Transfer Function and Frequency Response

Consider the general form of a differential equation for a continuous-time system

$$\sum_{k=0}^{N} a_k \, \mathrm{y}^{(k)}(t) = \sum_{k=0}^{M} b_k \, \mathrm{x}^{(k)}(t) \ . \tag{0.1}$$

where

1. The a's and b's are constants,
2. $N \geq M$,
3. $a_N \neq 0$

and 4. The notation $\mathrm{x}^{(k)}(t)$ means the kth derivative of $\mathrm{x}(t)$ with respect to time and, if k is negative, that indicates integration instead of differentiation.

If $\mathrm{x}(t) = Xe^{st}$, $\mathrm{y}(t)$ has the form $\mathrm{y}(t) = Ye^{st}$ where X and Y are complex constants. Then, in the differential equation, the kth derivatives take the forms $\mathrm{x}^{(k)}(t) = s^k X e^{st}$ and $\mathrm{y}^{(k)}(t) = s^k Y e^{st}$. Then (0.1) can be written in the form

$$\sum_{k=0}^{N} a_k s^k Y e^{st} = \sum_{k=0}^{M} b_k s^k X e^{st} \ .$$

The Xe^{st} and Ye^{st} can be factored out leading to

$$Ye^{st} \sum_{k=0}^{N} a_k s^k = Xe^{st} \sum_{k=0}^{M} b_k s^k \Rightarrow \frac{Y}{X} = \frac{\sum_{k=0}^{M} b_k s^k}{\sum_{k=0}^{N} a_k s^k} \ .$$

This ratio Y/X is a ratio of polynomials in s. It is called the **transfer function** and is conventionally given the symbol H.

$$\boxed{\mathrm{H}(s) = \frac{\sum_{k=0}^{M} b_k s^k}{\sum_{k=0}^{N} a_k s^k} = \frac{b_M s^M + \cdots + b_2 s^2 + b_1 s + b_0}{a_N s^N + \cdots + a_2 s^2 + a_1 s + a_0}} \ . \tag{0.2}$$

The transfer function can then be written directly from the differential equation and, if the differential equation describes the system, so does the transfer function. Functions like (0.2) in the form of a ratio of polynomials are called **rational functions**.

We can specialize the complex exponential input signal to a complex sinusoid by letting $s = j\omega = j2\pi f$, with ω and f real. The input signal is then $\mathrm{x}(t) = Xe^{j\omega t}$. The response signal is $\mathrm{y}(t) = Ye^{j\omega t} = \mathrm{H}(j\omega)Xe^{j\omega t}$. X and Y are of the forms

$$X = |X|e^{j\measuredangle X} = |X|\left[\cos(\measuredangle X) + j\sin(\measuredangle X)\right]$$

and

$$Y = |Y|e^{j\measuredangle Y} = |Y|\left[\cos(\measuredangle Y) + j\sin(\measuredangle Y)\right].$$

H is of the form

$$\mathrm{H}(j\omega) = |\mathrm{H}(j\omega)|e^{j\measuredangle \mathrm{H}(j\omega)}.$$

Therefore

$$|Y|e^{j\measuredangle Y} = |\mathrm{H}(j\omega)|e^{j\measuredangle \mathrm{H}(j\omega)}|X|e^{j\measuredangle X} = |\mathrm{H}(j\omega)||X|e^{j(\measuredangle \mathrm{H}(j\omega) + \measuredangle X)}$$

and

$$|Y| = |\mathrm{H}(j\omega)||X| \quad \text{and} \quad \measuredangle Y = \measuredangle \mathrm{H}(j\omega) + \measuredangle X.$$

The real part of $\mathrm{x}(t)$ is

$$\mathrm{Re}(\mathrm{x}(t)) = \mathrm{Re}\left(|X|e^{j(\omega t + \measuredangle X)}\right) = |X|\cos(\omega t + \measuredangle X)$$

and the real part of $\mathrm{y}(t)$ is

$$\mathrm{Re}(\mathrm{y}(t)) = \mathrm{Re}\left(|X||\mathrm{H}(j\omega)|e^{j(\omega t + \measuredangle \mathrm{H}(j\omega) + \measuredangle X)}\right) = |X||\mathrm{H}(j\omega)|\cos(\omega t + \measuredangle \mathrm{H}(j\omega) + \measuredangle X)$$

We showed in Chapter 4 that the real part of the complex input signal produces the real part of the response signal. Therefore a real sinusoidal input signal at a radian frequency ω produces a real sinusoidal response also at the radian frequency ω. The magnitude of the response is $|X||\mathrm{H}(j\omega)|$ and the phase of the response is $\measuredangle \mathrm{H}(j\omega) + \measuredangle X$. $\mathrm{H}(j\omega)$ is known as the **frequency response** of the system.

