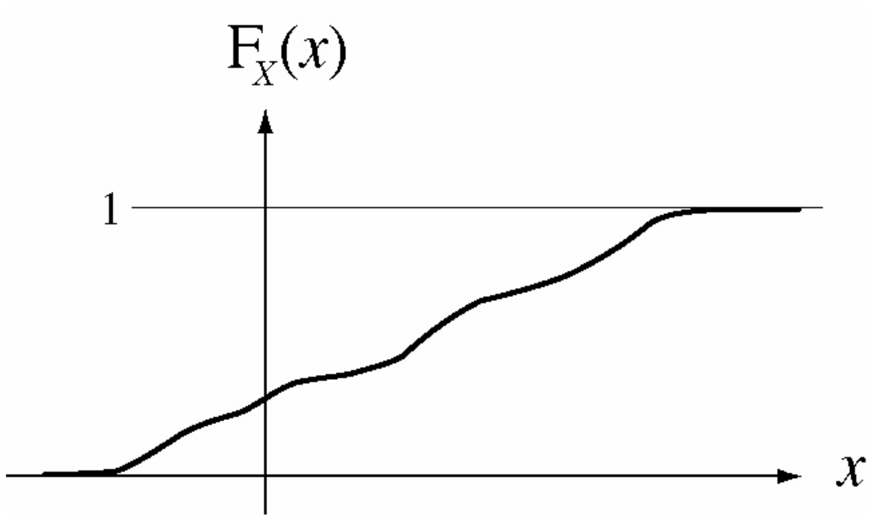
Continuous Random Variables

Cumulative Distribution Functions

A possible CDF for a continuous random variable



Probability Density Functions

The derivative of the CDF is the **probability density function** (**pdf**), $\mathrm{f}_X(x) \equiv \dfrac{d}{dx}\left(\mathrm{F}_X(x)\right)$.

Probability density can also be defined by

$$\mathrm{f}_X(x)\,dx = \mathrm{P}\left[x < X \le x + dx\right]$$

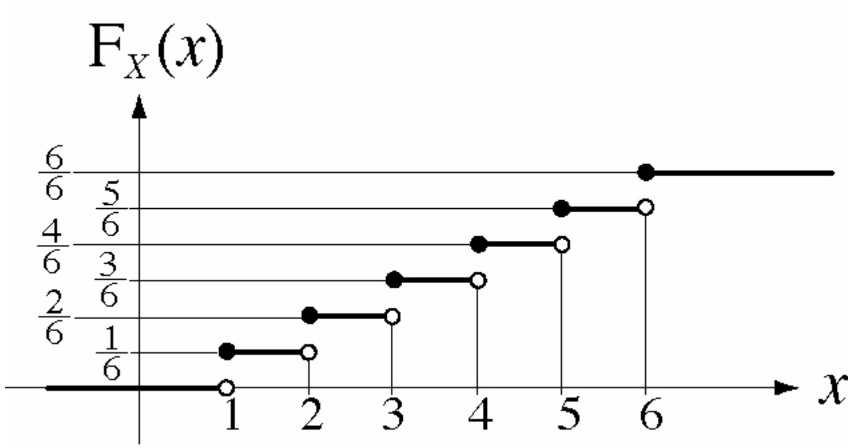
Properties

$$\mathrm{f}_X(x) \ge 0 \quad , \quad -\infty < x < +\infty \qquad\qquad \int_{-\infty}^{\infty} \mathrm{f}_X(x)\,dx = 1$$

$$\mathrm{F}_X(x) = \int_{-\infty}^{x} \mathrm{f}_X(\lambda)\,d\lambda \qquad\qquad \mathrm{P}\left[x_1 < X \le x_2\right] = \int_{x_1}^{x_2} \mathrm{f}_X(x)\,dx$$

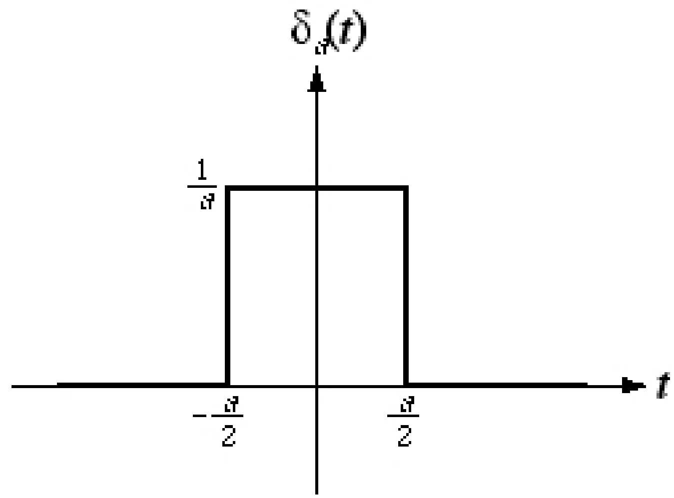
Probability Density Functions

We can also apply the concept of a pdf to a discrete random variable if we allow the use of the **impulse**. Consider the CDF for tossing a die illustrated below. If pdf is the derivative of CDF what does the pdf for tossing a die look like? It is zero everywhere except at the points $x = 1, 2, 3, 4, 5$ or 6. At those points, strictly speaking mathematically, the derivative does not exist.

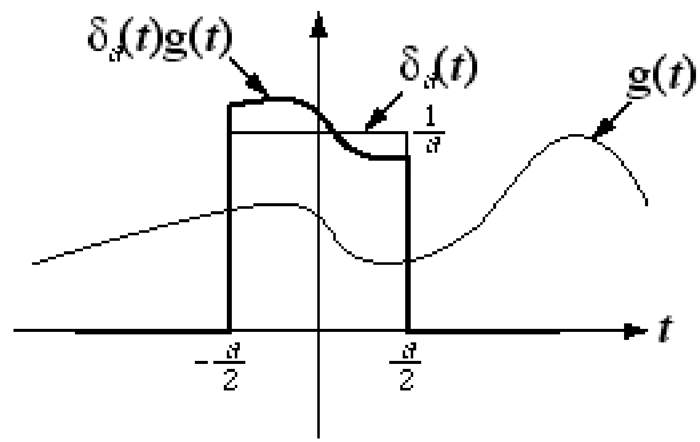


Introduction to the Impulse

Define a function $\delta_a(t) = \begin{cases} 1/a & , \ |t| < a/2 \\ 0 & , \ |t| > a/2 \end{cases}$



Let another function g(t) be finite and continuous at $t = 0$.



