Stochastic Processes

Definition

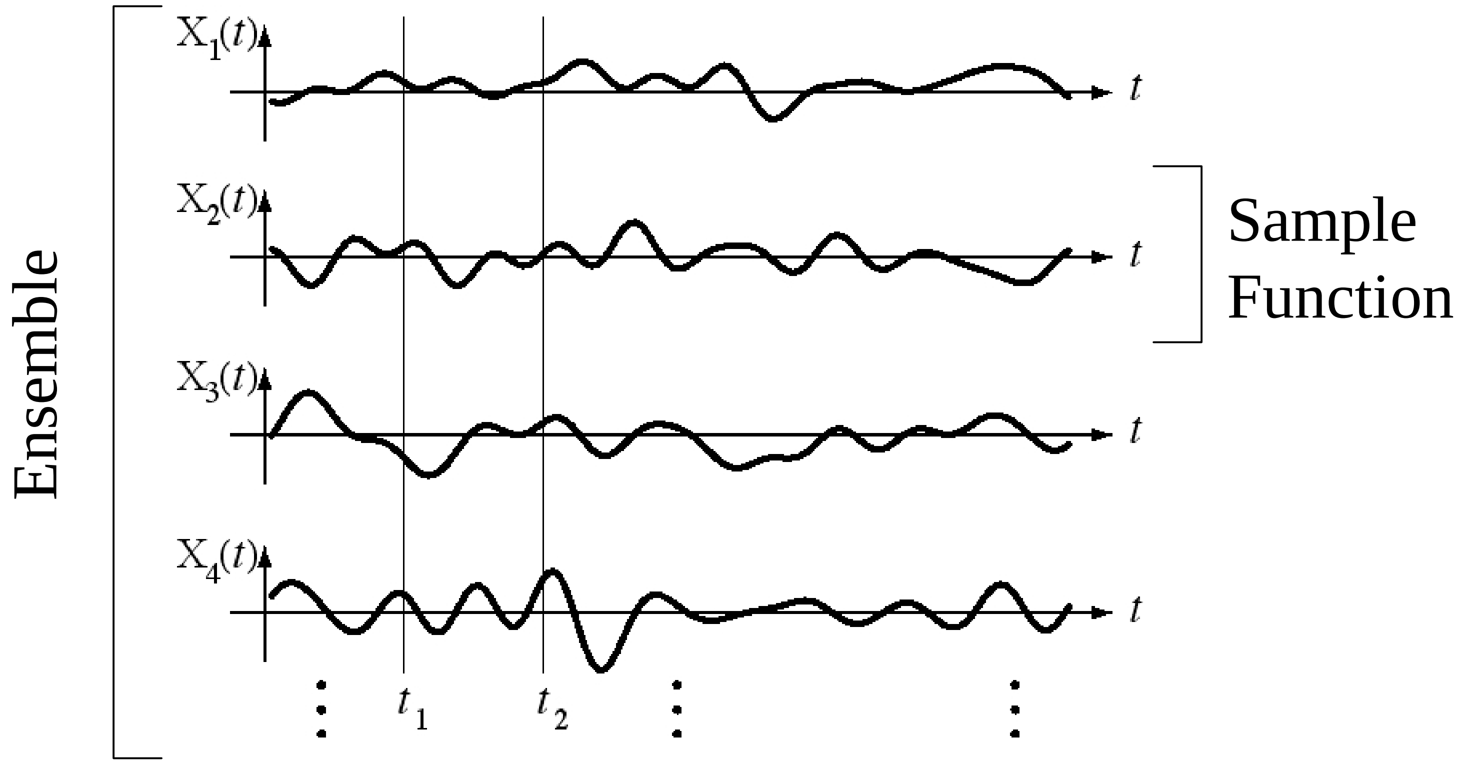
A random variable is a number $X(\zeta)$ assigned to every outcome of an experiment.

A **stochastic process** is the assignment of a function of t $X(t,\zeta)$ to each outcome of an experiment.

The set of functions $\{X(t,\zeta_1), X(t,\zeta_2), \cdots, X(t,\zeta_N)\}$ corresponding to the *N* outcomes of an experiment is called an **ensemble** and each member $X(t,\zeta_i)$ is called a **sample function** of the stochastic process.

A common convention in the notation describing stochastic processes is to write the sample functions as functions of *t* only and to indicate the stochastic process by $X(t)$ instead of $X(t,\zeta)$ and any particular sample function by $X_i(t)$ instead of $X(t,\zeta_i)$.