Example   Frequency response of a continuous-time system

A continuous-time system is described by the differential equation

$$\mathrm{y}''(t) + 5\,\mathrm{y}'(t) + 2\,\mathrm{y}(t) = 3\mathrm{x}''(t).$$

Find and graph the magnitude and phase of its frequency response.

The differential equation is in the general form

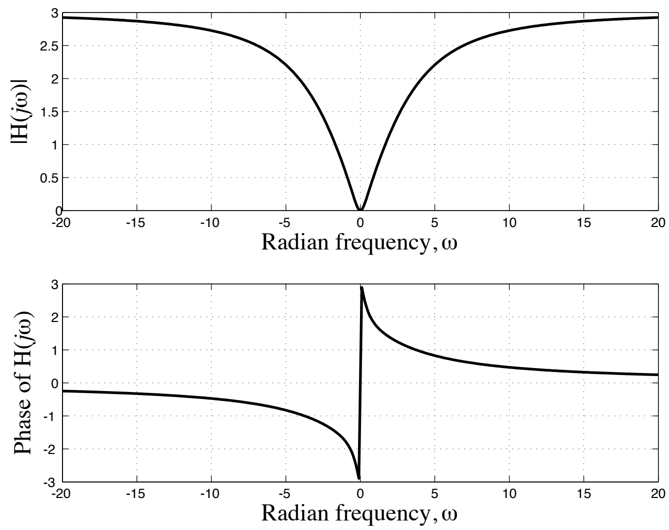
$$\sum_{k=0}^{N} a_k\,\mathrm{y}^{(k)}(t) = \sum_{k=0}^{M} b_k\,\mathrm{x}^{(k)}(t)$$

where, in this case, $N = M = 2$, $a_2 = 1$, $a_1 = 5$, $a_0 = 2$, $b_2 = 3$, $b_1 = 0$ and $b_0 = 0$. The transfer function is

$$\mathrm{H}(s) = \frac{\sum_{k=0}^{2} b_k s^k}{\sum_{k=0}^{2} a_k s^k} = \frac{b_2 s^2 + b_1 s + b_0}{a_2 s^2 + a_1 s + a_0} = \frac{3s^2}{s^2 + 5s + 2}$$

The frequency response is

$$\mathrm{H}(j\omega) = \frac{3(j\omega)^2}{(j\omega)^2 + j5\omega + 2}$$



Magnitude and phase of frequency response

These graphs were generated by the following MATLAB code.

```
wmax = 20 ;                  %      Maximum radian frequency magnitude for graph
dw = 0.1 ;                   %      Spacing between frequencies in graph
w = [-wmax:dw:wmax]' ;       %      Vector of frequencies for graph

%      Compute the frequency response

H = 3*(j*w).^2./((j*w).^2 + j*5*w + 2) ;

%      Graph and annotate the frequency response

subplot(2,1,1) ; p = plot(w,abs(H),'k') ; set(p,'LineWidth',2) ;
grid on ;
xlabel('Radian frequency, {\omega}','FontSize',18,'FontName','Times') ;
ylabel('|H({\itj}{\omega})|','FontSize',18,'FontName','Times') ;
subplot(2,1,2) ; p = plot(w,angle(H),'k') ; set(p,'LineWidth',2) ;
grid on ;
xlabel('Radian frequency, {\omega}','FontSize',18,'FontName','Times') ;
ylabel('Phase of H({\itj}{\omega})','FontSize',18,'FontName','Times') ;
```

MATLAB also has some handy functions for doing frequency-response analysis in the `control` toolbox. The command

```
H = freqs(num,den,w) ;
```

accepts the two vectors `num` and `den` and interprets them as the coefficients of the powers of s in the numerator and denominator of the transfer function $\mathrm{H}(s)$ starting with the highest power and going all the way to the zero power, not skipping any. It returns in `H` the complex frequency response at the radian frequencies in the vector `w`. This command is demonstrated below for a vector of 9 frequencies.

```
>> num = [3 0 0] ; den = [1 5 2] ;
>> w = [-20:5:20]' ; H = freqs(num,den,w) ;
>> [w,abs(H),angle(H)]
ans =
  -20.0000    2.9242   -0.2462
  -15.0000    2.8690   -0.3244
  -10.0000    2.7268   -0.4718
   -5.0000    2.2078   -0.8270
         0         0         0
    5.0000    2.2078    0.8270
   10.0000    2.7268    0.4718
   15.0000    2.8690    0.3244
   20.0000    2.9242    0.2462
>>
```