} + v \psi_t - v^2 \psi_{xx} &= l := L\psi \end{aligned}$$

where $\psi \in L_2[0, T] \times H_0^1(\Omega) := \Psi$.

$L: \Psi \rightarrow \mathbb{R}$

$L^*: \mathbb{R}^* \rightarrow \Psi^* = \mathbb{R} \rightarrow \Psi$.

The operator L and the adjoint operator L^* are related by:

$$\langle \lambda, L\psi \rangle = \langle L^*\lambda, \psi \rangle$$

where $\langle \cdot, \cdot \rangle$ is the L^2 inner product. For ψ_{tt} , integration by parts twice gives:

$$\int_0^T \int_{\Omega} \lambda \psi_{tt} dxdt = \int_0^T \int_{\Omega} \psi \lambda_{tt} dxdt$$

For ψ_{xx} , integration by parts twice gives:

$$\int_0^T \int_{\Omega} \lambda \psi_{xx} dxdt = \int_0^T \int_{\Omega} \psi \lambda_{xx} dxdt$$

For ψ_t , integration by parts once gives:

$$\int_0^T \int_{\Omega} \lambda \psi_t dxdt = - \int_0^T \int_{\Omega} \psi \lambda_t dxdt$$

So the Adjoint operator will be:

$$L^*\lambda = \lambda_{tt} - v \lambda_t - v^2 \lambda_{xx} \quad (13)$$

Then the Adjoint PDE is:

$$\begin{aligned} L^*\lambda &= \frac{\partial \text{inegrand}(J)}{\partial \delta} \\ \lambda_{tt} - v \lambda_t - v^2 \lambda_{xx} &= \delta^* - \delta_r \end{aligned} \quad (14)$$

The sensitivity and adjoint functions are used in the differentiation of the map $P \mapsto \delta(P)$.

$$\lim_{\varepsilon \rightarrow 0^+} \frac{J(P^* + \varepsilon l) - J(P^*)}{\varepsilon} \geq 0 \quad (15)$$

The numerator terms are:

$$J(P^*) = \frac{1}{2} \int_0^T \int_{\Omega} [|\delta^* - \delta_r|^2 + (P^*)^2] dxdt$$

$$J(P^* + \varepsilon l) = \frac{1}{2} \int_0^T \int_{\Omega} [|\delta^{\varepsilon*} - \delta_r|^2 + (P^* + \varepsilon l)^2] dxdt$$

Then the limit (15) becomes:

$$\begin{aligned} &= \lim_{\varepsilon \rightarrow 0^+} \frac{1}{2} \int_0^T \int_{\Omega} \left[\frac{(\delta^{\varepsilon*})^2 - (\delta^*)^2}{\varepsilon} - \frac{2\delta_r(\delta^{\varepsilon*} - \delta^*)}{\varepsilon} \right. \\ &\quad \left. + 2P^*l + \varepsilon l^2 \right] dxdt \\ &= \lim_{\varepsilon \rightarrow 0^+} \frac{1}{2} \int_0^T \int_{\Omega} \left[\frac{(\delta^{\varepsilon*} - \delta^*)}{\varepsilon} (\delta^{\varepsilon*} + \delta^*) - \frac{2\delta_r(\delta^{\varepsilon*} - \delta^*)}{\varepsilon} \right. \\ &\quad \left. + 2P^*l + \varepsilon l^2 \right] dxdt \\ &= \frac{1}{2} \int_0^T \int_{\Omega} [\psi(2\delta^*) - 2\delta_r\psi + 2P^*l] dxdt \\ &= \int_0^T \int_{\Omega} [\psi(\delta^* - \delta_r) + P^*l] dxdt \end{aligned}$$

Then from (14) we get:

$$\begin{aligned} \int_0^T \int_{\Omega} [\psi(L^* \lambda) + P^* l] dxdt &= 0 \\ \int_0^T \int_{\Omega} [\lambda L \psi + P^* l] dxdt &= 0 \\ \int_0^T \int_{\Omega} [\lambda l + P^* l] dxdt &= 0 \\ \int_0^T \int_{\Omega} [l(\lambda + P^*)] dxdt &= 0 \end{aligned}$$

So the optimal control becomes:

$$P^* = -\lambda$$

Let $P \in [0, M]$ is the admissible control set U_{ad} , then the optimal control is:

$$P^* = \min(\max(-\lambda, 0), M) \tag{16}$$

where λ is computed by solving the coupled PDE system:

$$\begin{aligned} \lambda_{tt} - v \lambda_t - v^2 \lambda_{xx} &= \delta^* - \delta_r \\ \delta_{tt}^* + v \delta_t^* - v^2 \delta_{xx}^* &= \min(\max(-\lambda, 0), M) \end{aligned}$$