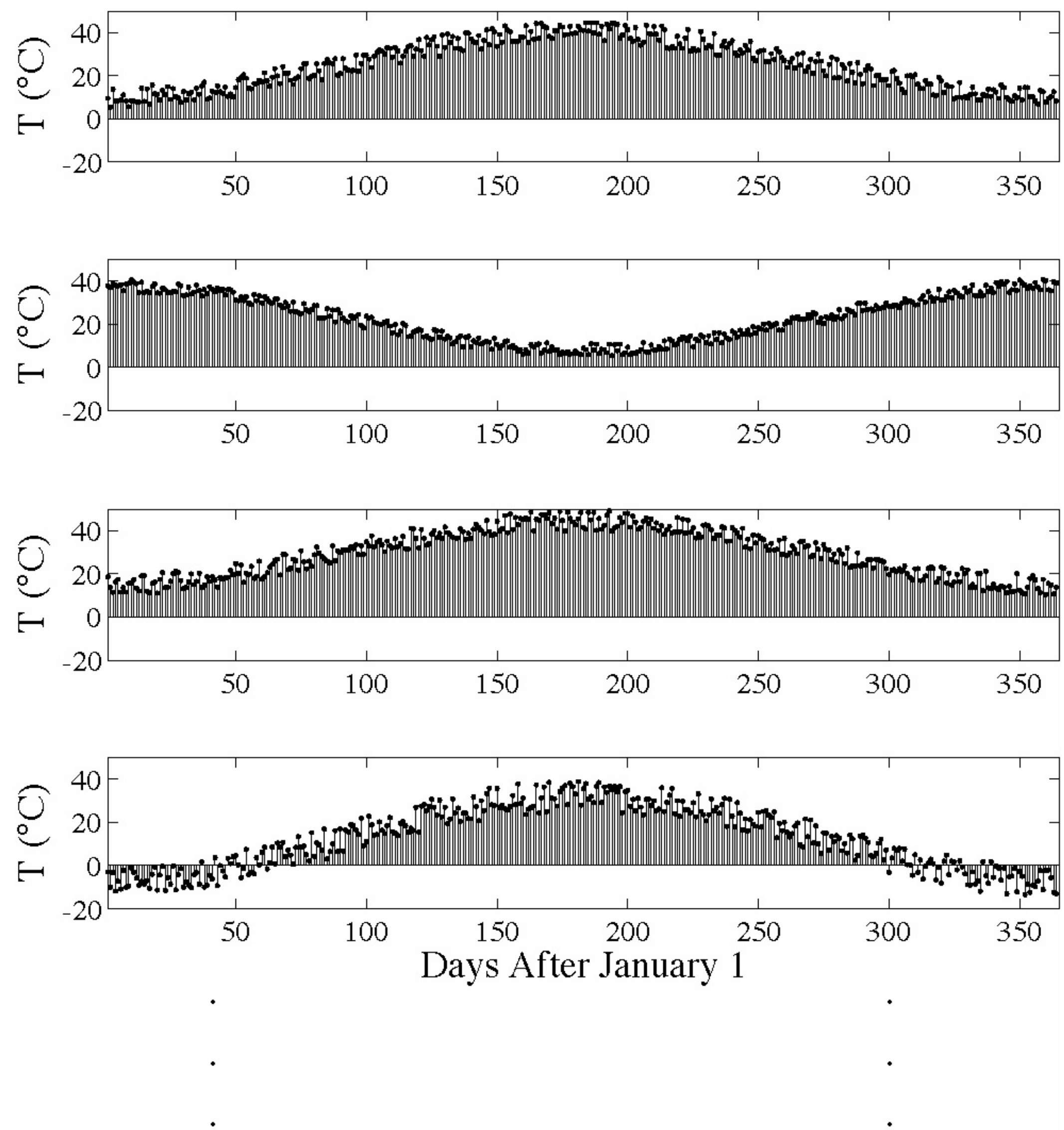
Definition



The values of $X(t)$ at a particular time t_1 define a random variable $X(t_1)$ or just X_1.