Introduction to the Impulse

The area under the product of the two functions is

$$A = \frac{1}{a}\int_{-a/2}^{a/2} \mathrm{g}(t)\,dt$$

As the width of $\delta_a(t)$ approaches zero,

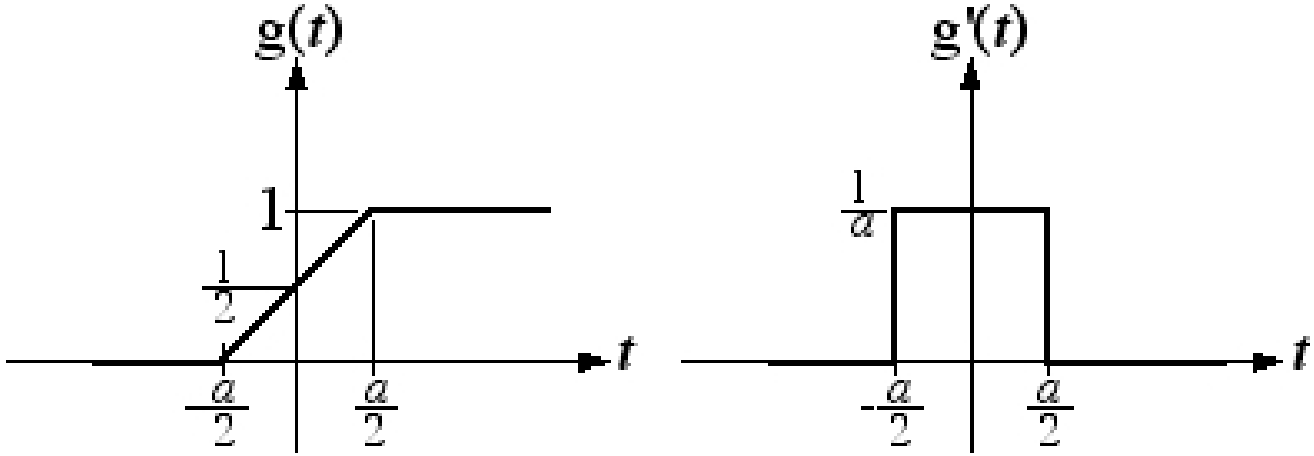
$$\lim_{a\to 0} A = \mathrm{g}(0)\lim_{a\to 0}\frac{1}{a}\int_{-a/2}^{a/2} dt = \mathrm{g}(0)\lim_{a\to 0}\frac{1}{a}(a) = \mathrm{g}(0)$$

The continuous-time unit impulse is implicitly defined by

$$\mathrm{g}(0) = \int_{-\infty}^{\infty} \delta(t)\,\mathrm{g}(t)\,dt$$

The Unit Step and Unit Impulse

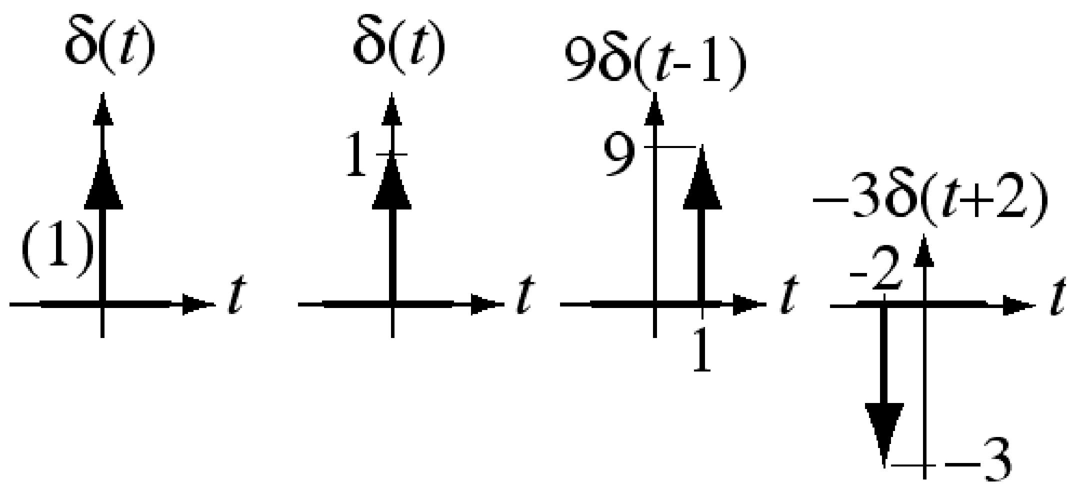
As a approaches zero, $\mathrm{g}(t)$ approaches a unit step and $\mathrm{g}'(t)$ approaches a unit impulse



The unit step is the integral of the unit impulse and the unit impulse is the **generalized derivative** of the unit step

Graphical Representation of the Impulse

The impulse is not a function in the ordinary sense because its value at the time of its occurrence is not defined. It is represented graphically by a vertical arrow. Its strength is either written beside it or is represented by its length.



Properties of the Impulse

The Sampling Property

$$\int_{-\infty}^{\infty} g(t)\delta(t-t_0)\,dt = g(t_0)$$

The sampling property "extracts" the value of a function at a point.

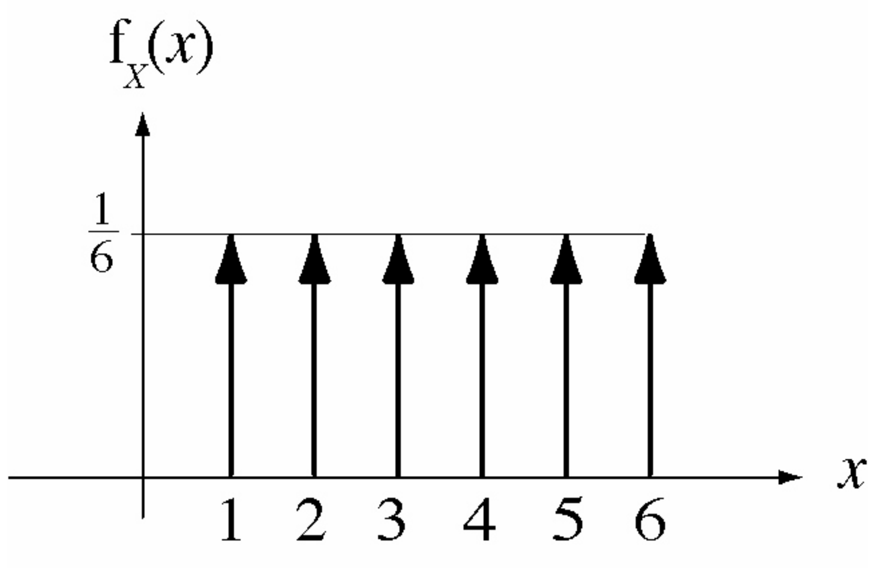
The Scaling Property

$$\delta(a(t-t_0)) = \frac{1}{|a|}\delta(t-t_0)$$

This property illustrates that the impulse is different from ordinary mathematical functions.

