Find the DTFT of $8\text{rect}(3(n-2)/13)$.

This function does not appear explicitly in the table of DTFT pairs.

But we know that it is a shifted version of $\text{rect}(3n/13)$ and that $\text{rect}(3n/13)$ is zero for all n except $-2 \le n < 3$. Therefore

$$\text{rect}(3n/13) = \text{u}[n+2] - \text{u}[n-3].$$

In the table we find

$$\text{u}[n-n_0] - \text{u}[n-n_1] \xleftrightarrow{\mathcal{F}} \frac{e^{-j\pi F(n_0+n_1)}}{e^{-j\pi F}}(n_1-n_0)\,\text{drcl}(F, n_1-n_0)$$

In our case $n_0 = -2$ and $n_1 = 3$. Then

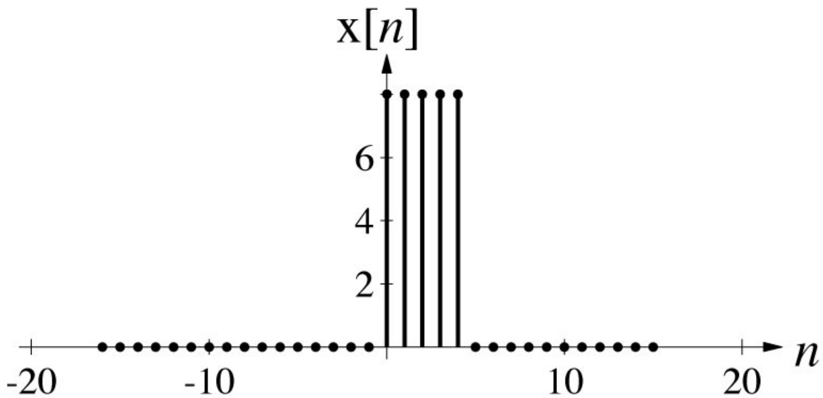
$$\text{rect}(3n/13) = \text{u}[n+2] - \text{u}[n-3] \xleftrightarrow{\mathcal{F}} \underbrace{\frac{e^{-j\pi F}}{e^{-j\pi F}}}_{=1} 5\,\text{drcl}(F,5)$$

