J Exercises

Exercise III.1 Prove that projectors are idempotent, that is, $P^2 = P$.

Exercise III.2 Prove that a normal matrix is Hermitian iff it has real eigenvalues.

Exercise III.3 Prove that $U(t) \stackrel{\text{def}}{=} \exp(-iHt/\hbar)$ is unitary.

Exercise III.4 Use spectral decomposition to show that $K = -i\log(U)$ is Hermitian for any unitary U, and thus $U = \exp(iK)$ for some Hermitian K.

Exercise III.5 Show that $[L, M]$ and $\{L, M\}$ are bilinear operators (linear in both of their arguments).

Exercise III.6 Show that $[L, M]$ is anticommutative, i.e., $[M, L] = -[L, M]$, and that $\{L, M\}$ is commutative.

Exercise III.7 Show that $LM = \frac{[L,M]+\{L,M\}}{2}$.

Exercise III.8 Show that the four Bell states are orthonormal.

Exercise III.9 Prove that $\frac{1}{\sqrt{2}}(|000\rangle + |111\rangle)$ is entangled.

Exercise III.10 What is the effect of Y (imaginary definition) on the computational basis vectors? What is its effect if you use the real definition (¶12, p. 119)?

Exercise III.11 Prove that I, X, Y, and Z are unitary. Use either the imaginary or real definition of Y (¶12, p. 119).

Exercise III.12 Show that the X, Y, Z and H gates are Hermitian (their own inverses) and prove your answers. Use either the imaginary or real definition of Y (¶12, p. 119).

Exercise III.13 Prove the following useful identities:

$$HXH = Z, HYH = -Y, HZH = X.$$

Exercise III.14 Show (using the real definition of Y, ¶12, p. 119):
$|0\rangle\langle 0| = \frac{1}{2}(I + Z), |0\rangle\langle 1| = \frac{1}{2}(X - Y), |1\rangle\langle 0| = \frac{1}{2}(X + Y), |1\rangle\langle 1| = \frac{1}{2}(I - Z)$.

Exercise III.15 Prove $|\beta_{xy}\rangle = (P \otimes I)|\beta_{00}\rangle$, where $xy = 00, 01, 11, 10$ for $P = I, X, Y, Z$, respectively.

Exercise III.16 Suppose that P is one of the Pauli operators, but you don't know which one. However, you are able to pick a 2-qubit state $|\psi_0\rangle$ and operate on it, $|\psi_1\rangle = (P \otimes I)|\psi_0\rangle$. Further, you are able to select a unitary operation U to apply to $|\psi_1\rangle$, and to measure the 2-qubit result, $|\psi_2\rangle = U|\psi_1\rangle$, in the computational basis. Select $|\psi_0\rangle$ and U so that you can determine with certainty the unknown Pauli operator P.

Exercise III.17 What is the matrix for CNOT in the standard basis? Prove your answer.

Exercise III.18 Show that CNOT does not violate the No-cloning Theorem by showing that, in general, $\text{CNOT}|\psi\rangle|0\rangle \neq |\psi\rangle|\psi\rangle$. Under what conditions does the equality hold?

Exercise III.19 What is the matrix for CCNOT in the standard basis? Prove your answer.

Exercise III.20 Use a single Toffoli gate to implement each of NOT, NAND, and XOR.

Exercise III.21 Use Toffoli gates to implement FAN-OUT. FAN-OUT would seem to violate the No-cloning Theorem, but it doesn't. Explain why.

Exercise III.22 Design a quantum circuit to transform $|000\rangle$ into the entangled state $\frac{1}{\sqrt{2}}(|000\rangle + |111\rangle)$.

Exercise III.23 Show that $|+\rangle, |-\rangle$ is an ON basis.

Exercise III.24 Prove:

$$\begin{aligned} |0\rangle &= \frac{1}{\sqrt{2}}(|+\rangle + |-\rangle), \\ |1\rangle &= \frac{1}{\sqrt{2}}(|+\rangle - |-\rangle). \end{aligned}$$

Exercise III.25 Prove that $Z|+\rangle = |-\rangle$ and $Z|-\rangle = |+\rangle$.