Probability Density Functions

Using the concept of the impulse, the pdf for tossing a die consists of 6 impulses, each of strength 1/6.



Probability Density Functions

The pdf's of the important discrete random variable types are

Bernoulli - $f_X(x) = (1-p)\delta(x) + p\delta(x-1)$

Binomial - $f_X(x) = \sum_{k=0}^{n} \binom{n}{k} p^k (1-p)^{n-k} \delta(x-k)$

Geometric - $f_X(x) = \sum_{k=1}^{\infty} p(1-p)^{k-1} \delta(x-k)$

Negative Binomial (Pascal) - $f_Y(y) = \sum_{k=r}^{\infty} \binom{k-1}{r-1} p^r (1-p)^{k-r} \delta(y-k)$

Poisson - $f_X(x) = \sum_{k=0}^{\infty} \frac{\alpha^k}{k!} e^{-\alpha} \delta(x-k)$

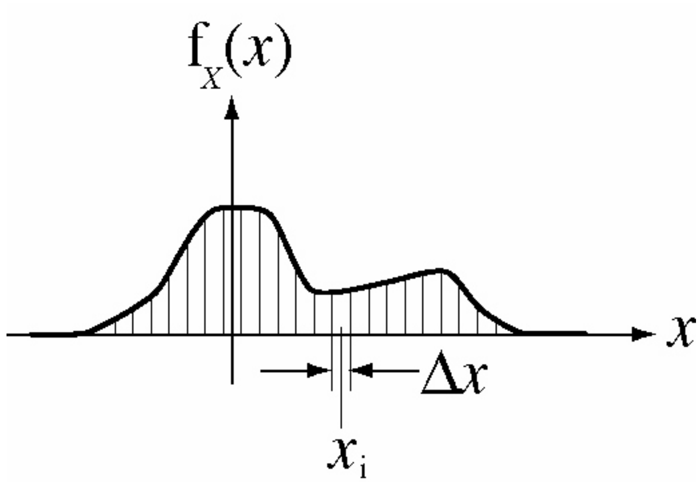
Expectation and Moments

For a discrete random variable, the expected value of X is

$$\mathrm{E}\left(X\right) = \sum_{i=1}^{M} \mathrm{P}\left(X = x_i\right) x_i$$

For a continuous random variable, the probability that X lies within some small range can be approximated by

$$\mathrm{P}\left[x_i - \frac{\Delta x}{2} < X \le x_i + \frac{\Delta x}{2}\right] \cong \mathrm{f}_X\left(x_i\right)\Delta x \ .$$

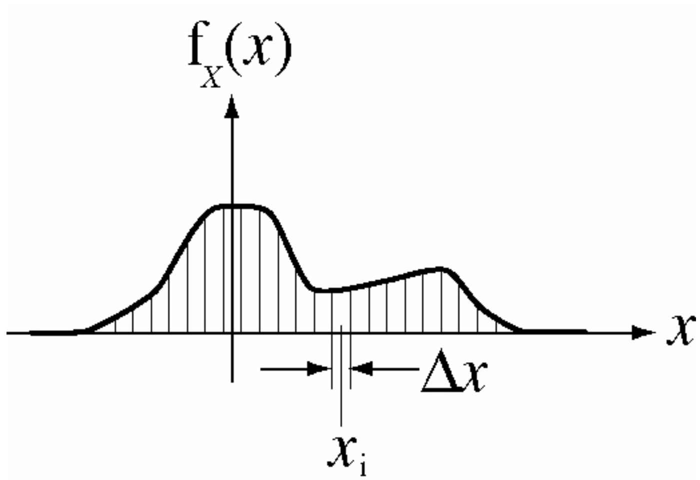


Expectation and Moments

The expected value is then approximated by

$$\mathrm{E}(X) = \sum_{i=1}^{M} x_i \, \mathrm{P}\left[x_i - \frac{\Delta x}{2} < X \le x_i + \frac{\Delta x}{2} \right] \cong \sum_{i=1}^{M} x_i \, \mathrm{f}_X(x_i) \Delta x$$

where M is now the number of subdivisions of width Δx of the range of the random variable.



Expectation and Moments

In the limit as Δx approaches zero,

$$\mathrm{E}(X) = \int_{-\infty}^{\infty} x\,\mathrm{f}_X(x)\,dx$$

Similarly

$$\mathrm{E}(\mathrm{g}(X)) = \int_{-\infty}^{\infty} \mathrm{g}(x)\,\mathrm{f}_X(x)\,dx$$

The *n*th moment of a continuous random variable is

$$\mathrm{E}(X^n) = \int_{-\infty}^{\infty} x^n\,\mathrm{f}_X(x)\,dx$$

Expectation and Moments

The first moment of a continuous random variable is its expected value $\mathrm{E}(X) = \int_{-\infty}^{\infty} x\,\mathrm{f}_X(x)\,dx$.

The second moment of a continuous random variable is its mean-squared value $\mathrm{E}(X^2) = \int_{-\infty}^{\infty} x^2\,\mathrm{f}_X(x)\,dx$.