The difficulty in solving these coupled PDEs arises from the fact that the state PDE has initial conditions while the adjoint PDE has final conditions. One method to solve such coupled systems is the forward backward sweep method explained in [12] and [13]. The steps of the forward backward sweep algorithm is as follows:

- 1) Start with an initial guess for the control P^* over the domain.
- 2) Using the state PDE initial conditions and the values for P^* , solve δ^* forward in time.
- 3) Using the adjoint PDE final conditions and the values for P^* and δ^* , solve λ backward in time.
- 4) Update P^* by entering the new δ^* and λ into the expression of the optimal control.
- 5) Check convergence. Stop if the difference is negligible between this iteration and the previous one, otherwise return to step 2.

Convergence and stability of this algorithm is discussed in [16]. Figure (6) shows the numerical solution of a controlled system for the initial disturbance shown in Figure (5).

IV. DISTRIBUTED OPTIMAL CONTROL OF THE CONSTANT VOLTAGE SWING PDE USING VOLTAGE AS THE CONTROL INPUT

Although constant Voltage is the assumption for the swing PDE in this case, we can still use the voltage as a control input if the deviation above or below a constant value is kept minimum. We need to assume that the voltage varies only within a narrow neighborhood around a constant value V_r because letting it vary freely violates the original assumption

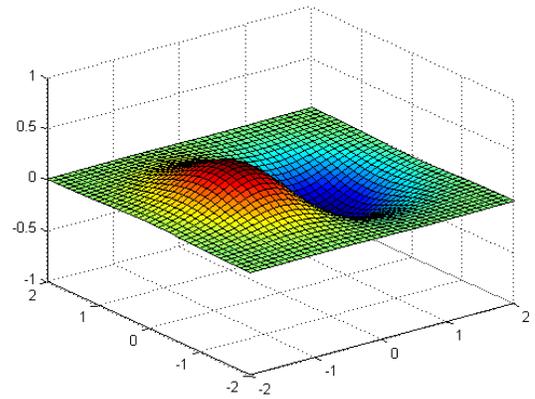


Fig. 5. Initial disturbance

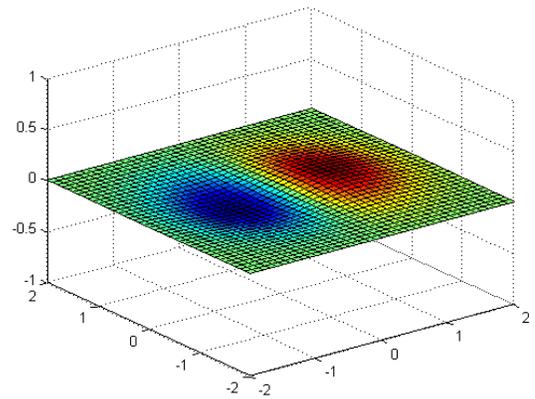


Fig. 6. Controlled System

in deriving the swing PDE (2), then the cost function to be minimized is :

$$J(V) = \frac{1}{2} \int_0^T \int_{\Omega} [|\delta(x,t) - \delta_r(x,t)|^2 + |V(x,t) - V_r|^2] dxdt \tag{17}$$

Subject to:

$$\begin{aligned} \frac{\partial^2 \delta}{\partial t^2} + v \frac{\partial \delta}{\partial t} - \frac{\omega V^2 \sin \theta}{2h|z|} \nabla^2 \delta &= \frac{\omega(p_m - GV^2)}{2h} \\ \delta(x,t) &= 0 \quad \text{for } \delta \in \partial \Omega \times [0, T] \\ \delta(x,0) &= \delta_i \end{aligned}$$

where $v, \omega, h, z, \theta, p_m, G$ are all constants. The sensitivity of the state with respect to the control input V is:

$$\begin{aligned} \psi &:= \lim_{\epsilon \rightarrow 0} \frac{\delta(V + \epsilon l) - \delta(V)}{\epsilon} \\ &= \lim_{\epsilon \rightarrow 0} \frac{\delta^\epsilon - \delta}{\epsilon} \end{aligned}$$

