Inverse z Transform by Synthetic Division

Consider first the transfer function

$$\mathrm{H}(z) = \frac{z}{z - 1/2} \quad , \quad |z| > 1/2 .$$

Dividing the denominator into the numerator by synthetic division,

$$\begin{array}{r|l} & 1 + 1/2z + 1/4z^2 \cdots \\ \hline z - 1/2 & z \\ & \underline{z - 1/2} \\ & \quad 1/2 \\ & \underline{\quad 1/2 - 1/4z} \\ & \qquad\quad 1/4z \\ & \qquad\quad \vdots \end{array}$$

The result of the synthetic division is the infinite series $\sum_{n=0}^{\infty} (1/2z)^n$. This geometric series coverges if $|1/2z| < 1$ or $|z| > 1/2$. The inverse transform (term by term) of the infinite series is

$$\mathrm{h}[n] = \delta[n] + (1/2)\delta[n-1] + (1/4)\delta[n-2] \cdots \text{ or } \mathrm{h}[n] = \sum_{m=0}^{\infty} (1/2)^m \, \delta[n-m].$$

The inverse transform can also be found directly from the tables as $\mathrm{h}[n] = (1/2)^n \, \mathrm{u}[n]$. These two results are equivalent. The inverse transform is a right-sided signal consistent with the region of convergence being the exterior of a circle of radius 1/2.

Now consider the transfer function

$$\mathrm{H}(z) = \frac{z}{z - 1/2} \quad , \quad |z| < 1/2 .$$

Dividing the denominator into the numerator by synthetic division,

$$
\begin{array}{r|l}
 & -2z-4z^2-8z^3\cdots \\
-1/2+z & z \\
 & \underline{z-2z^2} \\
 & 2z^2 \\
 & \underline{2z^2-4z^3} \\
 & 4z^3 \\
 & \vdots
\end{array}
$$

The result of the synthetic division is the infinite series $-\sum_{n=1}^{\infty}(2z)^n$ which can be written as $1-\sum_{n=0}^{\infty}(2z)^n$. The geometric series $\sum_{n=0}^{\infty}(2z)^n$ coverges if $|2z|<1$ or $|z|<1/2$. The inverse transform (term by term) of the infinite series is

$$
\mathrm{h}[n]=-2\delta[n+1]-4\delta[n+2]-8\delta[n+3]\cdots \text{ or } \mathrm{h}[n]=-\sum_{m=1}^{\infty}2^m\delta[n+m].
$$

The inverse transform can also be found directly from the tables as $\mathrm{h}[n]=-(1/2)^n\,\mathrm{u}[-n-1]$. These two results are equivalent. The inverse transform is a left-sided signal consistent with the region of convergence being the interior of a circle of radius 1/2.

The question that arose in class was "How can the two synthetic division results both be right when they seem to be so different?". The two synthetic division results are

$$
\frac{z}{z-1/2}=1+1/2z+1/4z^2\cdots \quad \text{and} \quad \frac{z}{z-1/2}=-2z-4z^2-8z^3\cdots
$$

Multiplying both sides of the first result by $z-1/2$ we get

$$
z=(z-1/2)\left(1+1/2z+1/4z^2\cdots\right).
$$

Multiplying out the right-hand side we get

$$
\begin{aligned} z=z+1/2+1/4z+1/8z^2+1/16z^3\cdots \\ -1/2-1/4z-1/8z^2-1/16z^3\cdots \end{aligned} \quad \text{or} \quad z=z+(1/2)\left[\sum_{n=0}^{\infty}(1/2z)^n-\sum_{n=0}^{\infty}(1/2z)^n\right]
$$

The right side is z plus and minus an infinite series that converges if $|z|>1/2$. That is, for $|z|>1/2$, the two sides must be equal and the synthetic division result is correct.

Multiplying both sides of the second result by $z-1/2$ we get

$$
z=(z-1/2)\left(-2z-4z^2-8z^3\cdots\right).
$$

Multiplying out the right-hand side we get

$$
\begin{aligned}
z = \quad & -2z^2 - 4z^3 - 8z^4 - 16z^5 - \cdots \\
& z + 2z^2 + 4z^3 + 8z^4 + 16z^5 + \cdots
\end{aligned}
\quad \text{or} \quad z = -z\sum_{n=1}^{\infty}(2z)^n + z + z\sum_{n=1}^{\infty}(2z)^n
$$

The summations can be re-written in the form of a geometric series

$$\sum_{n=1}^{\infty}(2z)^n = -1 + \sum_{n=0}^{\infty}(2z)^n \, .$$

Then

$$z = -z\left(-1 + \sum_{n=0}^{\infty}(2z)^n\right) + z + z\left(-1 + \sum_{n=0}^{\infty}(2z)^n\right) = z\left[1 - \sum_{n=0}^{\infty}(2z)^n + \sum_{n=0}^{\infty}(2z)^n\right]$$

The right side is z plus and minus an infinite series that converges if $|z| < 1/2$. That is, for $|z| < 1/2$, the two sides must be equal and the synthetic division result is correct.