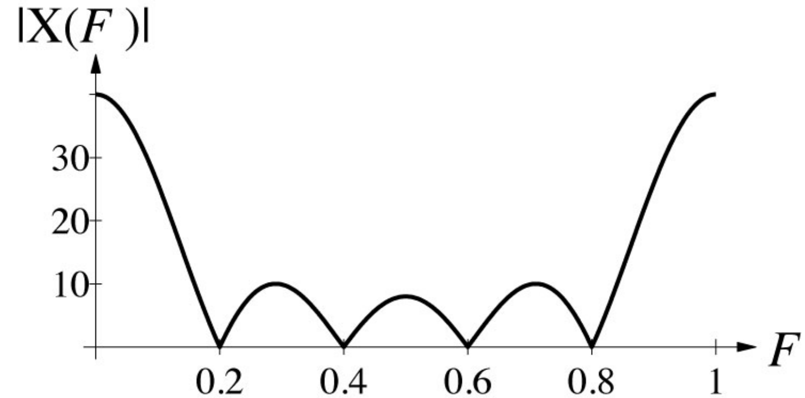
Using the time shifting property,

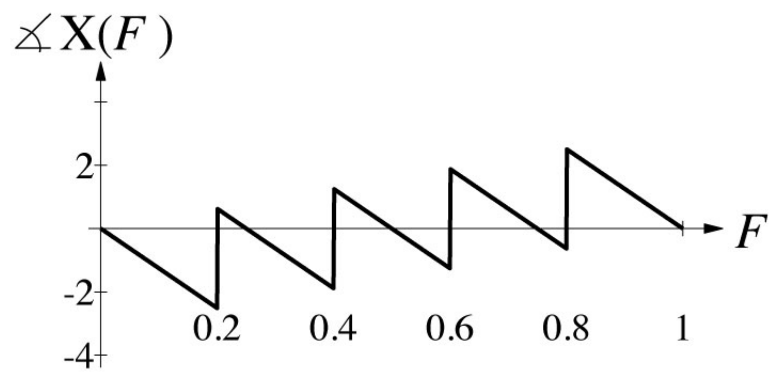
$$\text{rect}(3(n-2)/13) \xleftrightarrow{\mathcal{F}} 5\,\text{drcl}(F,5)\,e^{-j4\pi F}$$

Using the linearity property,

$$8\text{rect}(3(n-2)/13) \xleftrightarrow{\mathcal{F}} 40\,\text{drcl}(F,5)\,e^{-j4\pi F}$$


x[n]
6
4
2
-20
-10
10
20
n


|X(F)|
30
20
10
0.2
0.4
0.6
0.8
1
F


∡ X(F)
2
-2
-4
0.2
0.4
0.6
0.8
1
F

Find the DTFT of $8\text{rect}(3(n-2)/13)$ in the Ω form.

$$8\text{rect}(3(n-2)/13) \xleftrightarrow{\mathcal{F}} 40\,\text{drcl}(F,5)e^{-j4\pi F}$$

Let $F = \Omega/2\pi$. Then $8\text{rect}(3(n-2)/13) \xleftrightarrow{\mathcal{F}} 40\,\text{drcl}(\Omega/2\pi,5)e^{-j2\Omega}$

Find the DTFT of $8\text{rect}(3(n-2)/13)$ numerically using the DFT. The DTFT is defined by $\text{X}(F) = \sum_{n=-\infty}^{\infty} \text{x}[n]e^{-j2\pi Fn}$ and the DFT is defined by $\text{X}[k] = \sum_{n=0}^{N-1} \text{x}[n]e^{-j2\pi kn/N}$. In our case the function $8\text{rect}(3(n-2)/13)$ is non-zero only in the range $0 \le n < 5$. So $\text{X}(F) = \sum_{n=0}^{4} \text{x}[n]e^{-j2\pi Fn}$. It then follows that $\text{X}(k/N) = \sum_{n=0}^{N-1} \text{x}[n]e^{-j2\pi kn/N}$ with $N = 5$.

Numerically, $\{8,8,8,8,8\} \xleftrightarrow[5]{\mathcal{DFT}} \{40,0,0,0,0\}$. Analytically $\text{X}(k/5) = 8\sum_{n=0}^{4} e^{-j2\pi kn/5}$

$$= 8\frac{1-e^{-j10\pi k/5}}{1-e^{-j2\pi k/5}} = 8\frac{e^{-j\pi k}}{e^{-j\pi k/5}}\frac{\sin(2\pi k)}{\sin(2\pi k/5)} = 40e^{-j4\pi k/5}\,\text{drcl}(k/5,5) \text{ and}$$