The variance of a continuous random variable is its second central moment

$$\sigma_X^2 = \mathrm{E}\left(\left[X - \mathrm{E}(X)\right]^2\right) = \int_{-\infty}^{\infty} \left[x - \mathrm{E}(X)\right]^2 \mathrm{f}_X(x)\,dx$$

Expectation and Moments

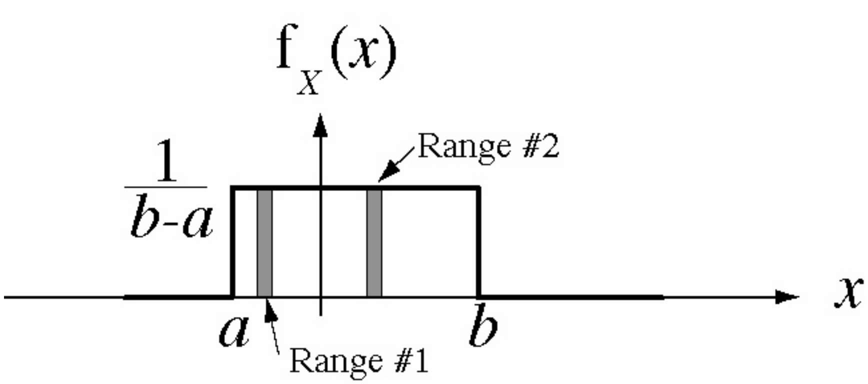
The properties of expectation for discrete random variables also hold true for continuous random variables.

$$\mathrm{E}(a) = a \quad , \quad \mathrm{E}(aX) = a\,\mathrm{E}(X) \quad , \quad \mathrm{E}\left(\sum_n X_n\right) = \sum_n \mathrm{E}(X_n)$$

$$\sigma_X^2 = \mathrm{E}(X^2) - \mathrm{E}^2(X) \quad , \quad \mathrm{Var}(aX + b) = a^2\,\mathrm{Var}(X)$$

Probability Density Functions

The **uniform** pdf



The probabilities of *X* being in Range #1 and *X* being in Range #2 are the same as long as both ranges have the same width and lie entirely between *a* and *b*

Probability Density Functions

The **exponential** pdf

The arrival times of photons in a light beam at a surface are random and have a Poisson distribution. Let the time between photons be T. The mean time between photons is $\bar{T}$. The probability that a photon arrives in any very short length of time Δt located randomly in time is

$$\mathrm{P}\left[\text{photon arrival during } \Delta \mathrm{t}\right] = \frac{\Delta t}{\bar{T}}$$

Let a photon arrive at time t_0. What is the probability that the next photon will arrive within the next t seconds?

Probability Density Functions

The exponential pdf

From one point of view, the probability that a photon arrives within the time range $t_0 + t < T \leq t_0 + t + \Delta t$ is

$$\mathrm{P}\left[t_0 + t < T < t_0 + t + \Delta t\right] = \mathrm{F}_T\left(t + \Delta t\right) - \mathrm{F}_T\left(t\right)$$

This probability is also the product of the probability of a photon arriving in any length of time Δt which is $\Delta t / \overline{T}$ and the probability that no photon arrives before that time which is

$$\mathrm{P}\left[\text{no photon before } t_0 + t\right] = 1 - \mathrm{F}_T\left(t\right)$$

Probability Density Functions

The exponential pdf

$$\mathrm{F}_T\left(t+\Delta t\right)-\mathrm{F}_T\left(t\right)=\left[1-\mathrm{F}_T\left(t\right)\right]\frac{\Delta t}{\bar{T}}$$

Dividing both sides by Δt and letting it approach zero,

$$\lim_{\Delta t\to 0}\frac{\mathrm{F}_T\left(\tau+\Delta\tau\right)-\mathrm{F}_T\left(\tau\right)}{\Delta t}=\frac{d}{dt}\left(\mathrm{F}_T\left(t\right)\right)=\frac{1-\mathrm{F}_T\left(t\right)}{\bar{T}}\quad,\quad t\geq 0$$

Solving the differential equation,

$$\mathrm{F}_T\left(t\right)=\left(1-e^{-t/\bar{T}}\right)\mathrm{u}\left(t\right)\Rightarrow \mathrm{f}_T\left(t\right)=\frac{e^{-t/\bar{T}}}{\bar{T}}\mathrm{u}\left(t\right)$$

Probability Density Functions

The **Erlang** pdf

The Erlang pdf is a generalization of the exponential pdf. It is the probability of the time between one event and the *k*th event later in time.

$$f_{T,k}(t) = \frac{t^{k-1} e^{-t/\overline{T}}}{\overline{T}^{k} (k-1)!} u(t) \quad , \quad k = 1, 2, 3, \ldots$$

Notice that, for $k = 1$,

$$f_{T,1}(t) = \frac{e^{-t/\overline{T}}}{\overline{T}} u(t)$$

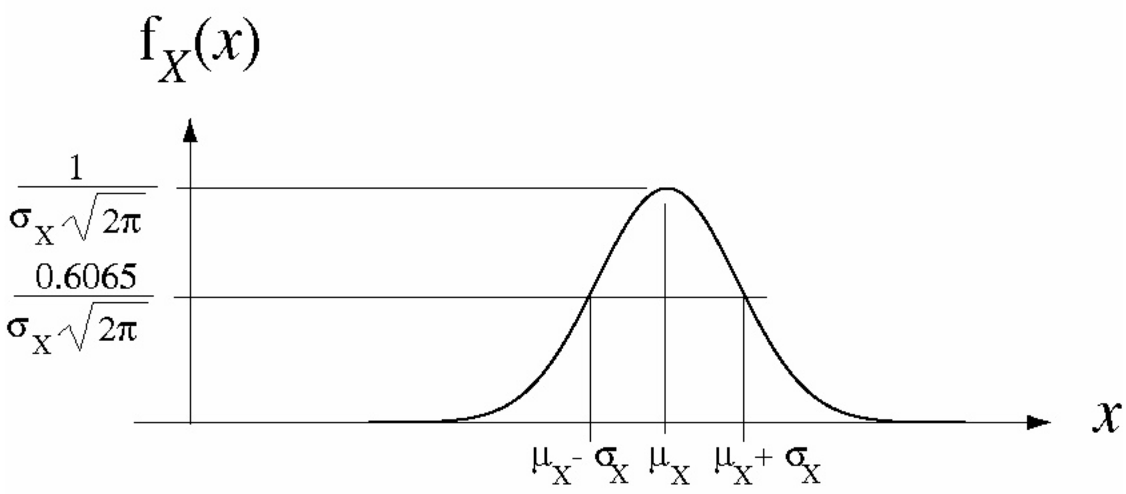
which is the same as the exponential pdf.