Exercise III.26 Prove:

$$\begin{aligned} H(a|0\rangle + b|1\rangle) &= a|+\rangle + b|-\rangle, \\ H(a|+\rangle + b|-\rangle) &= a|0\rangle + b|1\rangle. \end{aligned}$$

Exercise III.27 Prove $H = (X + Z)/\sqrt{2}$.

Exercise III.28 Prove Eq. III.18 (p. 126).

Exercise III.29 Show that three successive CNOTs, connected as in Fig. III.11 (p. 124), will swap two qubits.

Exercise III.30 Recall the conditional selection between two operators (¶14, p. 125): $|0\rangle\langle 0| \otimes U_0 + |1\rangle\langle 1| \otimes U_1$. Suppose the control bit is a superposition $|\chi\rangle = a|0\rangle + b|1\rangle$. Show that:

$$(|0\rangle\langle 0| \otimes U_0 + |1\rangle\langle 1| \otimes U_1)|\chi, \psi\rangle = a|0, U_0\psi\rangle + b|1, U_1\psi\rangle.$$

Exercise III.31 Show that the 1-bit full adder (Fig. III.15, p. 127) is correct.

Exercise III.32 Show that the operator U_f is unitary:

$$U_f|x, y\rangle \stackrel{\text{def}}{=} |x, y \oplus f(x)\rangle,$$

Exercise III.33 Verify the remaining superdense encoding transformations in Sec. C.6.a, ¶5 (p. 131).

Exercise III.34 Verify the remaining superdense decoding transformations in Sec. C.6.a, ¶9 (p. 132).

Exercise III.35 Complete the following step from the derivation of the Deutsch-Jozsa algorithm (Sec. D.1.b, ¶11, p. 144):

$$H|x\rangle = \sum_{z \in \mathbf{2}} \frac{1}{\sqrt{2}} (-1)^{xz} |z\rangle.$$

Exercise III.36 Show that $\text{CNOT}(H \otimes I) = (I \otimes H) C_Z H^{\otimes 2}$, where C_Z is the controlled-Z gate.

Exercise III.37 Show that the Fourier transform matrix (¶11, p. 151, Sec. D.3.a) in unitary.

Exercise III.38 Prove the claim in ¶29, p. 168 (Sec. D.4.b).

Exercise III.39 Prove the claim in ¶32, p. 168 (Sec. D.4.b).

Exercise III.40 Design a quantum gate array for the following syndrome extraction operator (¶3, p. 177, in Sec. D.5.d, p. 177):

$$S|x_3, x_2, x_1, 0, 0, 0\rangle \stackrel{\text{def}}{=} |x_3, x_2, x_1, x_1 \oplus x_2, x_1 \oplus x_3, x_2 \oplus x_3\rangle.$$

Exercise III.41 Design a quantum gate array to apply the appropriate error correction for the extracted syndrome as given in ¶4, p. 178 (Sec. D.5.d, p. 177):

bit flipped	syndrome	error correction
none	$\|000\rangle$	$I \otimes I \otimes I$
1	$\|110\rangle$	$I \otimes I \otimes X$
2	$\|101\rangle$	$I \otimes X \otimes I$
3	$\|011\rangle$	$X \otimes I \otimes I$

Exercise III.42 Prove that $A_a A_a = \mathbf{1}$ (Sec. F.1.b).

Exercise III.43 Prove that $A_{ab,c} = \mathbf{1} + a^\dagger a b^\dagger b(c + c^\dagger - \mathbf{1}) = \mathbf{1} + N_a N_b (A_c - \mathbf{1})$ is a correct definition of CCNOT by showing how it transforms the quantum register $|a, b, c\rangle$ (Sec. F.1.b).

Exercise III.44 Show that the following definition of Feynman's switch is unitary (Sec. F.1.b):

$$q^\dagger c p + r^\dagger c^\dagger p + p^\dagger c^\dagger q + p^\dagger c r.$$