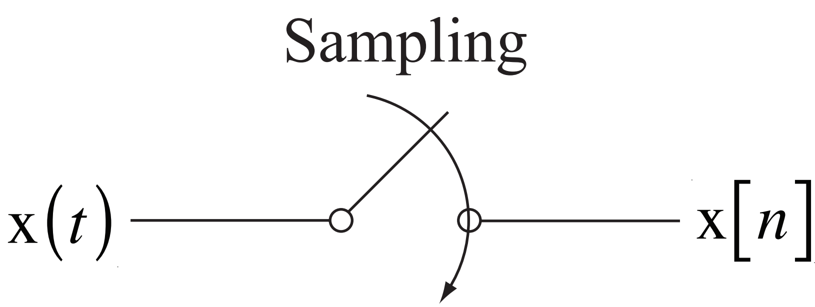
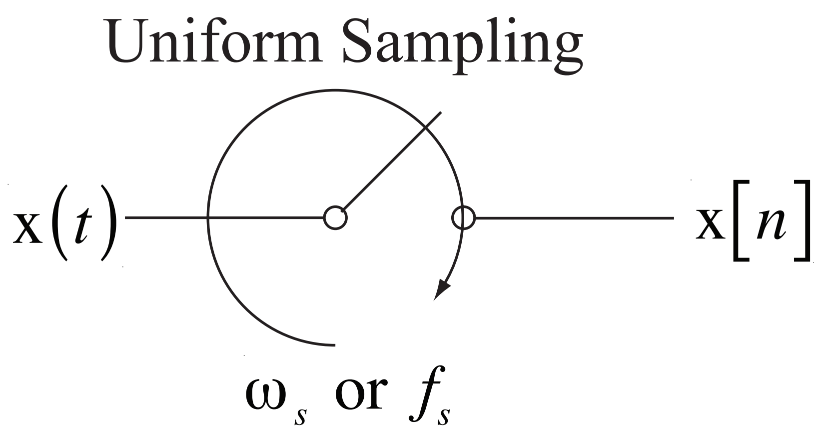
Mathematical Description of Discrete-Time Signals

Sampling and Discrete Time

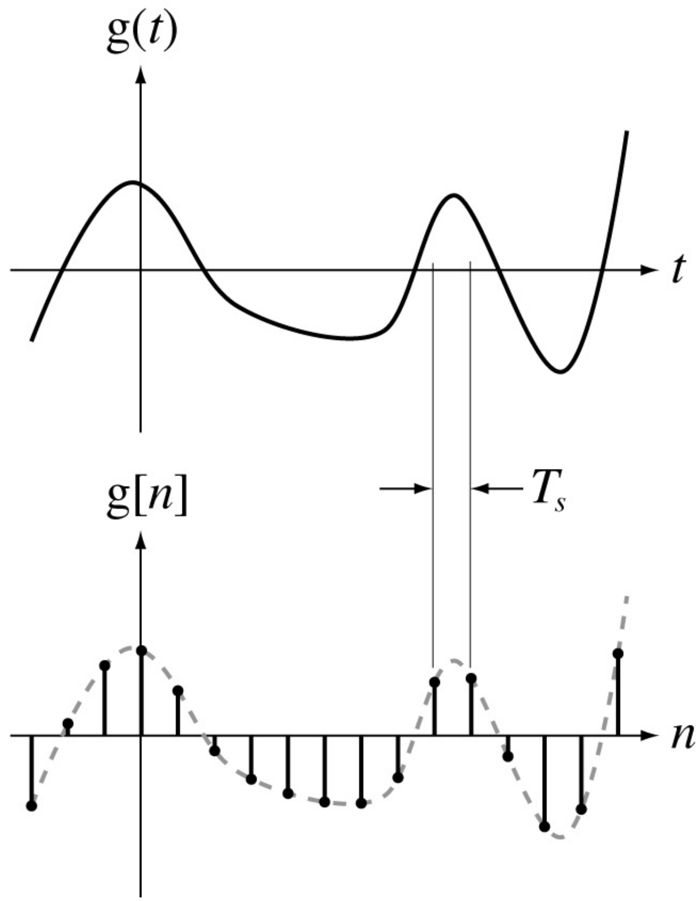
Sampling is the acquisition of the values of a continuous-time signal at discrete points in time. $\mathrm{x}(t)$ is a continuous-time signal, $\mathrm{x}[n]$ is a discrete-time signal.

$$\mathrm{x}[n] = \mathrm{x}(nT_s) \text{ where } T_s \text{ is the time between samples}$$