Probability Density Functions

The **Gaussian** pdf

$$\mathrm{f}_X(x) = \frac{1}{\sigma_X \sqrt{2\pi}} e^{-(x-\mu_X)^2/2\sigma_X^2}$$

$$\mu_X = \mathrm{E}(X) \text{ and } \sigma_X = \sqrt{\mathrm{E}\left(\left[X - \mathrm{E}(X)\right]^2\right)}$$



Probability Density Functions

The Gaussian pdf

Its maximum value occurs at the mean value of its argument.

It is symmetrical about the mean value.

The points of maximum absolute slope occur at one standard deviation above and below the mean.

Its maximum value is inversely proportional to its standard deviation.

The limit as the standard deviation approaches zero is a unit impulse.

$$\delta\left(x-\mu_x\right)=\lim_{\sigma_X \to 0} \frac{1}{\sigma_X \sqrt{2\pi}} e^{-\left(x-\mu_X\right)^2/2\sigma_X^2}$$

Probability Density Functions

The normal pdf is a Gaussian pdf with a mean of zero and a variance of one.

$$f_X(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}$$

The central moments of the Gaussian pdf are

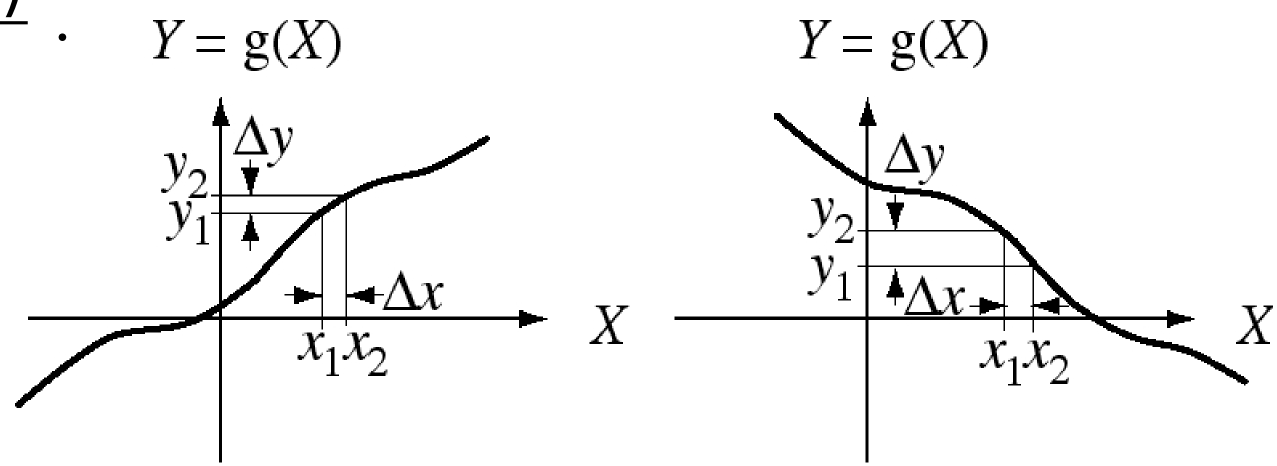
$$E\left(\left[X - E(X)\right]^n\right) = \begin{cases} 0 & , \ n \text{ odd} \\ 1 \cdot 3 \cdot 5 \ldots (n-1)\sigma_X^n & , \ n \text{ even} \end{cases}$$

Functions of a Random Variable

Let X and Y be continuous random variables and let $Y = \mathrm{g}\left(X\right)$.

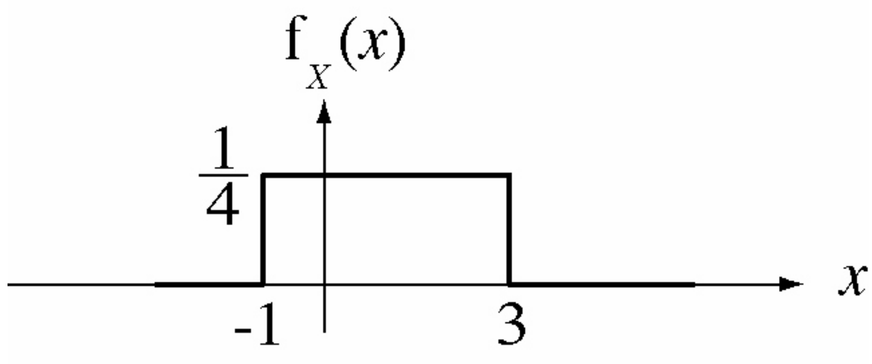
Also, let the function g be invertible, meaning that an inverse function $X = \mathrm{g}^{-1}\left(Y\right)$ exists and is single-valued as in the illustrations below. Then it can be shown that the pdf's of X and Y are related by

$$\mathrm{f}_Y\left(y\right) = \frac{\mathrm{f}_X\left(\mathrm{g}^{-1}\left(y\right)\right)}{\left|dy / dx\right|} .$$



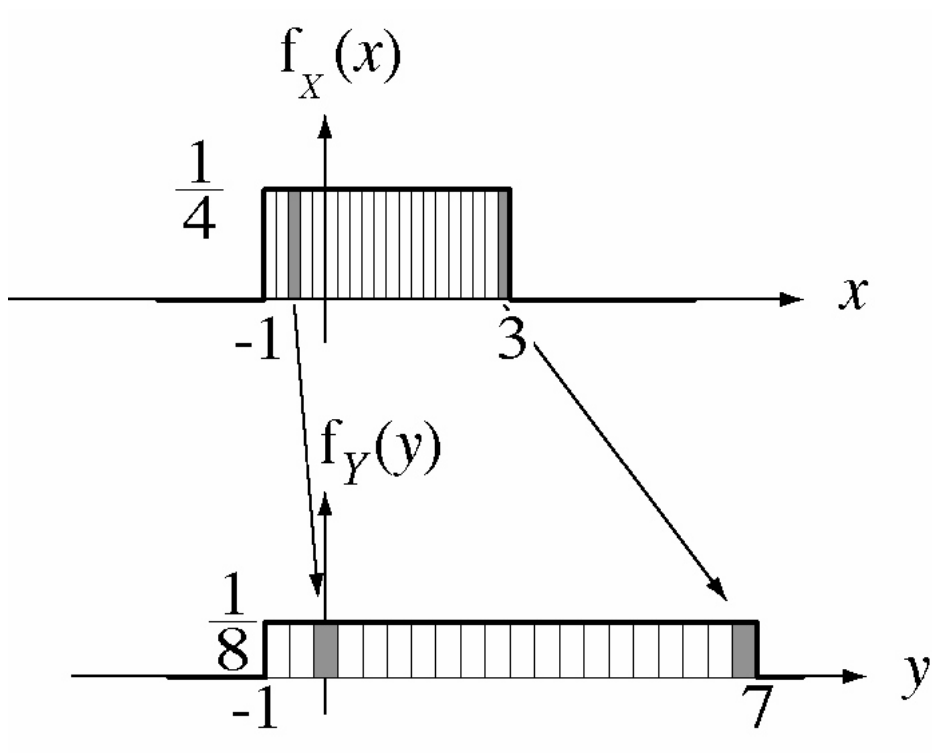
Functions of a Random Variable

Let the pdf of X be $\mathrm{f}_X(x) = (1/4)\mathrm{rect}((x-1)/4)$ and let $Y = 2X + 1$. Then $X = \mathrm{g}^{-1}(Y) = \dfrac{Y-1}{2}$ and $\dfrac{dY}{dX} = 2$.