k	0	1	2	3	4
$\text{X}(k/5)$	40	0	0	0	0

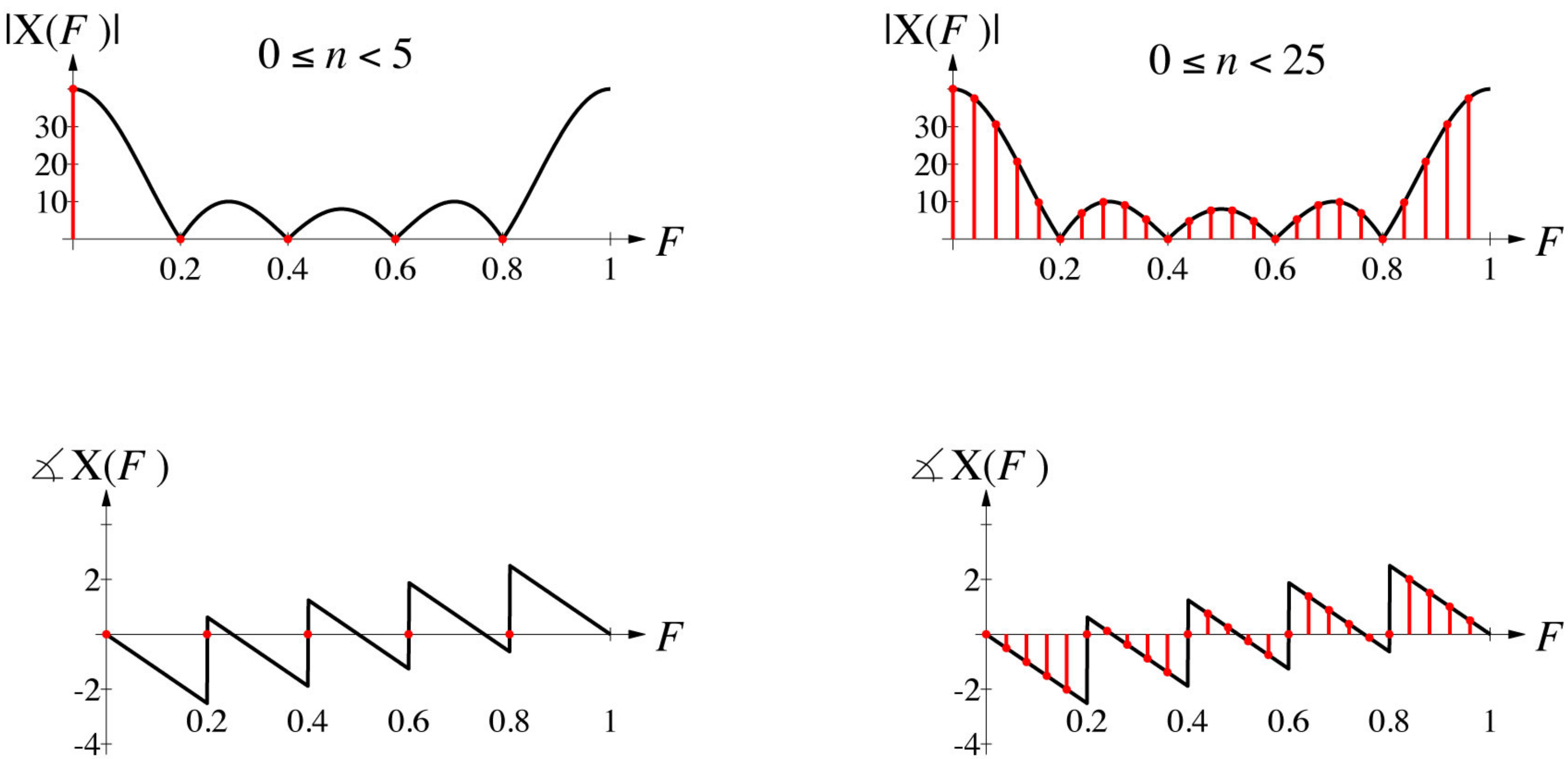
which agrees with the numerical result but the resolution in F is poor.

We can improve the resolution by using a larger N. Let $N = 25$. Then

$$\mathrm{X}(k/25) = 8\sum_{n=0}^{24} \mathrm{x}[n] e^{-j2\pi kn/25} = 8\sum_{n=0}^{4} e^{-j2\pi kn/25} = 8\frac{1-e^{-j2\pi kn/5}}{1-e^{-j2\pi kn/25}}$$

$$\mathrm{X}(k/25) = 8e^{-j4\pi kn/25}\frac{\sin(2\pi kn/5)}{\sin(2\pi kn/25)} = 40e^{-j4\pi kn/25}\,\mathrm{drcl}(k/25,5)$$

Now we have resolved values of the DTFT every 1/25 in F.



Find the DTFT of $3\text{sinc}(n/24)\cos(2\pi n/8)$.