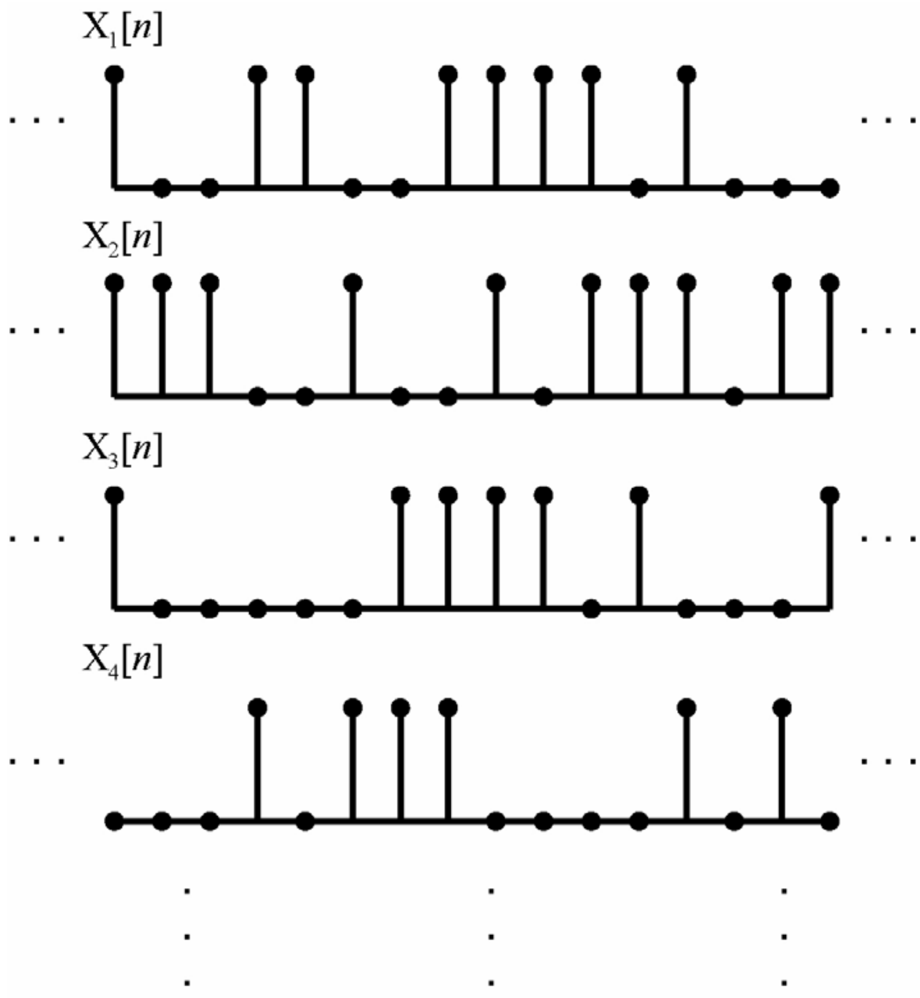
Example of a Stochastic Process

Suppose we place a temperature sensor at every airport control tower in the world and record the temperature at noon every day for a year. Then we have a **discrete-time, continuous-value** (**DTCV**) stochastic process.