For the control input $V + \epsilon l$ we have:

$$\frac{\partial^2 \delta^\epsilon}{\partial t^2} + v \frac{\partial \delta^\epsilon}{\partial t} - \frac{\omega(V + \epsilon l)^2 \sin \theta}{2h|z|} \nabla^2 \delta^\epsilon = \frac{\omega(p_m - G(V + \epsilon l)^2)}{2h} \quad (18)$$

while for the control input V we have:

$$\frac{\partial^2 \delta}{\partial t^2} + v \frac{\partial \delta}{\partial t} - \frac{\omega V^2 \sin \theta}{2h|z|} \nabla^2 \delta = \frac{\omega(p_m - GV^2)}{2h} \quad (19)$$

Subtracting (19) from (18) and dividing both sides by ϵ yields:

$$\left(\frac{\delta^\epsilon - \delta}{\epsilon}\right)_{tt} + v \left(\frac{\delta^\epsilon - \delta}{\epsilon}\right)_t - \frac{\omega V^2 \sin \theta}{2h|z|} \nabla^2 \left(\frac{\delta^\epsilon - \delta}{\epsilon}\right) - \frac{\omega l(2V + \epsilon l) \sin \theta}{2h|z|} \nabla^2 \delta^\epsilon = -\frac{\omega G l(2V + \epsilon l)}{2h}$$

$$\psi_{tt} + v \psi_t - \frac{\omega V^2 \sin \theta}{2h|z|} \nabla^2 \psi - \frac{\omega l(2V) \sin \theta}{2h|z|} \nabla^2 \delta = -\frac{2\omega G l V}{2h}$$

Defining the operator $L\psi := \psi_{tt} + v\psi_t - \frac{\omega V^2 \sin \theta}{2h|z|} \nabla^2 \psi$ we have:

$$L\psi = \frac{\omega l(2V) \sin \theta}{2h|z|} \nabla^2 \delta - \frac{2\omega G l V}{2h} \quad (20)$$

Then the adjoint operator will be:

$$\begin{aligned} L^* \lambda &= \lambda_{tt} - v \lambda_t - \frac{\omega V^2 \sin \theta}{2h|z|} \nabla^2 \lambda \\ &= \delta^* - \delta_r \end{aligned} \quad (21)$$

Now the sensitivity and adjoint functions will be used in the differentiation of the map $V \mapsto \delta(V)$.

$$\lim_{\epsilon \rightarrow 0^+} \frac{J(V^* + \epsilon l) - J(V^*)}{\epsilon} \geq 0 \quad (22)$$

where

$$\begin{aligned} J(V^*) &= \frac{1}{2} \int_0^T \int_\Omega [|\delta^* - \delta_r|^2 + |V^* - V_r|^2] dxdt \\ J(V^* + \epsilon l) &= \frac{1}{2} \int_0^T \int_\Omega [|\delta^{\epsilon*} - \delta_r|^2 + |V^* - V_r + \epsilon l|^2] dxdt \end{aligned}$$

Then the limit (22) becomes:

$$\begin{aligned} &\lim_{\epsilon \rightarrow 0^+} \frac{1}{2} \int_0^T \int_\Omega \left[\frac{(\delta^{\epsilon*})^2 - (\delta^*)^2}{\epsilon} - \frac{2\delta_r(\delta^{\epsilon*} - \delta^*)}{\epsilon} + 2V^* l + \epsilon l^2 \right] dxdt \\ &= \lim_{\epsilon \rightarrow 0^+} \frac{1}{2} \int_0^T \int_\Omega \left[\frac{(\delta^{\epsilon*} - \delta^*)}{\epsilon} (\delta^{\epsilon*} + \delta^*) - \frac{2\delta_r(\delta^{\epsilon*} - \delta^*)}{\epsilon} + 2(V^* - V_r)l + \epsilon l^2 \right] dxdt \\ &= \frac{1}{2} \int_0^T \int_\Omega [\psi(2\delta^*) - 2\delta_r \psi + 2(V^* - V_r)l] dxdt \\ &= \int_0^T \int_\Omega [\psi(\delta^* - \delta_r) + (V^* - V_r)l] dxdt \\ &= \int_0^T \int_\Omega [\psi L^* \lambda + (V^* - V_r)l] dxdt \\ &= \int_0^T \int_\Omega [\lambda L \psi + (V^* - V_r)l] dxdt \\ &= \int_0^T \int_\Omega \left[\lambda \left(\frac{\omega l 2V^* \sin \theta}{2h|z|} \nabla^2 \delta - \frac{2\omega G l V^*}{2h} \right) + (V^* - V_r)l \right] dxdt \\ &= \int_0^T \int_\Omega l \left[\lambda V^* \left(\frac{\omega \sin \theta}{h|z|} \nabla^2 \delta - \frac{\omega G}{h} \right) + (V^* - V_r) \right] dxdt \end{aligned}$$

