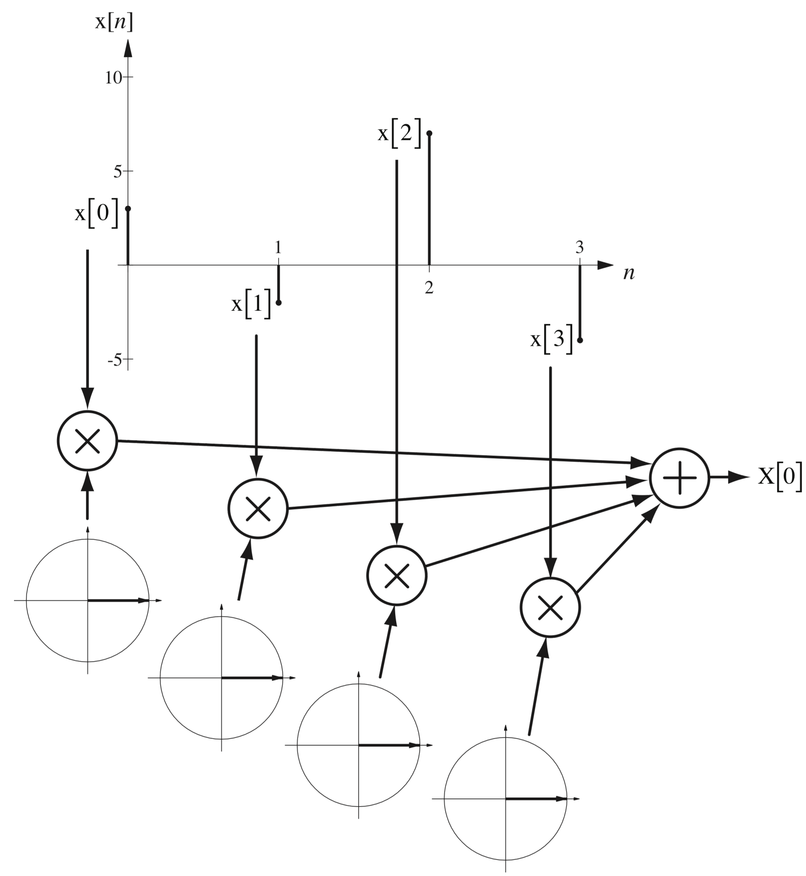
Four-Point DFT

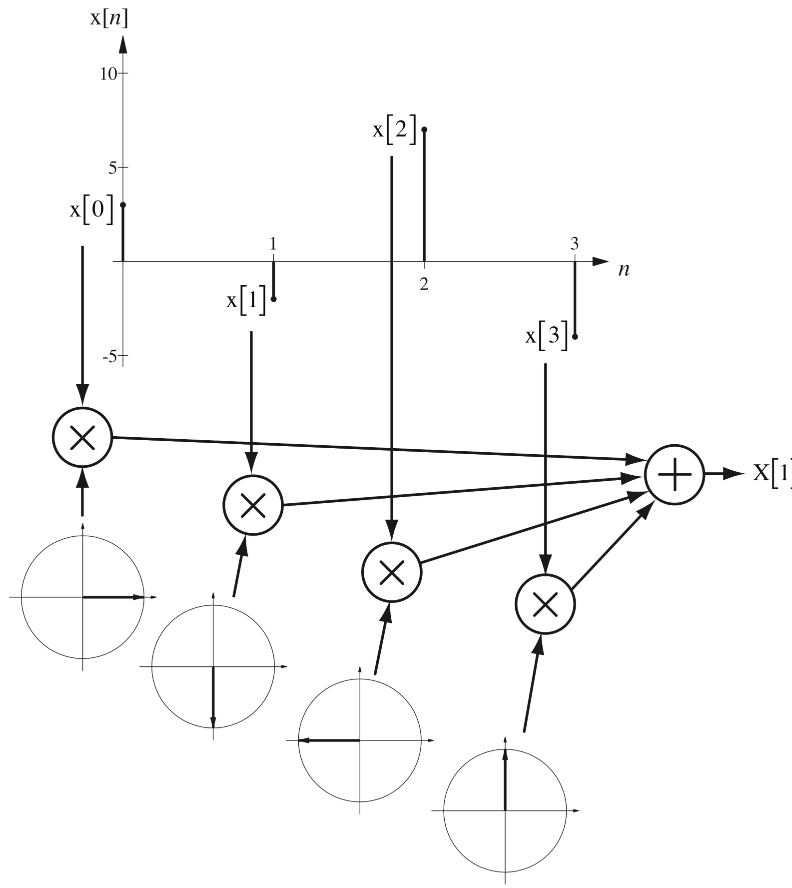
$$\mathrm{X}[k]=\sum_{n=0}^{N-1}\mathrm{x}[n]e^{-j2\pi kn/N}\Rightarrow \mathrm{X}[k]=\sum_{n=0}^{3}\mathrm{x}[n]e^{-j\pi kn/2}$$