From the DTFT table of Fourier pairs

$$\text{sinc}(n/w) \xleftrightarrow{\mathcal{F}} w\,\text{rect}(wF) * \delta_1(F) \Rightarrow 3\text{sinc}(n/24) \xleftrightarrow{\mathcal{F}} 72\text{rect}(24F) * \delta_1(F)$$

$$\cos(2\pi F_0 n) \xleftrightarrow{\mathcal{F}} (1/2)\left[\delta_1(F - F_0) + \delta_1(F + F_0)\right]$$

$$\cos(2\pi n/8) \xleftrightarrow{\mathcal{F}} (1/2)\left[\delta_1(F - 1/8) + \delta_1(F + 1/8)\right]$$

Using $\text{x}[n]\text{y}[n] \xleftrightarrow{\mathcal{F}} \text{X}(F) \circledast \text{Y}(F)$,

$$3\text{sinc}(n/24)\cos(2\pi n/8) \xleftrightarrow{\mathcal{F}} \left[72\text{rect}(24F) * \delta_1(F)\right] \circledast (1/2)\left[\delta_1(F - 1/8) + \delta_1(F + 1/8)\right]$$

Periodic convolution of two periodic functions with a common period is equivalent to aperiodic convolution of either function with one period of the other function. Therefore

$$3\text{sinc}(n/24)\cos(2\pi n/8) \xleftrightarrow{\mathcal{F}} \left[72\text{rect}(24F) * \delta_1(F)\right] * (1/2)\left[\delta(F - 1/8) + \delta(F + 1/8)\right]$$

$$3\text{sinc}(n/24)\cos(2\pi n/8) \xleftrightarrow{\mathcal{F}} 36\left[\text{rect}(24F) * \delta_1(F) * \delta(F - 1/8) + \text{rect}(24F) * \delta_1(F) * \delta(F + 1/8)\right]$$

$$3\text{sinc}(n/24)\cos(2\pi n/8) \xleftrightarrow{\mathcal{F}} 36\left[\text{rect}(24F) * \delta_1(F - 1/8) + \text{rect}(24F) * \delta_1(F + 1/8)\right]$$

$$3\text{sinc}(n/24)\cos(2\pi n/8) \xleftrightarrow{\mathcal{F}} 36\text{rect}(24F) * \left[\delta_1(F - 1/8) + \delta_1(F + 1/8)\right]$$

Convert to the Ω form.

Using the scaling property of convolution, If $\text{y}(t) = \text{x}(t) * \text{h}(t)$, then $\text{y}(at) = |a|\text{x}(at) * \text{h}(at)$.

$$3\text{sinc}(n/24)\cos(2\pi n/8) \xleftrightarrow{\mathcal{F}} (1/2\pi)36\text{rect}(24\Omega/2\pi) * \left[\delta_1(\Omega/2\pi - 1/8) + \delta_1(\Omega/2\pi + 1/8)\right]$$

Using the scaling property of the periodic impulse, $\delta_a(x/b) = b\delta_{ab}(x)$

$$3\text{sinc}(n/24)\cos(2\pi n/8) \xleftrightarrow{\mathcal{F}} (1/2\pi)36\text{rect}(12\Omega/\pi) * 2\pi\left[\delta_{2\pi}(\Omega - \pi/4) + \delta_{2\pi}(\Omega + \pi/4)\right]$$

$$3\text{sinc}(n/24)\cos(2\pi n/8) \xleftrightarrow{\mathcal{F}} 36\text{rect}(12\Omega/\pi) * \left[\delta_{2\pi}(\Omega - \pi/4) + \delta_{2\pi}(\Omega + \pi/4)\right]$$

Find the DTFT of $-7(-0.4)^{n-2}\,\mathrm{u}[n-2]*\delta_8[n]$.