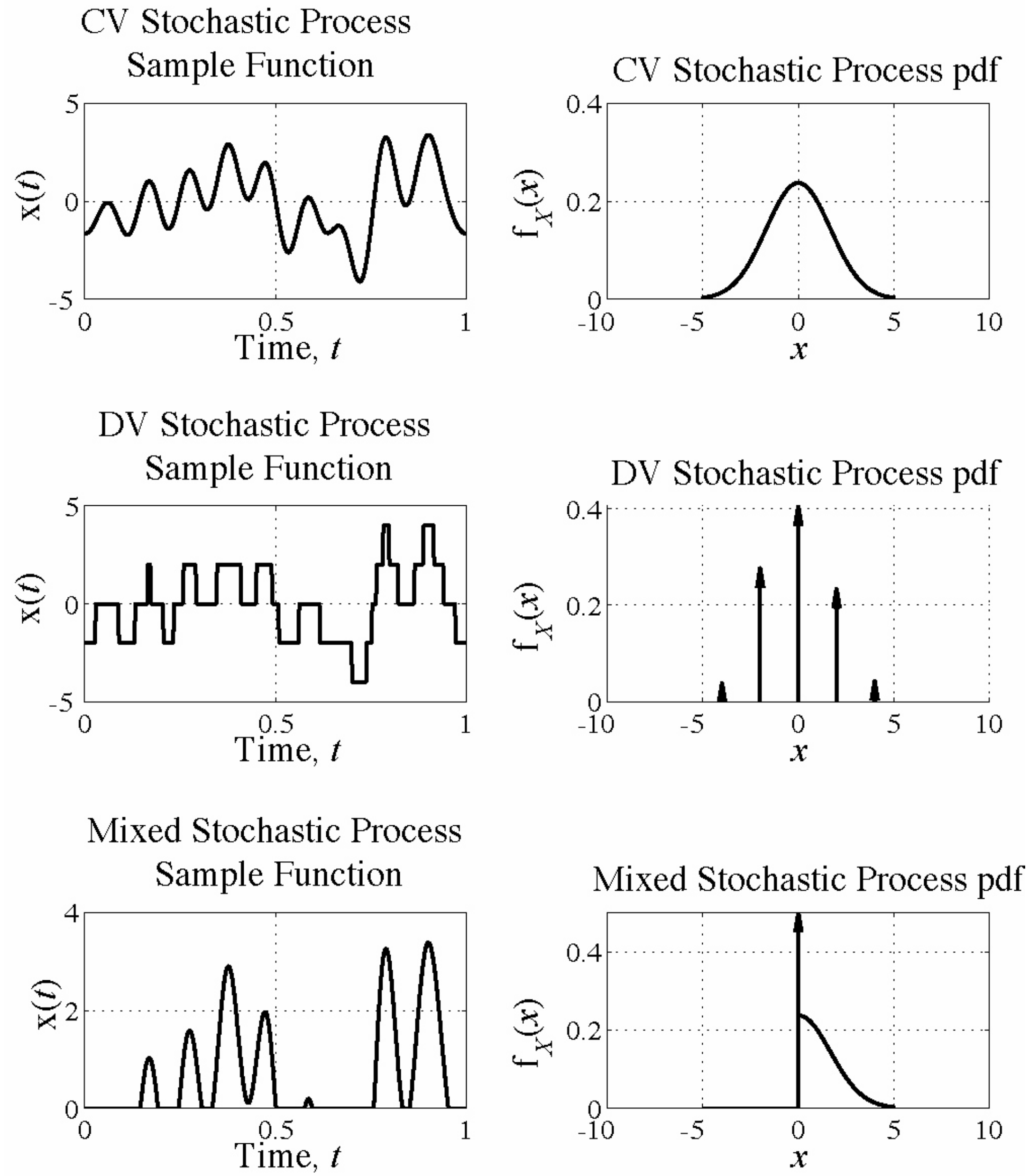
Example of a Stochastic Process

Suppose there is a large number of people, each flipping a fair coin every minute. If we assign the value 1 to a head and the value 0 to a tail we have a **discrete-time, discrete-value** (**DTDV**) stochastic process



Continuous-Value vs. Discrete-Value

A continuous-value (CV) random process has a pdf with no impulses. A discrete-value (DV) random process has a pdf consisting only of impulses. A mixed random process has a pdf with impulses, but not just impulses.



Deterministic vs. Non-Deterministic

A random process is **deterministic** if a sample function can be described by a mathematical function such that its future values can be computed. The randomness is in the ensemble, not in the time functions. For example, let the sample functions be of the form,

$$\mathrm{X}(t) = A\cos(2\pi f_0 t + \theta)$$

and let the parameter θ be random over the ensemble but constant for any particular sample function.

All other random processes are **non-deterministic**.

Stationarity

If all the multivariate statistical descriptors of a random process are not functions of time, the random process is said to be **strict-sense stationary** (**SSS**).

A random process is **wide-sense stationary** (**WSS**) if

$\mathrm{E}\left(\mathrm{X}\left(t_1\right)\right)$ is independent of the choice of t_1

and

$\mathrm{E}\left(\mathrm{X}\left(t_1\right)\mathrm{X}\left(t_2\right)\right)$ depends only on the difference between t_1 and t_2

Ergodicity

If all of the sample functions of a random process have the same statistical properties the random process is said to be **ergodic**. The most important consequence of ergodicity is that ensemble moments can be replaced by time moments.

$$\mathrm{E}\left(X^n\right) = \lim_{T\to\infty}\frac{1}{T}\int_{-T/2}^{T/2} X^n\left(t\right)dt$$

Every ergodic random process is also stationary.

Measurement of Process Parameters

The mean value of an ergodic random process can be estimated by

$$\overline{X} = \frac{1}{T}\int_0^T \mathrm{X}(t)\,dt$$

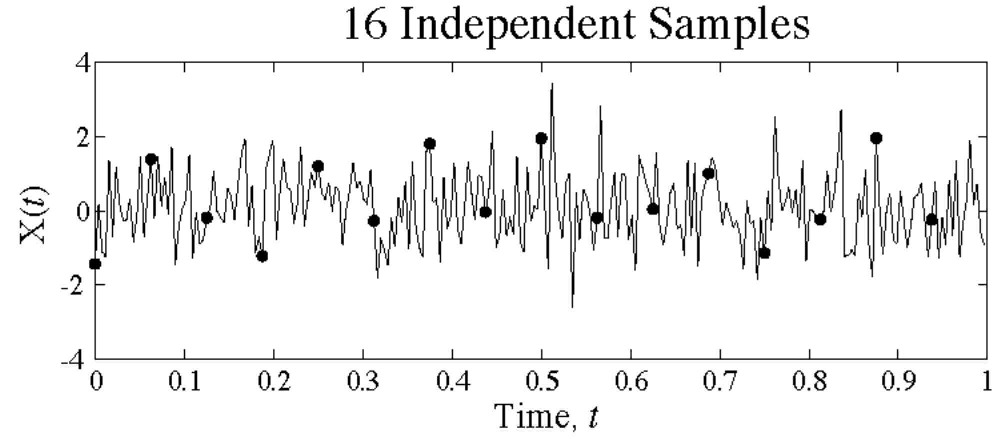
where $\mathrm{X}(t)$ is a sample function of that random process. In practical situations, this function is usually not known. Instead samples from it are known. Then the estimate of the mean value would be