$$\mathrm{X}[0]=\sum_{n=0}^{3}\mathrm{x}[n]=3-2+7-4=4$$



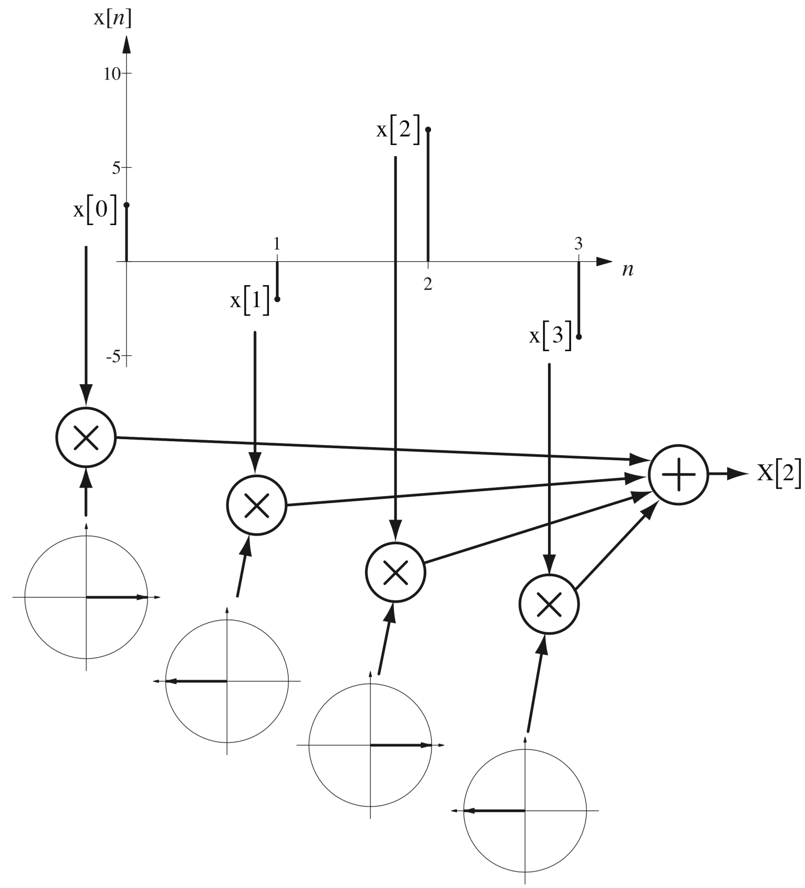
Four-Point DFT

$$\mathrm{X}[1] = \sum_{n=0}^{3} \mathrm{x}[n] e^{-j\pi n/2} = 3 - 2e^{-j\pi/2} + 7e^{-j\pi} - 4e^{-j3\pi/2} = 3 + j2 - 7 - j4 = -4 - j2$$