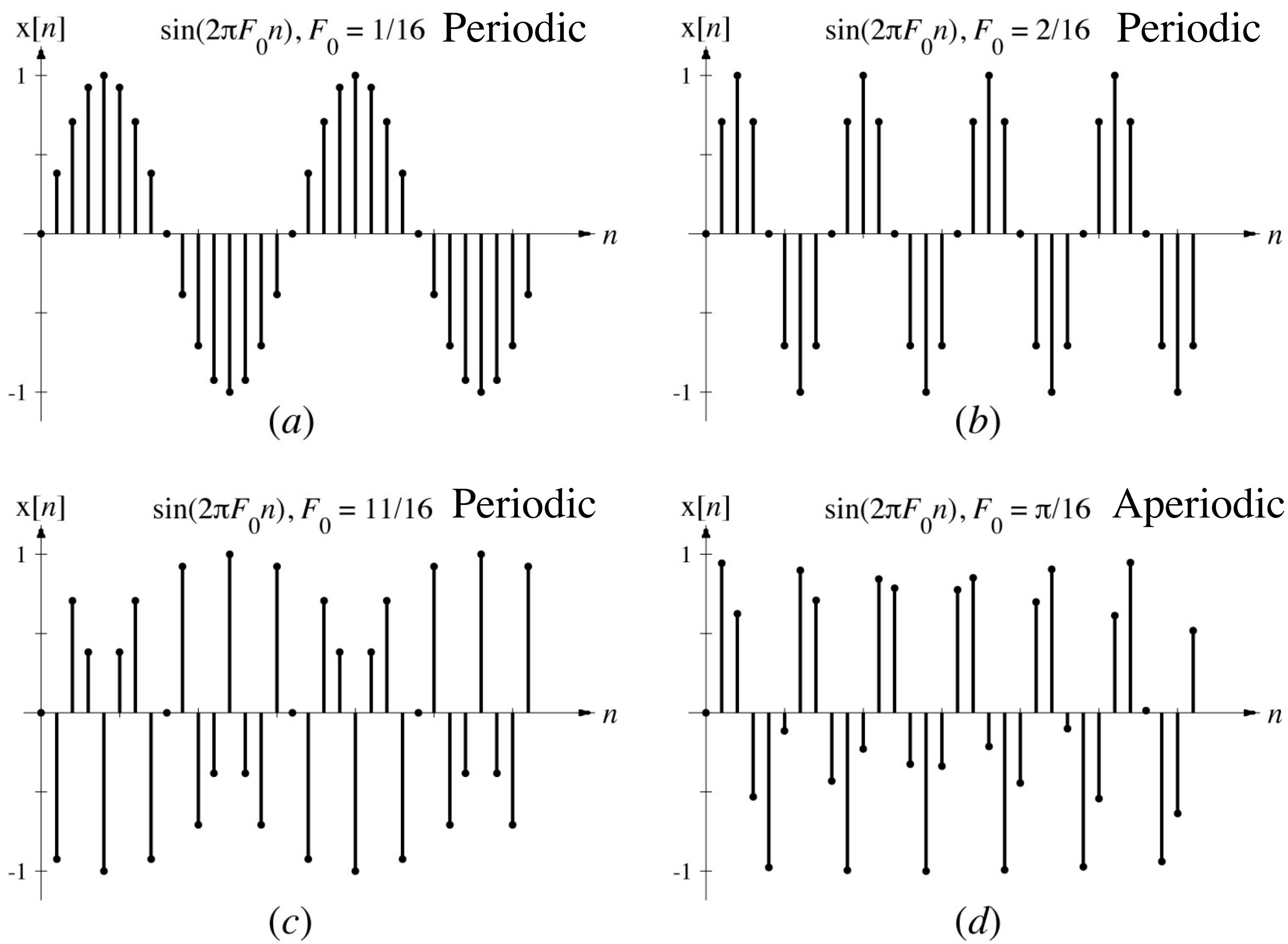
Sampling and Discrete Time



Sinusoids

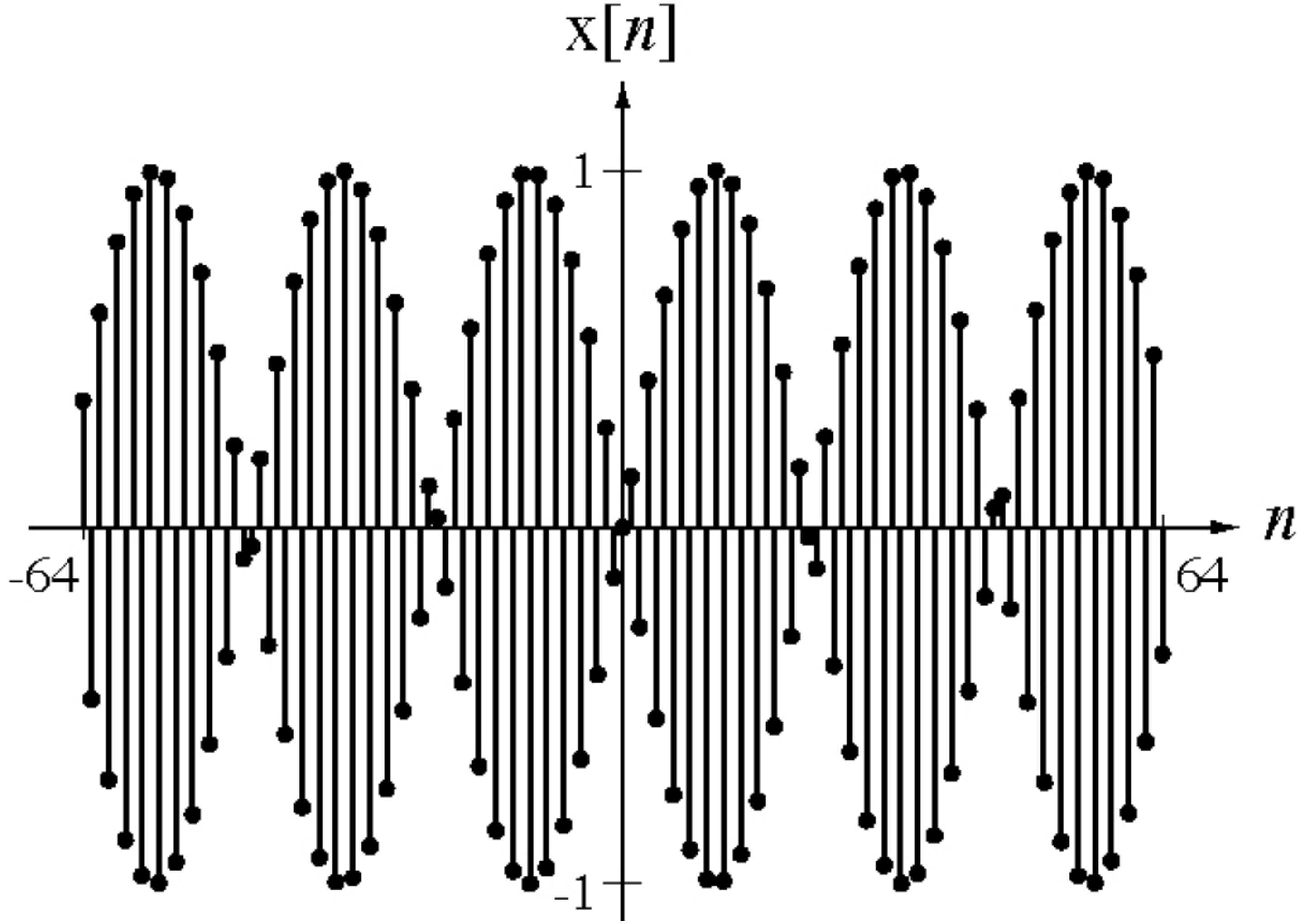
Unlike a continuous-time sinusoid, a discrete-time sinusoid is not necessarily periodic. If it is periodic, its period must be an integer. If a sinusoid has the form $\mathrm{g}[n] = A\cos(2\pi F_0 n + \theta)$ then F_0 must be a ratio of integers (a rational number) for $\mathrm{g}[n]$ to be periodic. If F_0 is rational in the form q / N_0 $(q$ and N_0 integers$)$ in which all common factors in q and N_0 have already been cancelled, then the fundamental period of the sinusoid is N_0, not N_0 / q (unless $q = 1$). Therefore, the general form of a periodic sinusoid with fundamental period N_0 is $\mathrm{g}[n] = A\cos(2\pi nq / N_0 + \theta)$.

Sinusoids


x[n]
sin(2πF0n), F0 = 1/16 Periodic
1
-1
n
(a)
x[n]
sin(2πF0n), F0 = 2/16 Periodic
1
-1
n
(b)
x[n]
sin(2πF0n), F0 = 11/16 Periodic
1
-1
n
(c)
x[n]
sin(2πF0n), F0 = π/16 Aperiodic
1
-1
n
(d)

Sinusoids

An Aperiodic Sinusoid



Sinusoids

Two sinusoids whose analytical expressions look different,

$$g_1[n] = A\cos(2\pi F_{01} n + \theta) \quad \text{and} \quad g_2[n] = A\cos(2\pi F_{02} n + \theta)$$