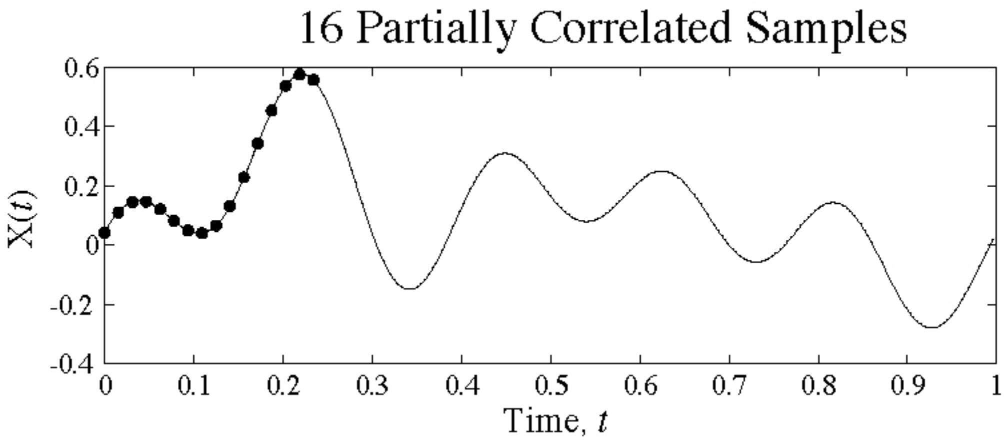
$$\overline{X} = \frac{1}{N}\sum_{i=1}^{N} X_i$$

where X_i is a sample from $\mathrm{X}(t)$.

Measurement of Process Parameters

To make a good estimate, the samples from the random process should be independent.





Correlation Functions

The Correlation Function

If X(t) is a sample function of one stochastic CT process and Y(t) is a sample function from another stochastic CT process and

$$X_1 = \mathrm{X}(t_1) \text{ and } Y_2 = \mathrm{Y}(t_2)$$

then

$$\mathrm{R}_{XY}(t_1, t_2) = \mathrm{E}\left(X_1 Y_2^*\right) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} X_1 Y_2^* \, \mathrm{f}_{XY}(x_1, y_2; t_1, t_2)\, dx_1 dy_2$$

is the correlation function relating X and Y. For stationary stochastic continuous-time processes this can be simplified to

$$\mathrm{R}_{XY}(\tau) = \mathrm{E}\left(\mathrm{X}(t)\mathrm{Y}^*(t+\tau)\right)$$

If the stochastic process is also ergodic then the *time* correlation function is

$$\mathrm{R}_{XY}(\tau) = \lim_{T\to\infty}\frac{1}{T}\int_{-T/2}^{T/2} \mathrm{X}(t)\mathrm{Y}^*(t+\tau)\,dt = \left\langle \mathrm{X}(t)\mathrm{Y}^*(t+\tau)\right\rangle = \mathrm{R}_{XY}(\tau)$$

Autocorrelation

If X and Y represent the same stochastic CT process then the correlation function becomes the special case called **autocorrelation**.

$$\mathrm{R}_X(\tau) = \mathrm{E}\left[\mathrm{X}(t)\mathrm{X}^*(t+\tau)\right]$$

For an ergodic stochastic process,

$$\mathrm{R}_X(\tau) = \lim_{T\to\infty}\frac{1}{T}\int_{-T/2}^{T/2}\mathrm{X}(t)\mathrm{X}^*(t+\tau)\,dt = \left\langle \mathrm{X}(t)\mathrm{X}^*(t+\tau)\right\rangle = \mathrm{R}_X(\tau)$$

Autocorrelation

$$\mathrm{R}_X(t,t) = \underbrace{\mathrm{E}\left(\mathrm{X}^2(t)\right)}_{\text{Mean-squared value of X}}$$

For WSS stochastic continuous-time processes

$$\mathrm{R}_X(0) = \underbrace{\mathrm{E}\left(\mathrm{X}^2(t)\right)}_{\text{Mean-squared value of X}} \text{ and } \mathrm{R}_X(0) = \lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{T/2} \mathrm{X}^2(t)\,dt = \underbrace{\left\langle \mathrm{X}^2(t) \right\rangle}_{\text{Average Signal Power of X}}$$

The Correlation Function

If $\mathrm{X}[n]$ is a sample function of one stochastic DT process and $\mathrm{Y}[n]$ is a sample function from another stochastic DT process and

$$X_1 = \mathrm{X}[n_1] \text{ and } Y_2 = \mathrm{Y}[n_2]$$

then

$$\mathrm{R}_{XY}[n_1, n_2] = \mathrm{E}\left(X_1 Y_2^*\right) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} X_1 Y_2^* \, \mathrm{f}_{XY}\left(x_1, y_2; n_1, n_2\right) dx_1 dy_2$$

is the correlation function relating *X* and *Y*. For stationary stochastic DT processes this can be simplified to

$$\mathrm{R}_{XY}[m] = \mathrm{E}\left(\mathrm{X}[n]\mathrm{Y}^*[n+m]\right)$$

If the stochastic DT process is also ergodic then the *time* correlation function is

$$\mathrm{R}_{XY}[m] = \lim_{N \to \infty} \frac{1}{2N} \sum_{n=-N}^{N-1} \mathrm{X}[n]\mathrm{Y}^*[n+m] = \left\langle \mathrm{X}[n]\mathrm{Y}^*[n+m] \right\rangle = \mathrm{R}_{XY}[m]$$