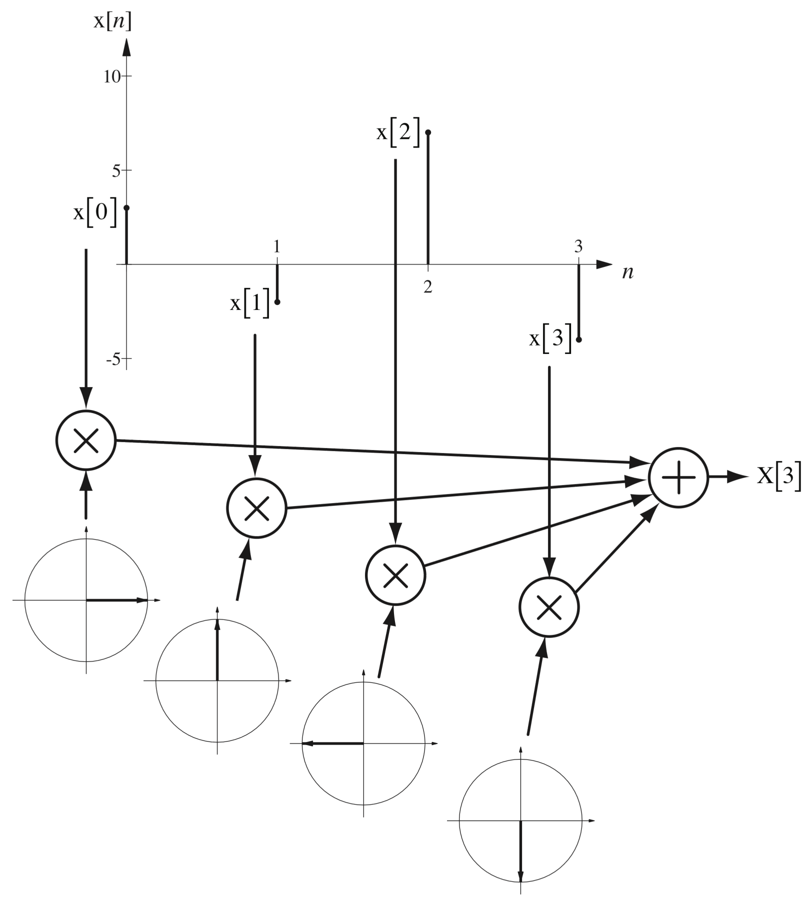
Four-Point DFT

$$\mathrm{X}[2]=\sum_{n=0}^{3}\mathrm{x}[n]e^{-j\pi n}=3-2e^{-j\pi}+7e^{-j2\pi}-4e^{-j3\pi}=3+2+7+4=16$$



Four-Point DFT

$$\mathrm{X}[3]=\sum_{n=0}^{3}\mathrm{x}[n]e^{-j3\pi n/2}=3-2e^{-j3\pi/2}+7e^{-j3\pi}-4e^{-j9\pi/2}=3-j2-7+j4=-4+j2=\mathrm{X}^*[1]$$



Find the DTFT of $\mathrm{x}[n] = 15(-0.3)^{n-1}\,\mathrm{u}[n-1]$.

$$\alpha^n\,\mathrm{u}[n] \xleftrightarrow{\mathcal{F}} \frac{1}{1-\alpha e^{-j\Omega}} \quad , \ |\alpha| < 1$$

$$(-0.3)^n\,\mathrm{u}[n] \xleftrightarrow{\mathcal{F}} \frac{1}{1+0.3e^{-j\Omega}}$$

$$(-0.3)^{n-1}\,\mathrm{u}[n-1] \xleftrightarrow{\mathcal{F}} \frac{e^{-j\Omega}}{1+0.3e^{-j\Omega}}$$

$$15(-0.3)^{n-1}\,\mathrm{u}[n-1] \xleftrightarrow{\mathcal{F}} \frac{15e^{-j\Omega}}{1+0.3e^{-j\Omega}}$$

Find the DTFT of $x[n]=\alpha^{|n|} \quad , \quad |\alpha|<1$

$$X\left(e^{j\Omega}\right)=\sum_{n=-\infty}^{\infty}\alpha^{|n|}e^{-j\Omega n}=\sum_{n=0}^{\infty}\alpha^{|n|}e^{-j\Omega n}+\sum_{n=-\infty}^{0}\alpha^{|n|}e^{-j\Omega n}-1$$

$$X\left(e^{j\Omega}\right)=\sum_{n=0}^{\infty}\alpha^{n}e^{-j\Omega n}+\sum_{n=0}^{\infty}\alpha^{|-n|}e^{+j\Omega n}-1$$

$$X\left(e^{j\Omega}\right)=\frac{1}{1-\alpha e^{-j\Omega}}+\frac{1}{1-\alpha e^{+j\Omega}}-1$$

$$X\left(e^{j\Omega}\right)=\frac{1-\alpha e^{+j\Omega}+1-\alpha e^{-j\Omega}-\left[1-2\alpha\cos(\Omega)+\alpha^{2}e^{-j2\Omega}\right]}{1-2\alpha\cos(\Omega)+\alpha^{2}e^{-j2\Omega}}$$

$$X\left(e^{j\Omega}\right)=\frac{1-\alpha^{2}}{1-2\alpha\cos(\Omega)+\alpha^{2}}$$