may actually be the same. If

$$F_{02} = F_{01} + m, \text{ where } m \text{ is an integer}$$

then (because n is discrete time and therefore an integer),

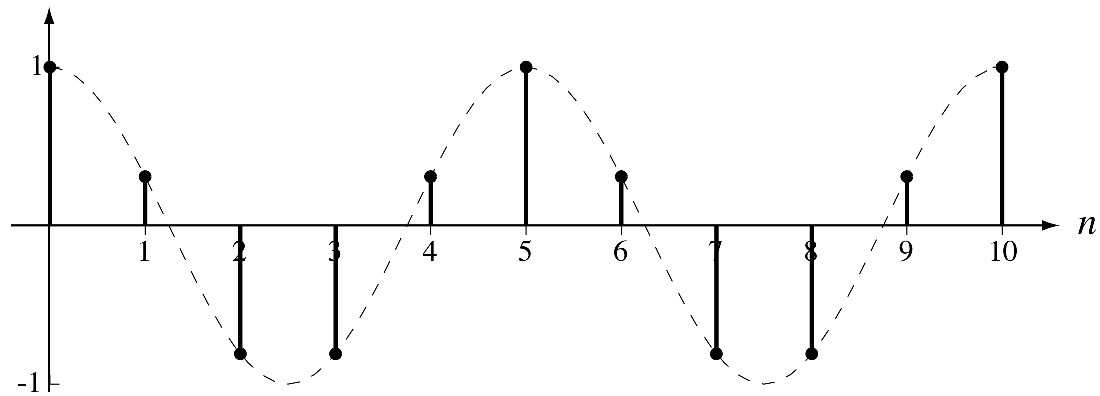
$$A\cos(2\pi F_{02} n + \theta) = A\cos(2\pi (F_{01} + m) n + \theta)$$

$$A\cos(2\pi F_{02} n + \theta) = A\cos\left(2\pi F_{01} n + \underbrace{2\pi m n}_{\substack{\text{Integer} \\ \text{Multiple} \\ \text{of } 2\pi}} + \theta\right) = A\cos(2\pi F_{01} n + \theta)$$

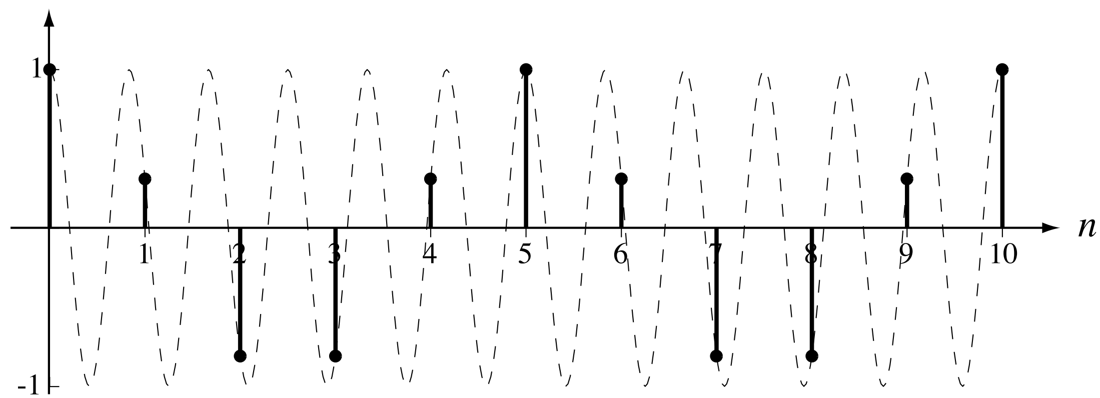
(Example on next slide)