Functions of a Random Variable

$$f_Y(y) = \frac{f_X((y-1)/2)}{2} = \frac{(1/4)\,\mathrm{rect}\left(\frac{(y-1)/2-1}{4}\right)}{2} = (1/8)\,\mathrm{rect}((y-3)/8)$$

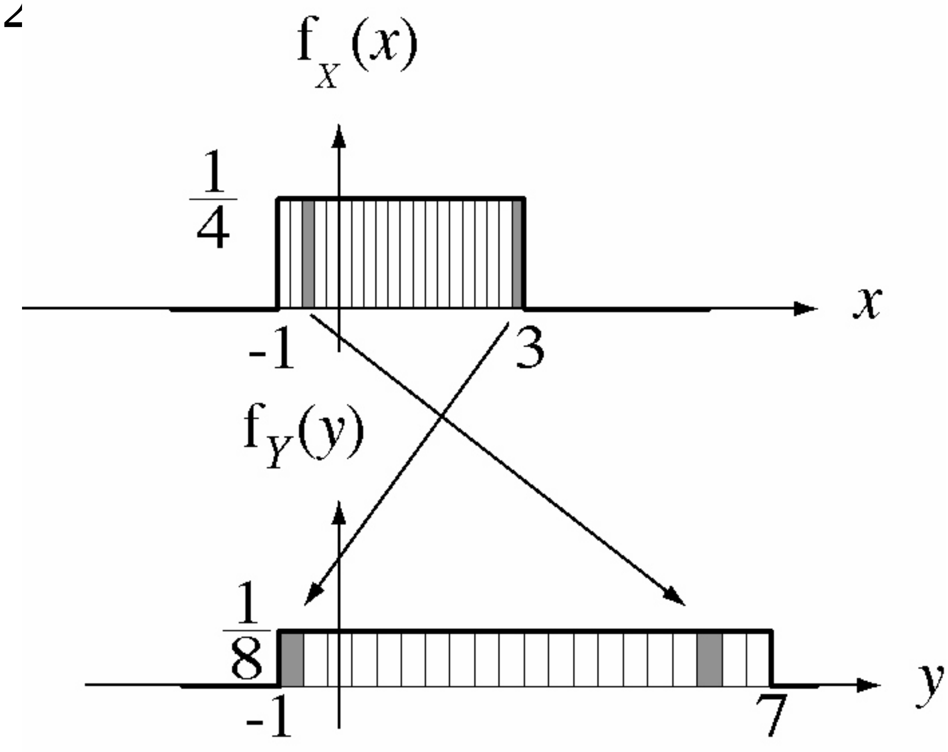


Functions of a Random Variable

Now let

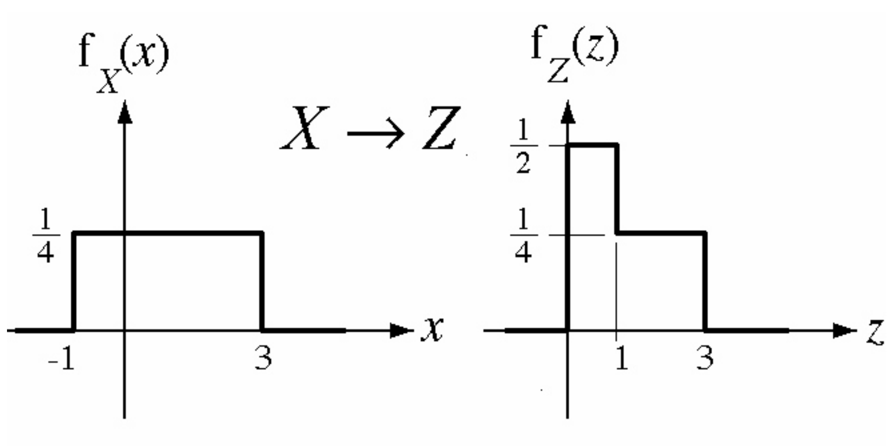
$$Y = -2X + 5 \Rightarrow X = g^{-1}(Y) = \frac{5-Y}{2} \quad , \quad \frac{dY}{dX} = -2 \ .$$

$$f_Y(y) = \frac{f_X\left(\left(5-y\right)/2\right)}{2} = \left(1/8\right)\text{rect}\left(\left(3-y\right)/8\right) = \left(1/8\right)\text{rect}\left(\left(y-3\right)/8\right)$$



Functions of a Random Variable

Now let $Y = g(X) = X^2$. This is more complicated because the event $\{1 < Y \le 4\}$ is caused by the event $\{1 < X \le 2\}$ but it is also caused by the event $\{-2 \le X < -1\}$. If we make the transformation from the pdf of X to the pdf of Y in two steps the process is simpler to see. Let $Z = |X|$.