Find the DTFT of $\mathrm{x}[n] = \mathrm{tri}(n/12)\cos(2\pi n/4)$.

$$\mathrm{tri}(n/w) \xleftrightarrow{\mathcal{F}} w\,\mathrm{drcl}^2(F,w)$$

$$\mathrm{tri}(n/12) \xleftrightarrow{\mathcal{F}} 12\,\mathrm{drcl}^2(F,12)$$

$$\cos(2\pi n/N_0) \xleftrightarrow{\mathcal{F}} (1/2)\left[\delta_1(F-1/N_0)+\delta_1(F+1/N_0)\right]$$

$$\cos(2\pi n/4) \xleftrightarrow{\mathcal{F}} (1/2)\left[\delta_1(F-1/4)+\delta_1(F+1/4)\right]$$

$$\mathrm{tri}(n/12)\cos(2\pi n/4) \xleftrightarrow{\mathcal{F}} 12\,\mathrm{drcl}^2(F,12) \circledast (1/2)\left[\delta_1(F-1/4)+\delta_1(F+1/4)\right]$$

$$\mathrm{tri}(n/12)\cos(2\pi n/4) \xleftrightarrow{\mathcal{F}} 6\,\mathrm{drcl}^2(F,12) * \left[\delta(F-1/4)+\delta(F+1/4)\right]$$

$$\mathrm{tri}(n/12)\cos(2\pi n/4) \xleftrightarrow{\mathcal{F}} 6\left[\mathrm{drcl}^2(F-1/4,12)+\mathrm{drcl}^2(F+1/4,12)\right]$$

$$\mathrm{tri}(n/12)\cos(2\pi n/4) \xleftrightarrow{\mathcal{F}} 6\left[\mathrm{drcl}^2(\Omega/2\pi-1/4,12)+\mathrm{drcl}^2(\Omega/2\pi+1/4,12)\right]$$

Find the DTFT of $\mathrm{x}[n]=(0.8)^n \mathrm{u}[n]*\delta_2[n]$.

$$\alpha^n \mathrm{u}[n] \xleftrightarrow{\mathcal{F}} \frac{1}{1-\alpha e^{-j2\pi F}} \quad , \quad |\alpha|<1$$

$$(0.8)^n \mathrm{u}[n] \xleftrightarrow{\mathcal{F}} \frac{1}{1-0.8 e^{-j2\pi F}}$$

$$\delta_N[n] \xleftrightarrow{\mathcal{F}} (1/N)\delta_{1/N}(F)$$

$$\delta_2[n] \xleftrightarrow{\mathcal{F}} (1/2)\delta_{1/2}(F)$$

$$(0.8)^n \mathrm{u}[n]*\delta_2[n] \xleftrightarrow{\mathcal{F}} \frac{1/2}{1-0.8 e^{-j2\pi F}}\delta_{1/2}(F)$$

$$(0.8)^n \mathrm{u}[n]*\delta_2[n] \xleftrightarrow{\mathcal{F}} (1/2)\left[\frac{1}{1-0.8}\delta_1(F)+\frac{1}{1+0.8}\delta_1(F-1/2)\right]$$

$$(0.8)^n \mathrm{u}[n]*\delta_2[n] \xleftrightarrow{\mathcal{F}} (1/2)\left[\frac{1}{1-0.8}\delta_1(\Omega/2\pi)+\frac{1}{1+0.8}\delta_1(\Omega/2\pi-1/2)\right]$$

$$(0.8)^n \mathrm{u}[n]*\delta_2[n] \xleftrightarrow{\mathcal{F}} \pi\left[\frac{1}{1-0.8}\delta_{2\pi}(\Omega)+\frac{1}{1+0.8}\delta_{2\pi}(\Omega-\pi)\right]$$