Sinusoids

$g_1[n] = \cos\left(\frac{2\pi n}{5}\right)$

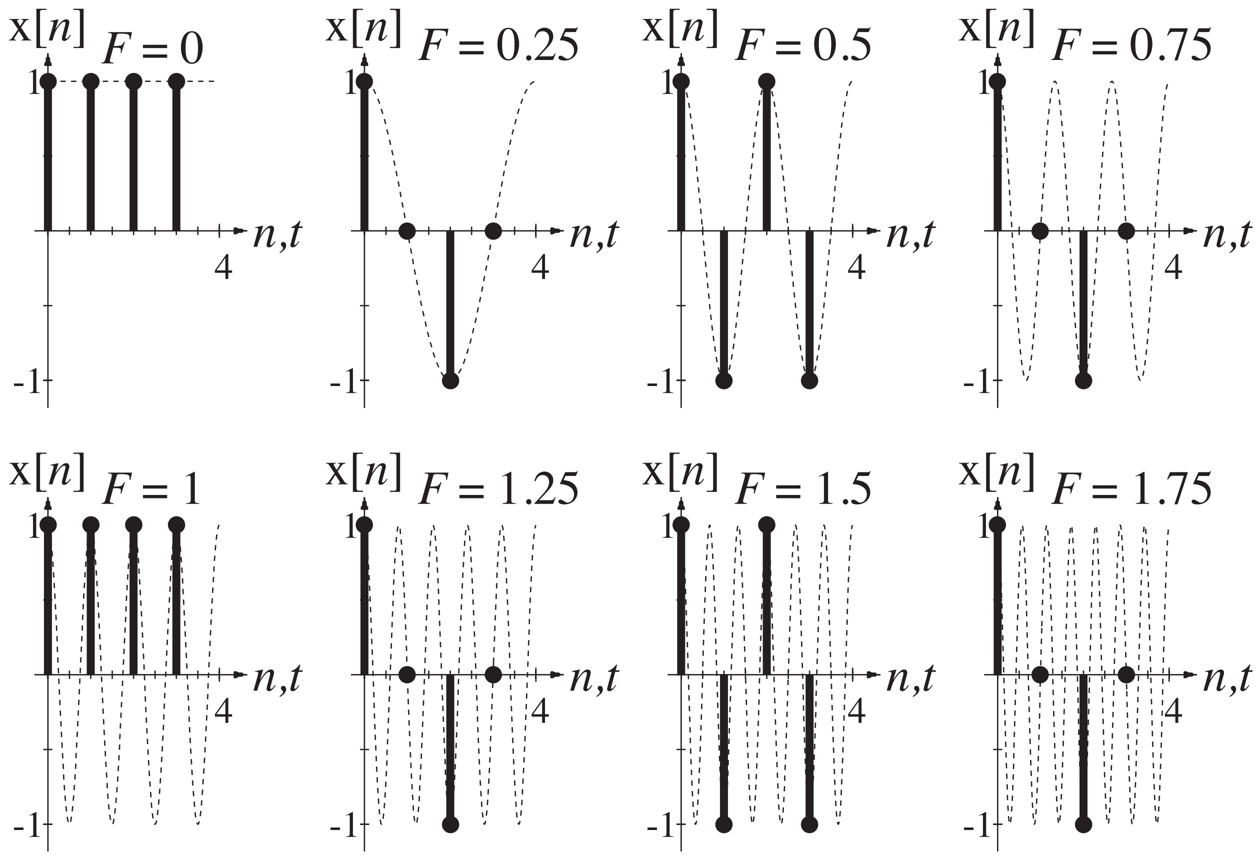


$g_2[n] = \cos\left(\frac{12\pi n}{5}\right)$



Sinusoids

$x[n] = \cos(2\pi F n)$ Dashed line is $x(t) = \cos(2\pi F t)$

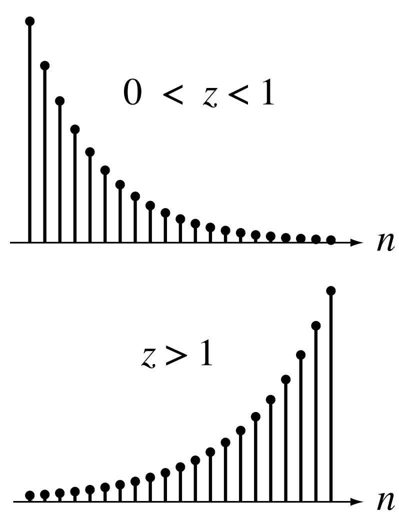


Exponentials

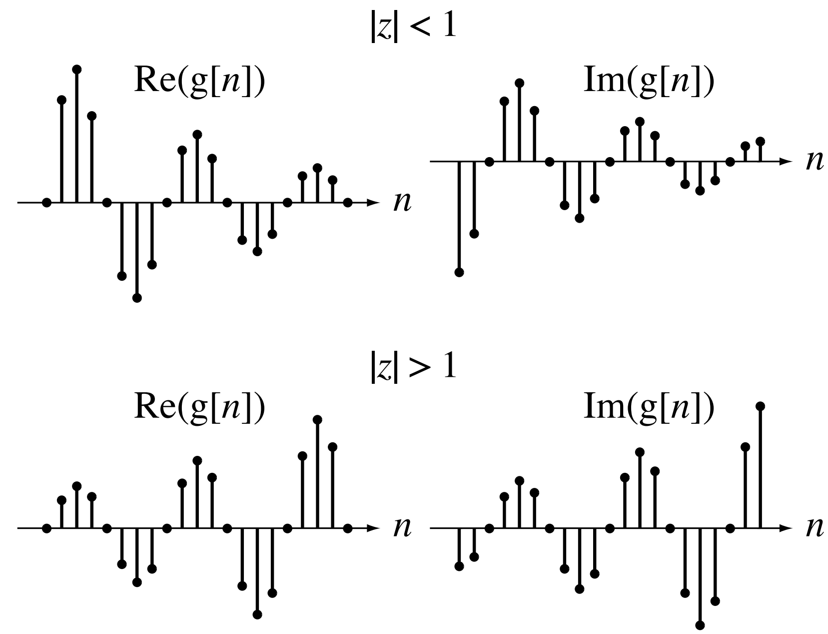
The form of the exponential is

$$\underbrace{\mathrm{x}[n] = Az^n}_{\text{Preferred}} \text{ or } \mathrm{x}[n] = Ae^{\beta n} \quad \text{where } z = e^{\beta}$$

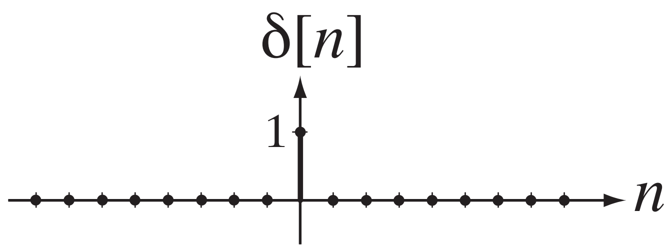
Real z



Complex z



The Unit Impulse Function



$$\delta[n] = \begin{cases} 1 \ , & n = 0 \\ 0 \ , & n \neq 0 \end{cases}$$

The discrete-time unit impulse (also known as the "**Kronecker delta function**") is a function in the ordinary sense (in contrast with the continuous-time unit impulse). It has a sampling property,

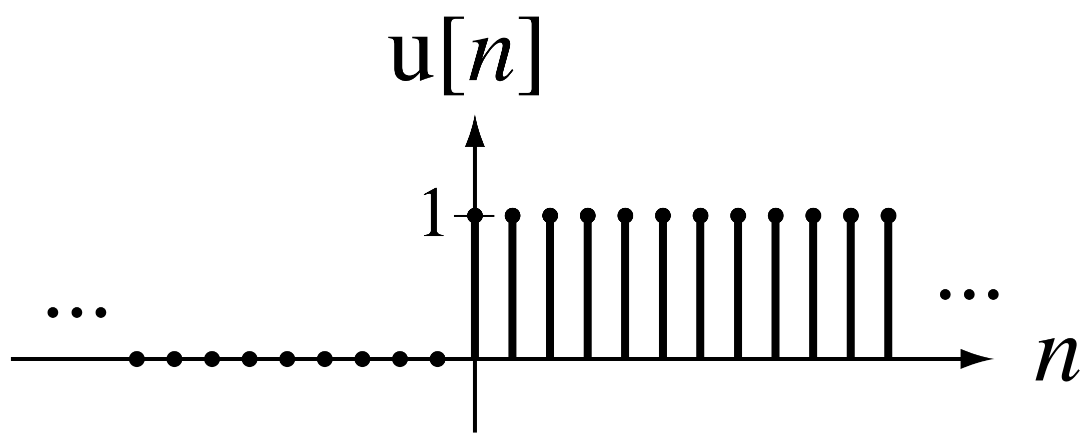
$$\sum_{n=-\infty}^{\infty} A\delta[n-n_0]\mathrm{x}[n] = A\,\mathrm{x}[n_0]$$

but no scaling property. That is,

$\delta[n] = \delta[an]$ for any non-zero, finite integer a.