Autocorrelation

If X and Y represent the same stochastic DT process then the correlation function becomes the special case called **autocorrelation**.

$$\mathrm{R}_X[m] = \mathrm{E}\left(\mathrm{X}[n]\mathrm{X}^*[n+m]\right)$$

For an ergodic stochastic DT process,

$$\mathrm{R}_X[m] = \lim_{N\to\infty} \frac{1}{2N} \sum_{n=-N}^{N-1} \mathrm{X}[n]\mathrm{X}^*[n+m] = \left\langle \mathrm{X}[n]\mathrm{X}^*[n+m] \right\rangle = \mathrm{R}_X[m]$$

Autocorrelation

$$\mathrm{R}_X[n,n] = \underbrace{\mathrm{E}\left(\mathrm{X}^2[n]\right)}_{\text{Mean-squared value of X}}$$

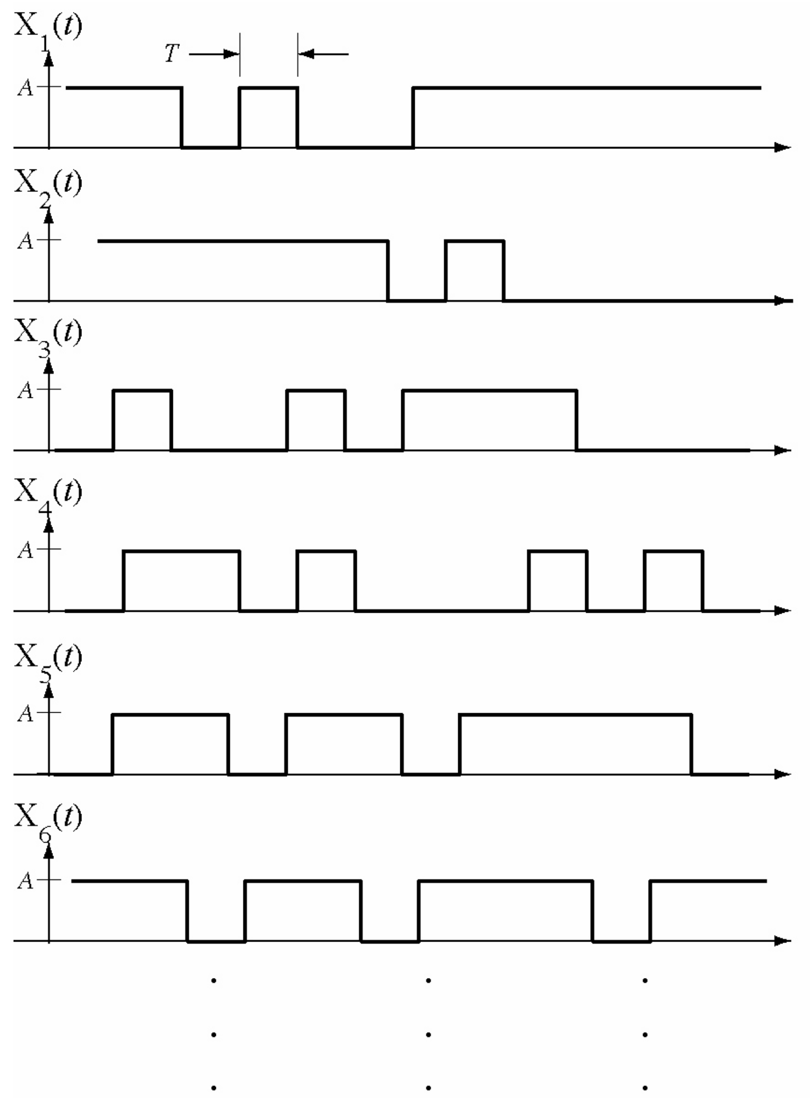
For WSS stochastic discrete-time processes

$$\mathrm{R}_X[0] = \underbrace{\mathrm{E}\left(\mathrm{X}^2[n]\right)}_{\text{Mean-squared value of X}} \quad \text{and} \quad \mathrm{R}_X[0] = \lim_{N\to\infty}\frac{1}{2N}\sum_{n=-N}^{N-1}\mathrm{X}^2[n] = \underbrace{\left\langle \mathrm{X}^2[n]\right\rangle}_{\text{Average Signal Power of X}}$$

Autocorrelation

Example

Let $X(t)$ be ergodic and let each sample function be a sequence of pulses of width T whose amplitudes are A and zero with equal probability and with uniformly-distributed pulse positions. Let the pulse amplitudes be independent of each other. Find the autocorrelation.