Functions of a Random Variable

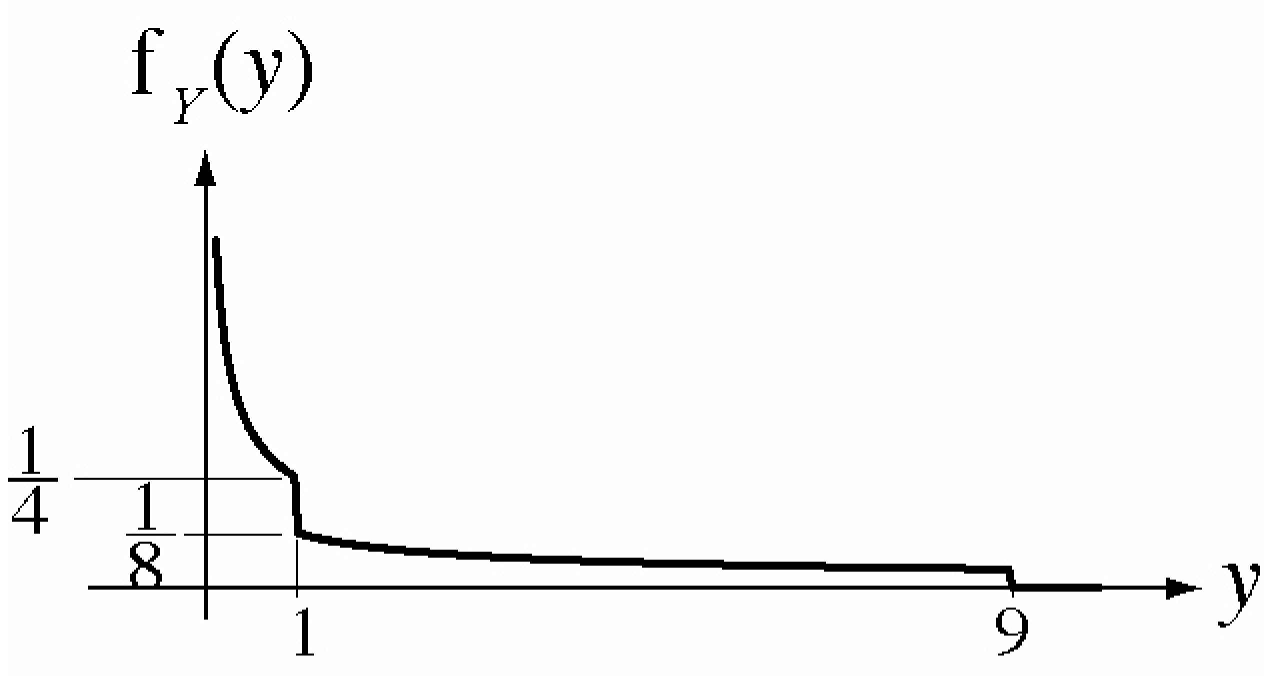
$$f_Z(z) = \frac{1}{4}\text{rect}(z - 1/2) + \frac{1}{4}\text{rect}\left(\frac{z - 3/2}{3}\right)$$

$$Z = \sqrt{Y} \quad , \quad Y \geq 0 \quad , \quad \frac{dY}{dZ} = 2Z = 2\sqrt{Y} \quad , \quad Y \geq 0$$

$$f_Y(y) = \begin{Bmatrix} \dfrac{f_Z\left(g^{-1}(y)\right)}{\left|dy / dz\right|} & , \quad y \geq 0 \\ 0 & , \quad y < 0 \end{Bmatrix} = \frac{\frac{1}{4}\text{rect}\left(\frac{\sqrt{y} - 3/2}{3}\right) + \frac{1}{4}\text{rect}\left(\sqrt{y} - 1/2\right)}{2\sqrt{y}}\text{u}(y)$$

$$f_Y(y) = \frac{1}{8\sqrt{y}}\left[\text{rect}\left(\frac{2\sqrt{y} - 3}{6}\right) + \text{rect}\left(\sqrt{y} - \frac{1}{2}\right)\right]\text{u}(y)$$

Functions of a Random Variable



$$\int_{-\infty}^{\infty} \mathrm{f}_Y(y)\,dy = 1$$

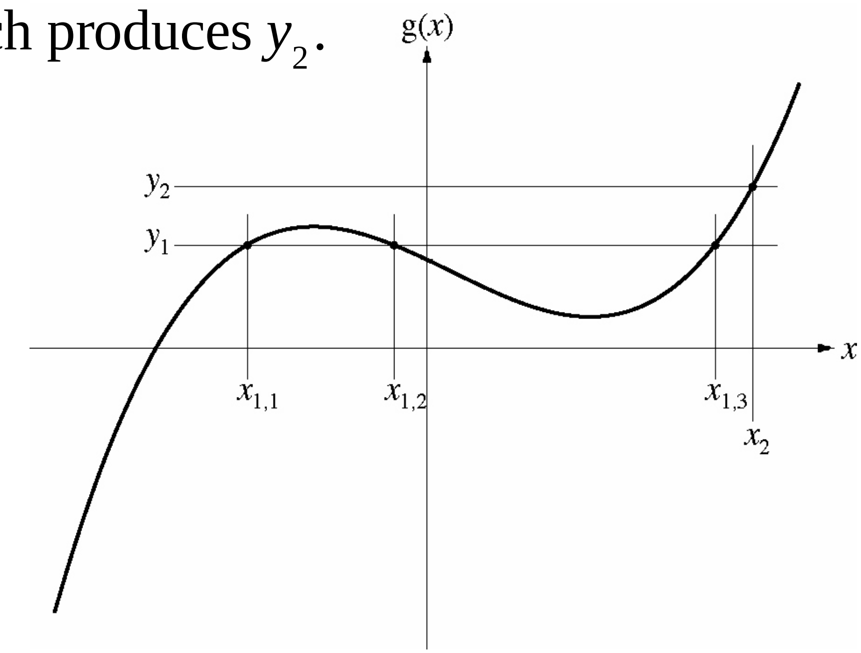
Functions of a Random Variable

In general, if $Y = \mathrm{g}(X)$ and the real solutions of this equation are $x_1, x_2, \cdots x_N$ then, for those ranges of Y for which there is a corresponding X through $Y = \mathrm{g}(X)$ we can find the pdf of Y.

Notice that for some ranges of X and Y there are multiple real solutions and for other ranges there may be fewer.

In this figure there are three real values of x which produce , y_1. But there is only one value of x which produces y_2.

Only the real solutions are used. So in some transformations, the transformation used may depend on the range of values of X and Y being considered.