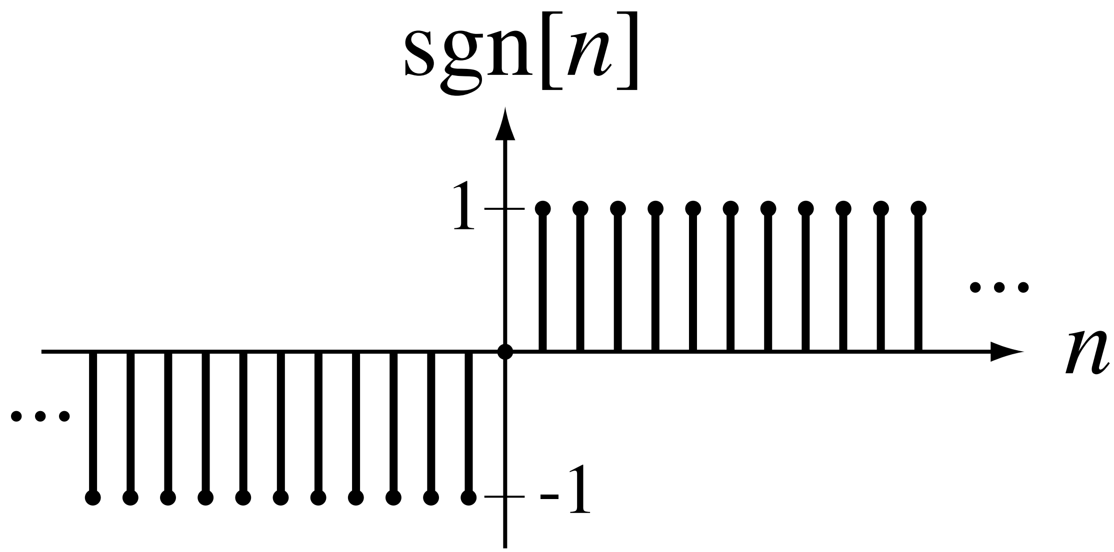
The Unit Sequence Function

$$\mathrm{u}[n] = \begin{cases} 1 \ , & n \geq 0 \\ 0 \ , & n < 0 \end{cases}$$



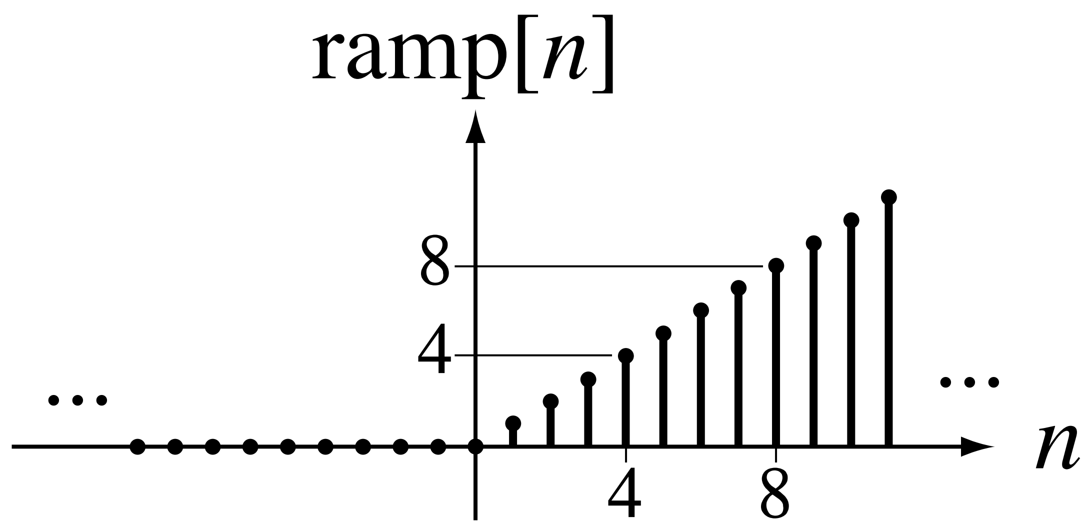
The Signum Function

$$\mathrm{sgn}[n] = \begin{cases} 1 & , \; n > 0 \\ 0 & , \; n = 0 \\ -1 & , \; n < 0 \end{cases} = 2\,\mathrm{u}[n] - \delta[n] - 1$$



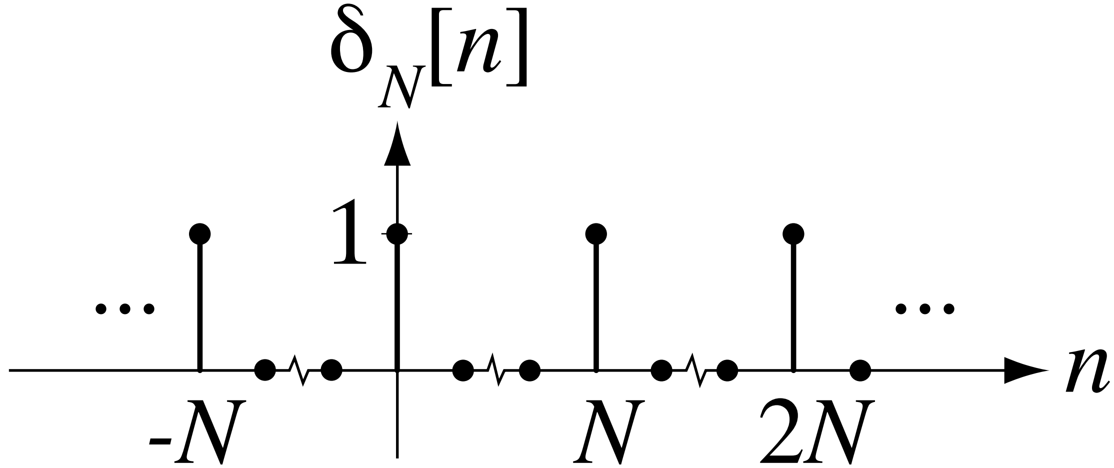
The Unit Ramp Function

$$\text{ramp}[n] = \begin{cases} n & , \ n \geq 0 \\ 0 & , \ n < 0 \end{cases} = n\,\text{u}[n] = \sum_{m=-\infty}^{n} \text{u}[m-1]$$



The Periodic Impulse Function

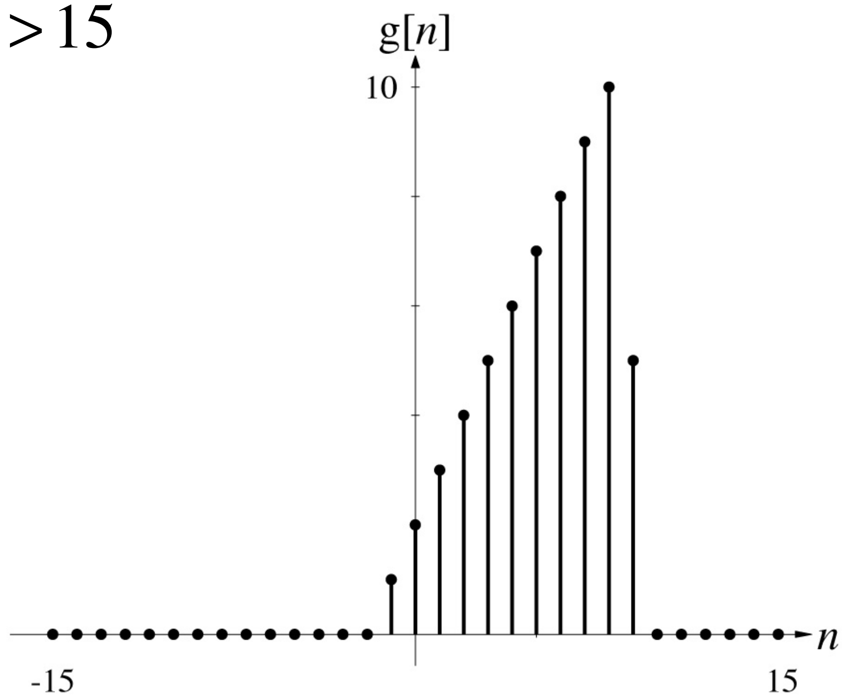
$$\delta_N[n] = \sum_{m=-\infty}^{\infty} \delta[n - mN]$$