Autocorrelation

Example

$$\mathrm{R}_X(\tau) = \mathrm{E}\left(\mathrm{X}(t)\mathrm{X}^*(t+\tau)\right) = \mathrm{E}\left(\mathrm{X}(t)\mathrm{X}(t+\tau)\right)$$

For delays τ whose magnitude is greater than the pulse width, the pulse amplitudes are uncorrelated and therefore

$$\mathrm{R}_X(\tau) = \mathrm{E}\left(\mathrm{X}(t)\right)\mathrm{E}\left(\mathrm{X}(t+\tau)\right) = \left(A/2\right)^2 = A^2/4 \quad , \quad |\tau| > T$$

Autocorrelation

For delays τ whose magnitude is less than the pulse width it is possible that the two times, separated by τ, are in the same pulse. When τ is zero that probability is one and as τ approaches the pulse width the probability approaches zero, linearly.

$$\mathrm{P}\left(t \text{ and } t+\tau \text{ are in the same pulse interval}\right) = \frac{T-|\tau|}{T} \quad , \quad |\tau| < t_a$$

When the two times are in the same pulse there are two possible cases of $\mathrm{X}(t)\mathrm{X}(t+\tau)$ $A \times A$ and 0×0 both with probability 1/2.

$$\mathrm{E}\left(\mathrm{X}(t)\mathrm{X}(t+\tau)\right) = A^2\,\mathrm{P}(A) + 0^2\,\mathrm{P}(0) = A^2/2$$

When the two times are not in the same pulse there are four possible cases of $\mathrm{X}(t)\mathrm{X}(t+\tau)$ $A \times A$, 0×0, 0×0, 0×0 each with probability 1/4.

$$\mathrm{E}\left(\mathrm{X}(t)\mathrm{X}(t+\tau)\right) = A^2\,\mathrm{P}(A)\mathrm{P}(A) + 0^2\,\mathrm{P}(A)\mathrm{P}(0) + 0^2\,\mathrm{P}(0)\mathrm{P}(A) + 0^2\,\mathrm{P}(0)\mathrm{P}(0) = A^2/4$$

Autocorrelation

For delays of magnitude less than the pulse width,

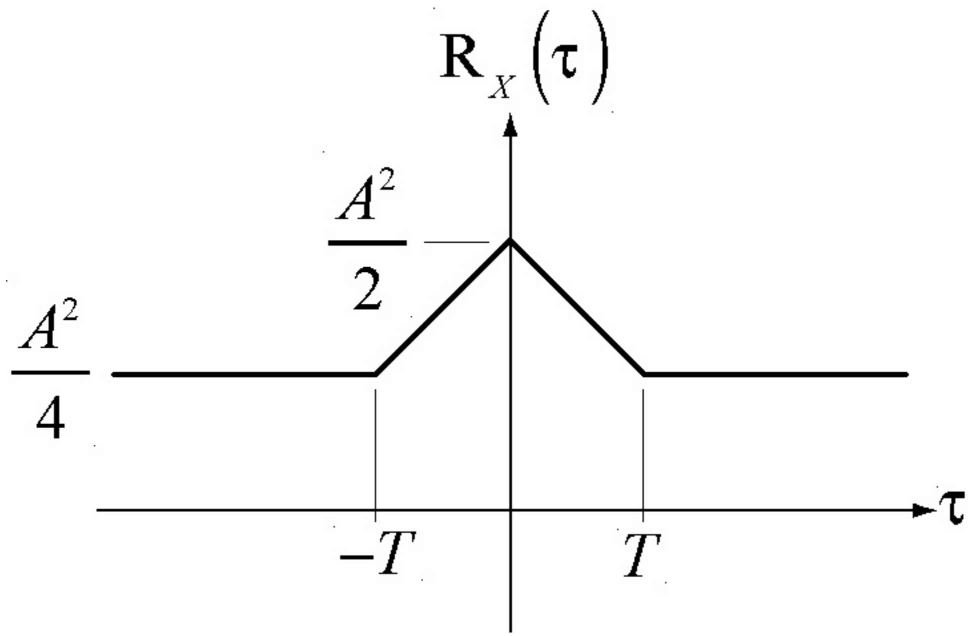
$$\mathrm{R}_X(\tau) = \mathrm{P}\begin{pmatrix}\text{times are in}\\ \text{same pulse}\end{pmatrix} \times A^2/2 + \mathrm{P}\begin{pmatrix}\text{times are not}\\ \text{in same pulse}\end{pmatrix} \times A^2/4$$