From the DTFT table $\alpha^n\,\mathrm{u}[n] \xleftrightarrow{\mathcal{F}} \dfrac{1}{1-\alpha e^{-j\Omega}}$, $|\alpha|<1$ and $\delta_N[n] \xleftrightarrow{\mathcal{F}} (1/N)\delta_{1/N}(F)$

$(-0.4)^n\,\mathrm{u}[n] \xleftrightarrow{\mathcal{F}} \dfrac{1}{1+0.4e^{-j\Omega}}$ and $\delta_8[n] \xleftrightarrow{\mathcal{F}} (1/8)\delta_{1/8}(F)$ or $(1/8)\delta_{1/8}(\Omega/2\pi) = (\pi/4)\delta_{\pi/4}(\Omega)$

Time shifting $\rightarrow (-0.4)^{n-2}\,\mathrm{u}[n-2] \xleftrightarrow{\mathcal{F}} \dfrac{e^{-j2\Omega}}{1+0.4e^{-j\Omega}}$

Linearity $\rightarrow -7(-0.4)^{n-2}\,\mathrm{u}[n-2] \xleftrightarrow{\mathcal{F}} -\dfrac{7e^{-j2\Omega}}{1+0.4e^{-j\Omega}}$

Using the property $\mathrm{x}[n]*\mathrm{y}[n] \xleftrightarrow{\mathcal{F}} \mathrm{X}(e^{j\Omega})\mathrm{Y}(e^{j\Omega})$

$$-7(-0.4)^{n-2}\,\mathrm{u}[n-2]*\delta_8[n] \xleftrightarrow{\mathcal{F}} -\frac{7e^{-j2\Omega}}{1+0.4e^{-j\Omega}}(\pi/4)\delta_{\pi/4}(\Omega)$$

$$-7(-0.4)^{n-2}\,\mathrm{u}[n-2]*\delta_8[n] \xleftrightarrow{\mathcal{F}} -\frac{(7\pi/4)e^{-j2\Omega}}{1+0.4e^{-j\Omega}}\delta_{\pi/4}(\Omega)$$

Convert to the F form.

$$-7(-0.4)^{n-2}\,\mathrm{u}[n-2]*\delta_8[n] \xleftrightarrow{\mathcal{F}} -\frac{(7\pi/4)e^{-j4\pi F}}{1+0.4e^{-j2\pi F}}\delta_{\pi/4}(2\pi F)$$

$$-7(-0.4)^{n-2}\,\mathrm{u}[n-2]*\delta_8[n] \xleftrightarrow{\mathcal{F}} -\frac{(7\pi/4)e^{-j4\pi F}}{1+0.4e^{-j2\pi F}}(1/2\pi)\delta_{1/8}(F)$$

$$-7(-0.4)^{n-2}\,\mathrm{u}[n-2]*\delta_8[n] \xleftrightarrow{\mathcal{F}} -\frac{(7/8)e^{-j4\pi F}}{1+0.4e^{-j2\pi F}}\delta_{1/8}(F)$$

A digital filter has a frequency response $\mathrm{H}\left(e^{j\Omega}\right)=\dfrac{1+e^{-j\Omega}+e^{-j2\Omega}}{3}$. Find its response to $\mathrm{x}[n]=15\sin(2\pi n/12)$.

The DTFT of the response is $\mathrm{Y}\left(e^{j\Omega}\right)=\mathrm{H}\left(e^{j\Omega}\right)\mathrm{X}\left(e^{j\Omega}\right)$

$$\mathrm{Y}\left(e^{j\Omega}\right)=\frac{1+e^{-j\Omega}+e^{-j2\Omega}}{3}\times j15\pi\left[\delta_{2\pi}(\Omega+\pi/6)-\delta_{2\pi}(\Omega-\pi/6)\right]$$

$$\mathrm{Y}\left(e^{j\Omega}\right)=j15\pi\left[\frac{1+e^{-j\Omega}+e^{-j2\Omega}}{3}\delta_{2\pi}(\Omega+\pi/6)-\frac{1+e^{-j\Omega}+e^{-j2\Omega}}{3}\delta_{2\pi}(\Omega-\pi/6)\right]$$