Scaling and Shifting Functions

Let $\mathrm{g}[n]$ be graphically defined by the graph below and

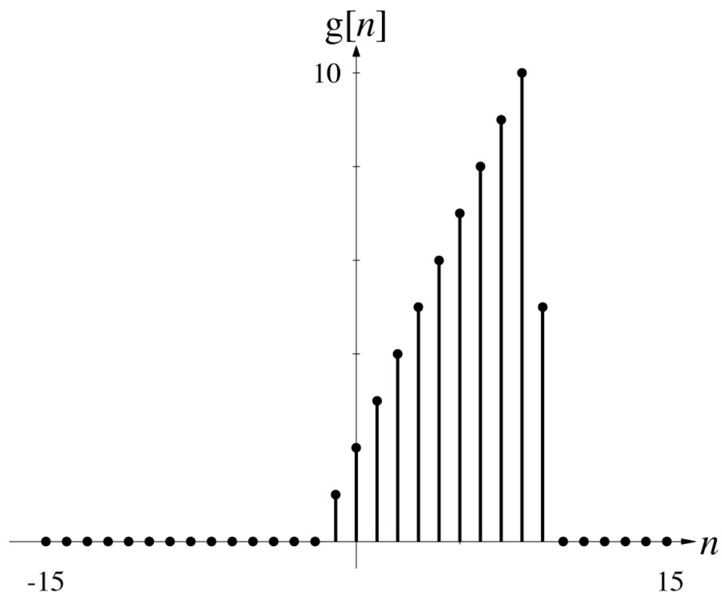
$\mathrm{g}[n] = 0 \ , |n| > 15$



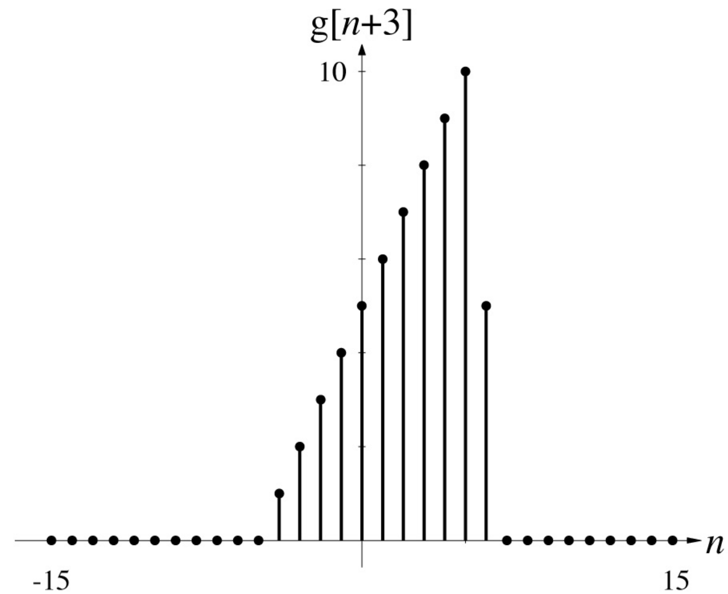
n	-1	0	1	2	3	4	5	6	7	8	9	10
$\mathrm{g}[n]$	1	2	3	4	5	6	7	8	9	10	5	0

Scaling and Shifting Functions

Time shifting $n \to n + n_0$, n_0 an integer



n	-1	0	1	2	3	4	5	6	7	8	9	10
$g[n]$	1	2	3	4	5	6	7	8	9	10	5	0



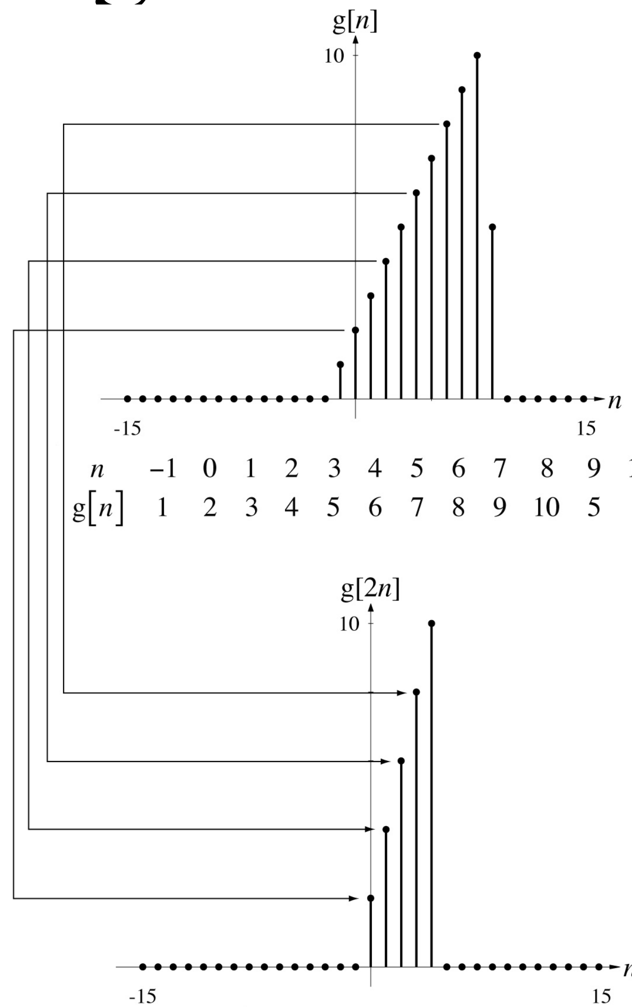
n	-4	-3	-2	-1	0	1	2	3	4	5	6	7
$n+3$	-1	0	1	2	3	4	5	6	7	8	9	10
$g[n+3]$	1	2	3	4	5	6	7	8	9	10	5	0

Scaling and Shifting Functions

Time compression

$$n \rightarrow Kn$$

K an integer > 1



n	-1	0	1	2	3	4	5	6	7	8	9	10
$g[n]$	1	2	3	4	5	6	7	8	9	10	5	0

n	0	1	2	3	4
$2n$	0	2	4	6	8
$g[2n]$	2	4	6	8	10

Scaling and Shifting Functions

Time expansion $n \to n / K, K > 1$

For all n such that n / K is an integer, $\mathrm{g}[n / K]$ is defined.

For all n such that n / K is not an integer, $\mathrm{g}[n / K]$ is <u>not defined</u>.

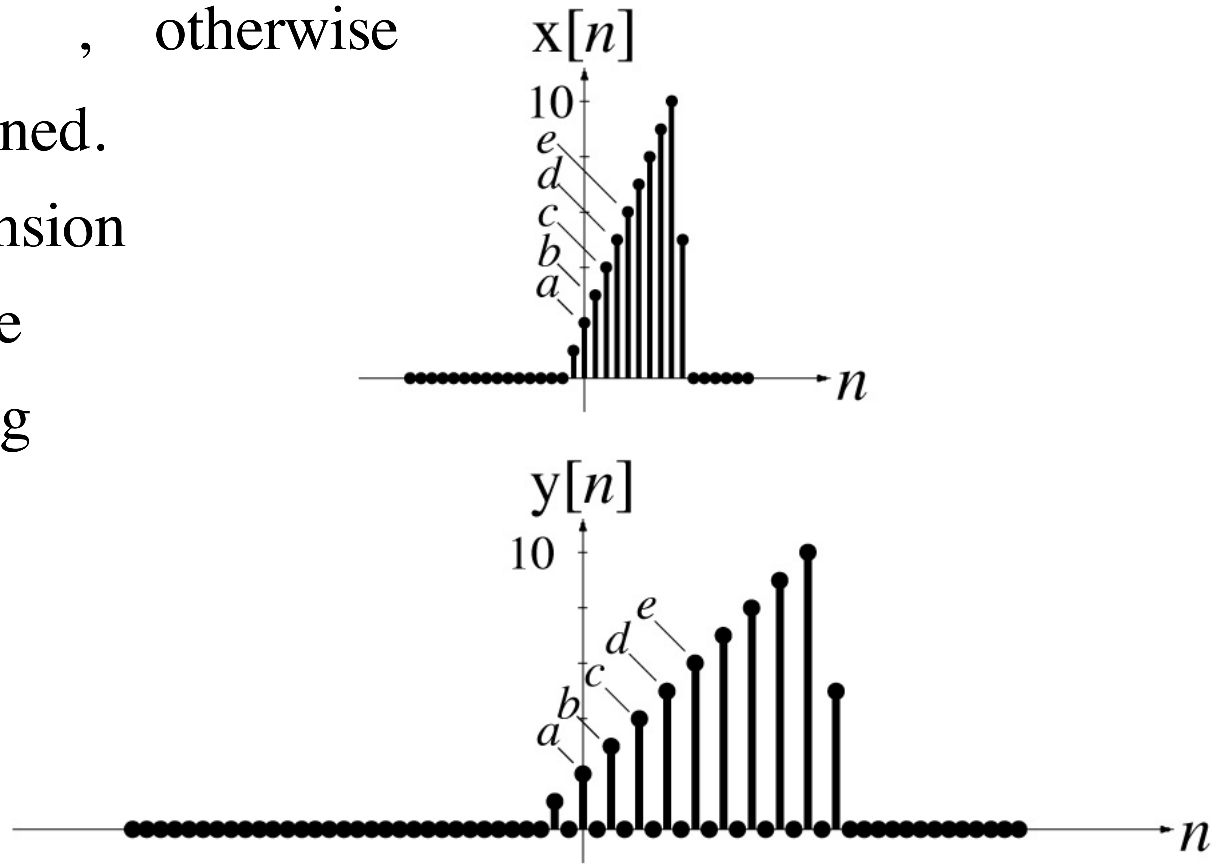
Scaling and Shifting Functions

There is a form of time expansion that is useful. Let

$$y[n] = \begin{cases} x[n/m] & , \quad n/m \text{ an integer} \\ 0 & , \quad \text{otherwise} \end{cases}$$