Find the inverse DTFT of $\mathrm{X}\left(e^{j\Omega}\right)=\mathrm{rect}\left(4\Omega/\pi\right)*\delta_{\pi}\left(\Omega\right)$.

$$\mathrm{x}[n]=\frac{1}{2\pi}\int_{2\pi}\left[\mathrm{rect}\left(4\Omega/\pi\right)*\delta_{\pi}\left(\Omega\right)\right]e^{j\Omega n}\,d\Omega$$

$$\mathrm{x}[n]=\frac{1}{2\pi}\left[\int_{-\pi/8}^{\pi/8}e^{j\Omega n}\,d\Omega+\int_{7\pi/8}^{9\pi/8}e^{j\Omega n}\,d\Omega\right]$$

$$\mathrm{x}[n]=\frac{1}{2\pi}\left\{\left[\frac{e^{j\Omega n}}{jn}\right]_{-\pi/8}^{\pi/8}+\left[\frac{e^{j\Omega n}}{jn}\right]_{7\pi/8}^{9\pi/8}\right\}=\frac{e^{j\pi n/8}-e^{-j\pi n/8}+e^{j9\pi n/8}-e^{j7\pi n/8}}{j2\pi n}$$

$$\mathrm{x}[n]=\frac{j2\sin\left(\pi n/8\right)+\overbrace{e^{j\pi n}}^{=(-1)^n}\overbrace{\left(e^{j\pi n/8}-e^{-j\pi n/8}\right)}^{=j2\sin(\pi n/8)}}{j2\pi n}=\frac{1+(-1)^n}{8}\,\frac{\sin\left(\pi n/8\right)}{\pi n/8}$$

$$\mathrm{x}[n]=\frac{1+(-1)^n}{8}\,\mathrm{sinc}\left(n/8\right)$$

The excitation of a digital filter with transfer function $\mathrm{H}(z) = \dfrac{z}{z-0.7}$ is a periodic signal $\mathrm{x}[n]$, one period of which is described by

n	0	1	2	3	4	5	6	7
$\mathrm{x}[n]$	3	10	–2	0	7	7	4	8

Find the response $\mathrm{y}[n]$ over that same time period.

Since the excitation is periodic we can find the response exactly using the DFT. The frequency response of the filter is $\mathrm{H}(e^{j\Omega}) = \dfrac{e^{j\Omega}}{e^{j\Omega}-0.7}$. The fundamental period of the excitation is $N_0 = 8$ and the harmonic response of the filter is therefore $\mathrm{H}(e^{j2\pi k/8}) = \dfrac{e^{j2\pi k/8}}{e^{j2\pi k/8}-0.7}$.

The DFT of $\mathrm{x}[n]$ is

k	0	1	2	3
$\mathrm{X}[k]$	37	$3.7782+j9.5355$	$8-j9$	$-11.7782-j2.4645$

k	4	5	6	7
$\mathrm{X}[k]$	-13	$-11.7782+j2.4645$	$8+j9$	$3.7782-j9.5355$

The corresponding values of the harmonic response are

k	0	1	2	3
$\mathrm{H}\left(e^{j2\pi k/8}\right)$	3.3333	$1.0099-j0.9898$	$0.6711-j0.4698$	$0.6028-j0.1996$

k	4	5	6	7
$\mathrm{H}\left(e^{j2\pi k/8}\right)$	0.5882	$0.6028+j0.1996$	$0.6711+j0.4698$	$1.0099+j0.9898$

Therefore the DFT of the response is $\mathrm{Y}[k]=\mathrm{H}\left(e^{j2\pi k/8}\right)\mathrm{X}[k]$

k	0	1	2	3
$\mathrm{Y}[k]$	123.33	$13.25+j5.89$	$1.14-j9.80$	$-7.59+j0.87$

k	4	5	6	7
$\mathrm{Y}[k]$	-7.65	$-7.59-j0.87$	$1.14+j9.80$	$13.25-j5.89$

$y[n]$ is the inverse DFT of $Y[k]$ which is

n	0	1	2	3
$y[n]$	16.1616	21.3131	12.9192	9.0434

n	4	5	6	7
$y[n]$	13.3304	16.3313	15.4319	18.8023