Using the equivalence property of the impulse and the periodicity of the complex sinusoid $e^{j\Omega}$

$$\mathrm{Y}\left(e^{j\Omega}\right)=j5\pi\left[\left(1+e^{j\pi/6}+e^{j\pi/3}\right)\delta_{2\pi}(\Omega+\pi/6)-\left(1+e^{-j\pi/6}+e^{-j\pi/3}\right)\delta_{2\pi}(\Omega-\pi/6)\right]$$

Using Euler's identity

$$\mathrm{Y}\left(e^{j\Omega}\right)=j5\pi\begin{bmatrix}(1+\cos(\pi/6)+\cos(\pi/3))\delta_{2\pi}(\Omega+\pi/6)-(1+\cos(-\pi/6)+\cos(-\pi/3))\delta_{2\pi}(\Omega-\pi/6)\\+(j\sin(\pi/6)+j\sin(\pi/3))\delta_{2\pi}(\Omega+\pi/6)-(j\sin(-\pi/6)+j\sin(-\pi/3))\delta_{2\pi}(\Omega-\pi/6)\end{bmatrix}$$

$$\mathrm{Y}\left(e^{j\Omega}\right)=j5\pi\left\{2.366\left[\delta_{2\pi}(\Omega+\pi/6)-\delta_{2\pi}(\Omega-\pi/6)\right]+j1.366\left[\delta_{2\pi}(\Omega+\pi/6)+\delta_{2\pi}(\Omega-\pi/6)\right]\right\}$$

$$\mathrm{y}[n]=11.83\sin(\pi n/6)-6.83\cos(\pi n/6)=13.66\cos(\pi n/6+2.618)$$

The signal $\mathrm{x}[n] = \mathrm{sinc}(n/21) * \delta_{100}[n]$ excites a system with transfer function $\mathrm{H}(z) = \dfrac{5}{1 - 0.8z^{-1}}$. Find and graph the response $\mathrm{y}[n]$ of the system.

$\mathrm{Y}(e^{j\Omega}) = \left[\mathrm{H}(z)_{z \to e^{j\Omega}}\right]\mathrm{X}(e^{j\Omega})$

Using $\mathrm{sinc}(n/w) \xleftrightarrow{\mathscr{F}} w\mathrm{rect}(wF) * \delta_1(F)$ and $\delta_N[n] \xleftrightarrow{\mathscr{F}} (1/N)\delta_{1/N}(F)$

and $\mathrm{x}[n] * \mathrm{y}[n] \xleftrightarrow{\mathscr{F}} \mathrm{X}(F)\mathrm{Y}(F)$

$\mathrm{sinc}(n/21) * \delta_{100}[n] \xleftrightarrow{\mathscr{F}} \left[21\mathrm{rect}(21F) * \delta_1(F)\right](1/100)\delta_{1/100}(F)$

Using the scaling properties of convolution and the periodic impulse,

$\mathrm{sinc}(n/21) * \delta_{100}[n] \xleftrightarrow{\mathscr{F}} \left[(1/2\pi)21\mathrm{rect}(21\Omega/2\pi) * \delta_1(\Omega/2\pi)\right](1/100)\delta_{1/100}(\Omega/2\pi)$

$\mathrm{sinc}(n/21) * \delta_{100}[n] \xleftrightarrow{\mathscr{F}} \left[(1/2\pi)21\mathrm{rect}(21\Omega/2\pi) * 2\pi\delta_{2\pi}(\Omega)\right](2\pi/100)\delta_{2\pi/100}(\Omega)$

$\mathrm{sinc}(n/21) * \delta_{100}[n] \xleftrightarrow{\mathscr{F}} (21\pi/50)\left[\mathrm{rect}(21\Omega/2\pi) * \delta_{2\pi}(\Omega)\right]\delta_{\pi/50}(\Omega)$