And then the optimal control input V^* in terms of the adjoint variable λ would be:

$$V^* = \frac{V_r}{\lambda \left(\frac{\omega \sin \theta}{h|z|} \nabla^2 \delta - \frac{\omega G}{h} \right) - 1} \quad (23)$$

And remember that the variation for V^* should be limited around V_r for practical results. Substituting the expression for V^* in (23) into the adjoint PDE (21), then substituting also in the state PDE (19) yields the coupled state and adjoint PDEs:

$$\begin{aligned} \lambda_{tt} - v \lambda_t - \frac{\omega V^{*2} \sin \theta}{2h|z|} \nabla^2 \lambda &= \delta^* - \delta_r \\ \delta_{tt}^* + v \delta_t^* - \frac{\omega V^{*2} \sin \theta}{2h|z|} \nabla^2 \delta^* &= \frac{\omega(p_m - GV^{*2})}{2h} \end{aligned} \quad (24)$$

V. CONCLUSIONS

In this paper, we generalized the electromechanical wave propagation model in electric power systems to account for the general and practical variant voltage case. The new PDE includes more complicated nonlinearities to the extent that linearization techniques don't capture the system behavior anymore. We designed two optimal controllers; one was for the case of using mechanical power and the second was for voltage control input. Due to page size limits we present the optimal control results for the variant voltage swing PDE in a different paper.

ACKNOWLEDGMENT

This work was supported in part by the National Science Foundation under grant No CNS-1239366, and in part by the Engineering Research Center Program of the National Science Foundation and the Department of Energy under NSF

Award Number EEC-1041877 and the CURENT Industry Partnership Program.

REFERENCES

- [1] Leung JSK, Hill DJ, Ni Y (2005) Global power system control using generator excitation, PSS, FACTS devices and capacitor switching. *International Journal Of Electrical Power And Energy Systems*, 2005, v. 27 n. 5-6, p. 448-464
- [2] Pavlos S. Georgilakis, "Technical challenges associated with the integration of wind power into power systems", *Renewable and Sustainable Energy Reviews*, Volume 12, Issue 3, April 2008, Pages 852-863.
- [3] P.W. Sauer, M. A. Pai, "Power System Dynamics and Stability", Pearson Education Pte. Ltd., Singapore, 1998.
- [4] W. D. Stevenson, Jr., "Elements of Power System Analysis", 4th ed. New York: McGraw-Hill, 1982, p. 378.
- [5] D. Mondal, A. Chakrabarti and A. Sengupta, "Power System Small Signal Stability Analysis and Control", Academic Press 2014, ISBN: 9780128005729
- [6] Momoh, James "Electric Systems, Dynamics, and Stability with Artificial Intelligence", CRC Press 1999, ISBN 978-0824702335.
- [7] P.M. Anderson, A.A. Fouad, "Power System Control and Stability", Iowa State University Press, Ames, IA, 1977.
- [8] Andersson, Goran, "Modelling and Analysis of Electric Power Systems", ETH Zurich, March 2004.
- [9] P. Kundur, "Power System Stability and Control", McGraw-Hill, New York, 1994
- [10] Thorp, J.S., Seyler, C.E., and Phadke, A.G., "Electromechanical wave propagation in large electric power systems", *IEEE Transactions on CAS*, Vol. 45, No. 6, June 1998, pp 614-622.
- [11] E. Kimbark, "Power system stability," in *Synchronous Machines*. New York: Dover, 1956, p. 241.
- [12] Lenhart S. and J. Workman, "Optimal Control Applied to Biological Models", Boca Raton, Chapman Hall/CRC, 2007, ISBN: 978-1584886402.
- [13] John Betts, "Practical Methods for Optimal Control Using Nonlinear Programming". SIAM, Philadelphia, 2001.
- [14] J. L. Lions. *Optimal Control of Systems Governed by Partial Differential Equations*. Springer-Verlag, Berlin, 1971.
- [15] Xunjing Li and Jiongmin Yong. *Optimal Control Theory for Infinite Dimensional Systems*. Birkhauser, Boston, 1995.
- [16] W. Hackbush. "A numerical method for solving parabolic equations with opposite orientations", *Computing*, 20(3):229-40, 1978.
- [17] L. S. Pontryagin, V. G. Boltyanskii, R. V. Gamkrelize, and E. F. Mishchenko. *The Mathematical Theory of Optimal Processes*, Wiley, New York, 1962.