$$= \frac{A^2}{2}\frac{T-|\tau|}{T} + \frac{A^2}{4}\left(1 - \frac{T-|\tau|}{T}\right) = \frac{A^2}{4}\left(\frac{2T-|\tau|}{T}\right)$$

and the overall autocorrelation function is

$$\mathrm{R}_X(\tau) = \frac{A^2}{4}\begin{cases}\dfrac{2T-|\tau|}{T} & , \ |\tau| \le T \\ 1 & , \ |\tau| > T\end{cases}$$

$$= \frac{A^2}{4}\left[\mathrm{tri}\left(\frac{\tau - T}{T}\right) + 1\right]$$



Properties of Autocorrelation

Autocorrelation is an even function

$$R_X(\tau) = R_X(-\tau) \text{ or } R_X[m] = R_X[-m]$$

The magnitude of the autocorrelation value is never greater than at zero delay.

$$|R_X(\tau)| \leq R_X(0) \text{ or } |R_X[m]| \leq R_X[0]$$

If X has a non-zero expected value then $R_X(\tau)$ or $R_X[m]$ will also and it will be the square of the expected value of X.

If X has a periodic component then $R_X(\tau)$ or $R_X[m]$ will also, with the same period.

Properties of Autocorrelation

If $\{X(t)\}$ is ergodic with zero mean and no periodic components then

$$\lim_{|\tau|\to\infty} R_X(\tau) = 0 \text{ or } \lim_{|m|\to\infty} R_X[m] = 0$$

Only autocorrelation functions for which

$$\mathcal{F}\{R_X(\tau)\} \geq 0 \text{ for all } f \text{ or } \mathcal{F}\{R_X[m]\} \geq 0 \text{ for all } \Omega$$

are possible

A time shift of a function does not affect its autocorrelation

Autocovariance

Autocovariance is similar to autocorrelation. Autocovariance is the autocorrelation of the time-varying part of a signal.

$$\mathrm{C}_X(\tau) = \mathrm{R}_X(\tau) - \mathrm{E}^2(X) \text{ or } \mathrm{C}_X[m] = \mathrm{R}_X[m] - \mathrm{E}^2(X)$$

Measurement of Autocorrelation

Autocorrelation of a real-valued WSS stochastic CT process can be estimated by

$$\hat{\mathrm{R}}_X(\tau) = \frac{1}{T-\tau}\int_0^{T-\tau} \mathrm{X}(t)\mathrm{X}(t+\tau)\,dt \ \ , \ \ 0 \le \tau << T$$

But because the function $\mathrm{X}(t)$ is usually unknown it is much more likely to be estimated from samples.

$$\hat{\mathrm{R}}_x(nT_s) = \frac{1}{N-n}\sum_{k=0}^{N-n-1} x[k]x[k+n] \ \ , \ \ n = 0,1,2,3,\ldots, M << N$$

where the x's are samples taken from the stochastic process X and the time between samples is T_s .

Measurement of Autocorrelation

The expected value of the autocorrelation estimate is

$$\mathrm{E}\left(\hat{\mathrm{R}}_X\left(nT_s\right)\right) = \mathrm{E}\left(\frac{1}{N-n}\sum_{k=0}^{N-n-1} X\left[k\right]X\left[k+n\right]\right)$$

$$= \frac{1}{N-n}\sum_{k=0}^{N-n-1} \mathrm{E}\left(X\left[k\right]X\left[k+n\right]\right)$$

$$\mathrm{E}\left(\hat{\mathrm{R}}_X\left(nT_s\right)\right) = \frac{1}{N-n}\sum_{k=0}^{N-n-1} \mathrm{R}_X\left(nT_s\right) = \mathrm{R}_X\left(nT_s\right)$$

This is an unbiased estimator.

Crosscorrelation

Properties

$$R_{XY}(\tau) = R_{YX}(-\tau) \quad \text{or} \quad R_{XY}[m] = R_{YX}[-m]$$

$$|R_{XY}(\tau)| \le \sqrt{R_X(0) R_Y(0)} \quad \text{or} \quad |R_{XY}[m]| \le \sqrt{R_X[0] R_Y[0]}$$

If two stochastic processes *X* and *Y* are statistically independent

$$R_{XY}(\tau) = E(X)E(Y^*) = R_{YX}(\tau) \quad \text{or} \quad R_{XY}[m] = E(X)E(Y^*) = R_{YX}[m]$$

If *X* is stationary CT and X' is its time derivative

$$R_{XX'}(\tau) = \frac{d}{d\tau}\left(R_X(\tau)\right) \qquad R_{X'}(\tau) = -\frac{d^2}{d\tau^2}\left(R_X(\tau)\right)$$

If $Z(t) = X(t) \pm Y(t)$ and *X* and *Y* are independent and at least one of them has a zero mean

$$R_Z(\tau) = R_X(\tau) + R_Y(\tau)$$