This result is periodic with period 2π, one period of which consists of set of impulses of strength $21\pi/50$ at radian frequencies $\pi k/50$ in the range $-1/2 < 21\pi k/100\pi < 1/2$ or $-50/21 < k < 50/21$ or, since k must be an integer, $-2 \le k \le 2$. This can be written

in the form $\mathrm{sinc}(n/21) * \delta_{100}[n] \xleftrightarrow{\mathscr{F}} \left[(21\pi/50)\sum_{k=-2}^{2}\delta(\Omega - \pi k/50)\right] * \delta_{2\pi}(\Omega)$

$$\text{sinc}(n/21) * \delta_{100}[n] \xleftrightarrow{\mathcal{F}} \left[(21\pi/50)\sum_{k=-2}^{2}\delta(\Omega - \pi k/50)\right] * \delta_{2\pi}(\Omega)$$

Then $\mathrm{Y}(e^{j\Omega}) = \dfrac{5}{1-0.8e^{-j\Omega}}\left[(21\pi/50)\sum_{k=-2}^{2}\delta(\Omega-\pi k/50)\right] * \delta_{2\pi}(\Omega)$

Using the equivalence property of the impulse,

$$\mathrm{Y}(e^{j\Omega}) = \left[(21\pi/10)\sum_{k=-2}^{2}\frac{\delta(\Omega-\pi k/50)}{1-0.8e^{-j\pi k/50}}\right] * \delta_{2\pi}(\Omega)$$

$$\mathrm{Y}(e^{j\Omega}) = 2.1\pi\begin{bmatrix}\dfrac{\delta(\Omega+2\pi/50)}{1-0.8e^{j2\pi/50}} + \dfrac{\delta(\Omega+\pi/50)}{1-0.8e^{j\pi/50}} + \dfrac{\delta(\Omega)}{1-0.8} \\ + \dfrac{\delta(\Omega-\pi/50)}{1-0.8e^{-j\pi/50}} + \dfrac{\delta(\Omega-2\pi/50)}{1-0.8e^{-j2\pi/50}}\end{bmatrix} * \delta_{2\pi}(\Omega)$$

$$\mathrm{Y}(e^{j\Omega}) = 2.1\left\{\begin{array}{l}3.921\pi\left[\delta(\Omega+2\pi/50)+\delta(\Omega-2\pi/50)\right] \\ +j1.906\pi\left[\delta(\Omega+2\pi/50)-\delta(\Omega-2\pi/50)\right] \\ +4.671\pi\left[\delta(\Omega-\pi/50)+\delta(\Omega-\pi/50)\right] \\ +j1.164\pi\left[\delta(\Omega-\pi/50)-\delta(\Omega-\pi/50)\right]+5\pi\delta(\Omega)\end{array}\right\} * \delta_{2\pi}(\Omega)$$

$$\mathrm{y}[n] = 2.1\begin{bmatrix}3.921\cos(2\pi n/50) - 1.906\sin(2\pi n/50) \\ +4.671\cos(\pi n/50) - 1.164\sin(\pi n/50) + 2.5\end{bmatrix}$$

$$y[n] = 2.1\begin{bmatrix} 3.921\cos(2\pi n/50) - 1.906\sin(2\pi n/50) \\ +4.671\cos(\pi n/50) - 1.164\sin(\pi n/50) + 2.5 \end{bmatrix}$$

Using $\mathrm{A}\cos(x) + B\sin(x) = \sqrt{A^2 + B^2}\cos\left(x - \tan^{-1}(B/A)\right)$

$$y[n] = 2.1\left[4.36\cos(2\pi n/50 - 0.452) + 4.814\cos(\pi n/50 - 0.244) + 2.5\right]$$

$$y[n] = 9.156\cos(2\pi n/50 - 0.452) + 10.11\cos(\pi n/50 - 0.244) + 5.25$$