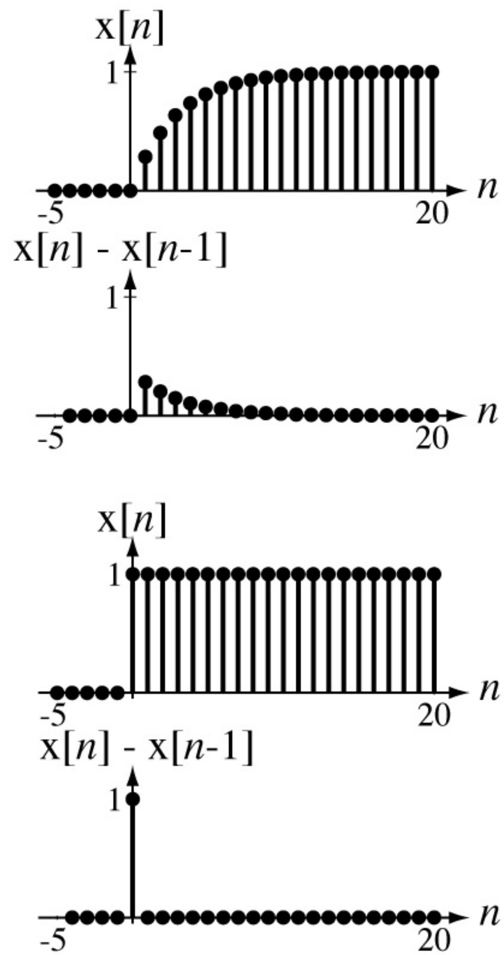
All values of y are defined.

This type of time expansion is actually used in some digital signal processing operations.