Functions of a Random Variable

For those ranges of Y for which there is a corresponding real X through $Y = \mathrm{g}(X)$

$$\mathrm{f}_Y(y) = \frac{\mathrm{f}_X(x_1)}{\left|\left(\frac{dY}{dX}\right)_{X=x_1}\right|} + \frac{\mathrm{f}_X(x_2)}{\left|\left(\frac{dY}{dX}\right)_{X=x_2}\right|} + \cdots + \frac{\mathrm{f}_X(x_N)}{\left|\left(\frac{dY}{dX}\right)_{X=x_N}\right|}$$

In the previous example $Y = \mathrm{g}(X) = X^2$ and $x_{1,2} = \pm\sqrt{y}$.

$$\mathrm{f}_Y(y) = \begin{Bmatrix} \dfrac{\mathrm{f}_X(\sqrt{y})}{2\sqrt{y}} + \dfrac{\mathrm{f}_X(-\sqrt{y})}{2\sqrt{y}} & , y \ge 0 \\ 0 & , y < 0 \end{Bmatrix} = \begin{cases} \dfrac{\mathrm{rect}\left(\dfrac{\sqrt{y}-1}{4}\right) + \mathrm{rect}\left(\dfrac{-\sqrt{y}-1}{4}\right)}{8\sqrt{y}} & , y \ge 0 \\ 0 & , y < 0 \end{cases}$$

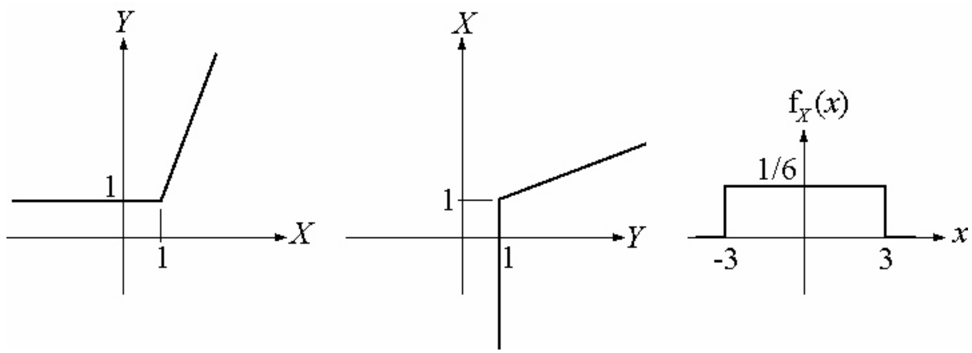
Functions of a Random Variable

One problem that arises when transforming continuous random variables occurs when the derivative is zero. This occurs in any type of transformation for which Y is constant for a non-zero range of X. Since division by zero is undefined, the formula

$$f_Y(y) = \frac{f_X(x_1)}{\left|(dy/dx)_{x=x_1}\right|} + \frac{f_X(x_2)}{\left|(dy/dx)_{x=x_2}\right|} + \cdots + \frac{f_X(x_N)}{\left|(dy/dx)_{x=x_N}\right|}$$

is not usable in the range of X in which Y is a constant. In these cases it is better to utilize a more fundamental relation between X and Y.

Functions of a Random Variable



Let $Y = \begin{cases} 3X - 2 & , \ X \geq 1 \\ 1 & , \ X < 1 \end{cases}$

and $f_X(x) = (1/6)\text{rect}(x/6)$.

We can say $P[Y = 1] = P[X < 1] = 2/3$. So Y has a probability of 2/3 of being exactly one. That means that there must be an impulse in $f_Y(y)$ at $y = 1$ and the strength of the impulse is 2/3. In the remaining range of Y we can use

$$f_Y(y) = \frac{f_X(x_1)}{\left|(dy/dx)_{x=x_1}\right|} + \frac{f_X(x_2)}{\left|(dy/dx)_{x=x_2}\right|} + \cdots + \frac{f_X(x_N)}{\left|(dy/dx)_{x=x_N}\right|}$$

Functions of a Random Variable

For this example, the pdf of Y would be

$$f_Y(y) = (2/3)\delta(y-1) + (1/18)\text{rect}((y+2)/18)\text{u}(y-1)$$

or, simplifying,

$$f_Y(y) = (2/3)\delta(y-1) + (1/18)\text{rect}((y-4)/6)$$

Conditional Probability

Distribution Function

$$\mathrm{F}_{X|A}(x) = \mathrm{P}\left[X \le x \mid A\right] = \frac{\mathrm{P}\left[\left(X \le x\right) \cap A\right]}{\mathrm{P}\left[A\right]}$$

This is the distribution function for x given that the condition A also occurs.

$$0 \le \mathrm{F}_{X|A}(x) \le 1 \quad , \quad -\infty < x < \infty$$

$$\mathrm{F}_{X|A}(-\infty) = 0 \quad \text{and} \quad \mathrm{F}_{X|A}(+\infty) = 1$$

$$\mathrm{P}\left[x_1 < X \le x_2 \mid A\right] = \mathrm{F}_{X|A}(x_2) - \mathrm{F}_{X|A}(x_1)$$

$\mathrm{F}_{X|A}(x)$ is a monotonic function of x.

Conditional Probability

Let the condition A be $A = \{X \le a\}$ where a is a constant.

$$\mathrm{F}_{X|A}(x) = \mathrm{P}[X \le x \mid X \le a] = \frac{\mathrm{P}[(X \le x) \cap (X \le a)]}{\mathrm{P}[X \le a]}$$

If $a \le x$ then $\mathrm{P}[(X \le x) \cap (X \le a)] = \mathrm{P}[X \le a]$ and

$$\mathrm{F}_{X|A}(x) = \mathrm{P}[(X \le x) \cap (X \le a)] = \frac{\mathrm{P}[X \le a]}{\mathrm{P}[X \le a]} = 1$$

If $a \ge x$ then $\mathrm{P}[(X \le x) \cap (X \le a)] = \mathrm{P}[X \le x]$ and

$$\mathrm{F}_{X|A}(x) = \mathrm{P}[(X \le x) \cap (X \le a)] = \frac{\mathrm{P}[X \le x]}{\mathrm{P}[X \le a]} = \frac{\mathrm{F}_X(x)}{\mathrm{F}_X(a)}$$

Conditional Probability

Conditional pdf

$$\mathrm{f}_{X|A}(x) = \frac{d}{dx}\left(\mathrm{F}_{X|A}(x)\right)$$

Conditional expected value of a function

$$\mathrm{E}\left(\mathrm{g}(X) \mid A\right) = \int_{-\infty}^{\infty} \mathrm{g}(x)\,\mathrm{f}_{X|A}(x)\,dx$$

Conditional mean

$$\mathrm{E}(X \mid A) = \int_{-\infty}^{\infty} x\,\mathrm{f}_{X|A}(x)\,dx$$