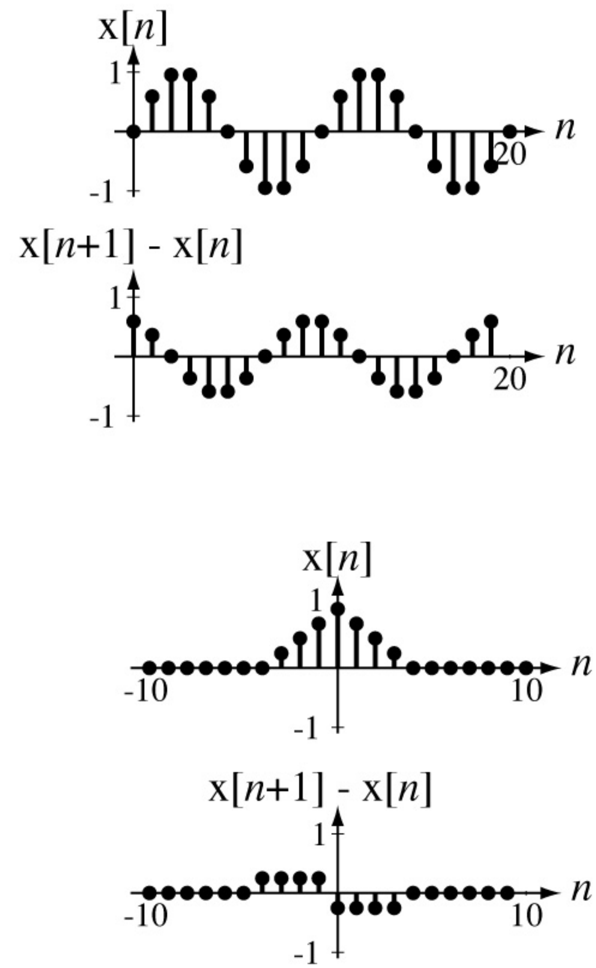


Differencing

Backward Differences

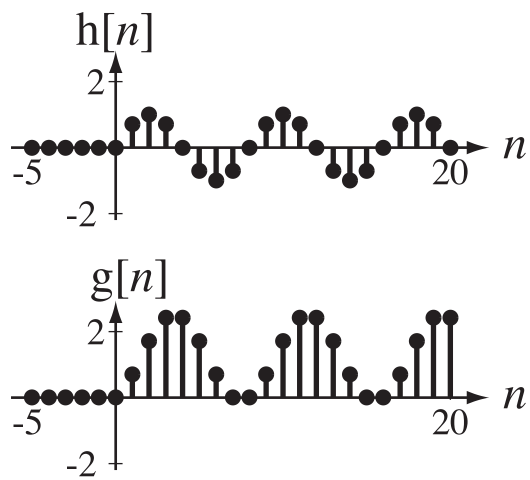


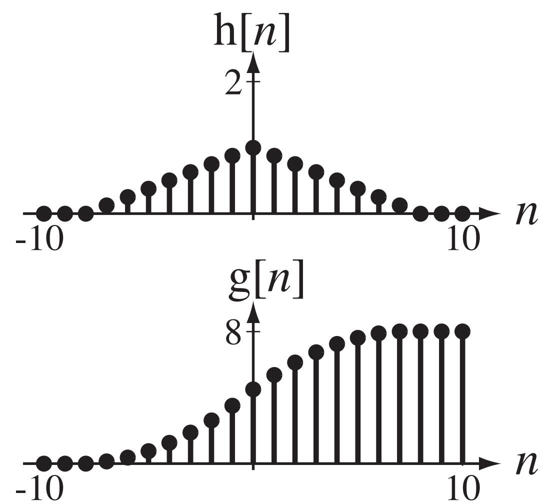
Forward Differences



Accumulation

$$g[n]=\sum_{m=-\infty}^{n} h[m]$$





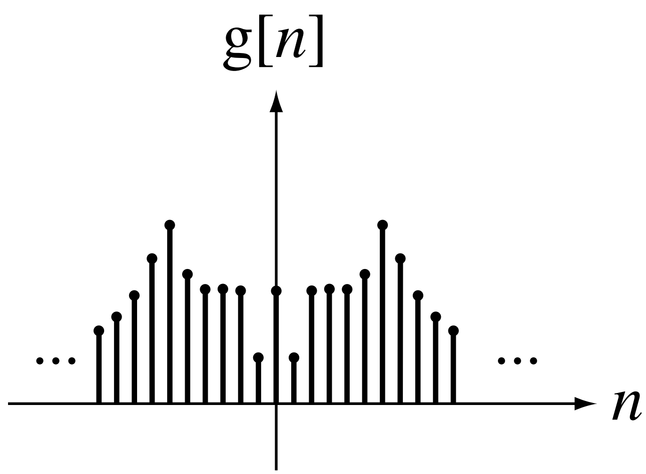
Even and Odd Signals

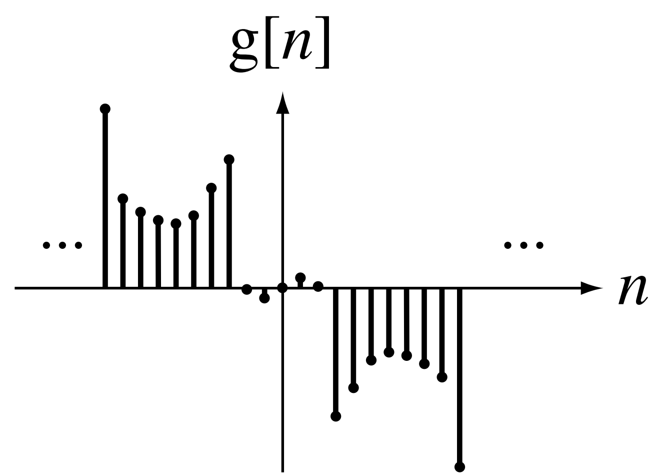
$$g[n] = g[-n]$$

$$g[n] = -g[-n]$$

Even Function

Odd Function



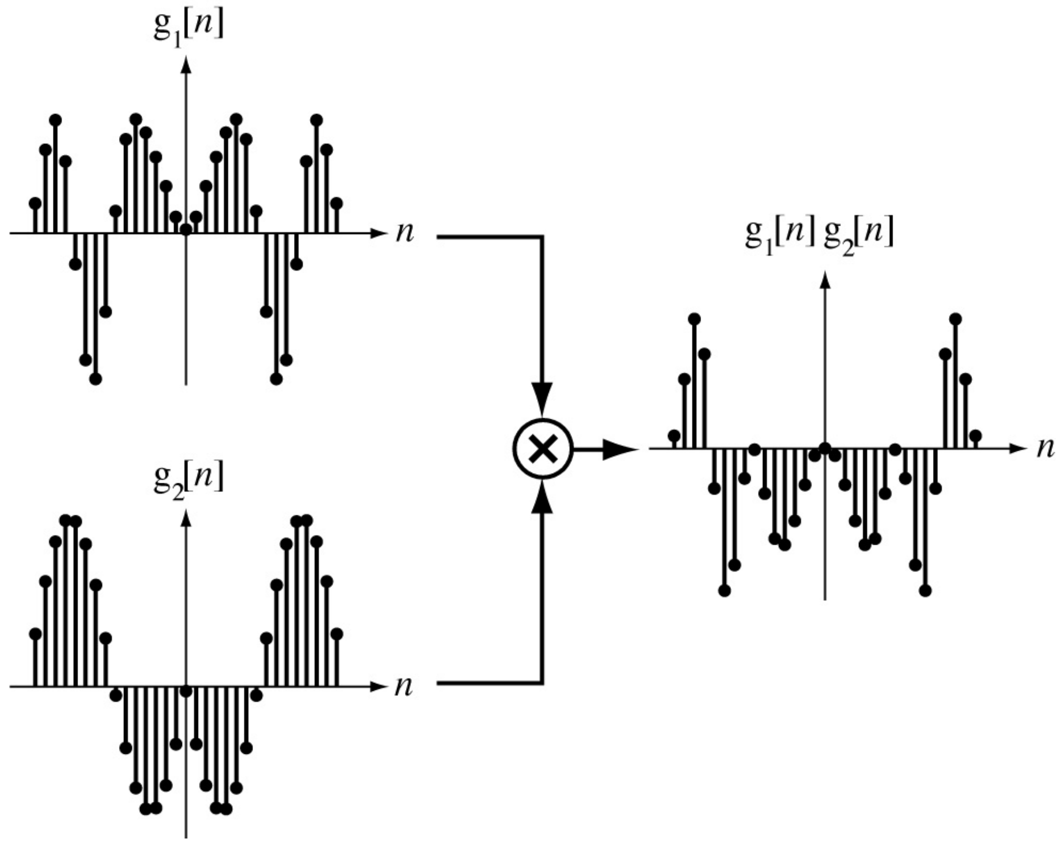


$$g_e[n] = \frac{g[n] + g[-n]}{2}$$

$$g_o[n] = \frac{g[n] - g[-n]}{2}$$

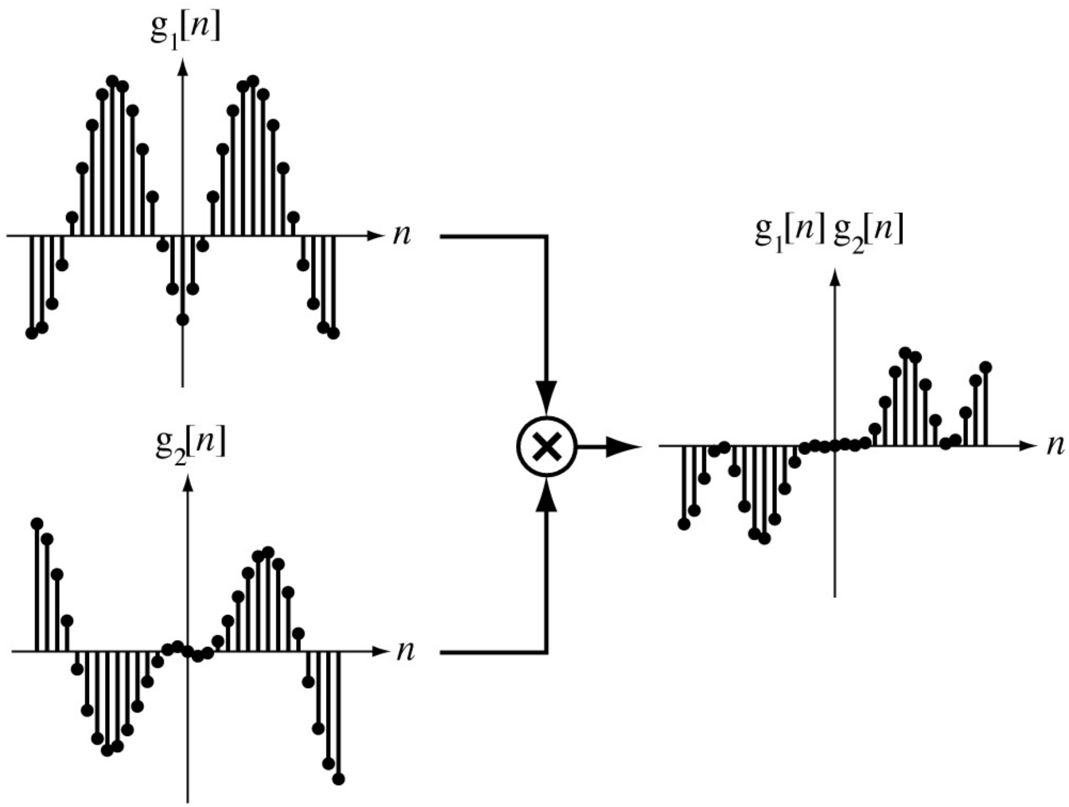
Products of Even and Odd Functions

Two Even Functions



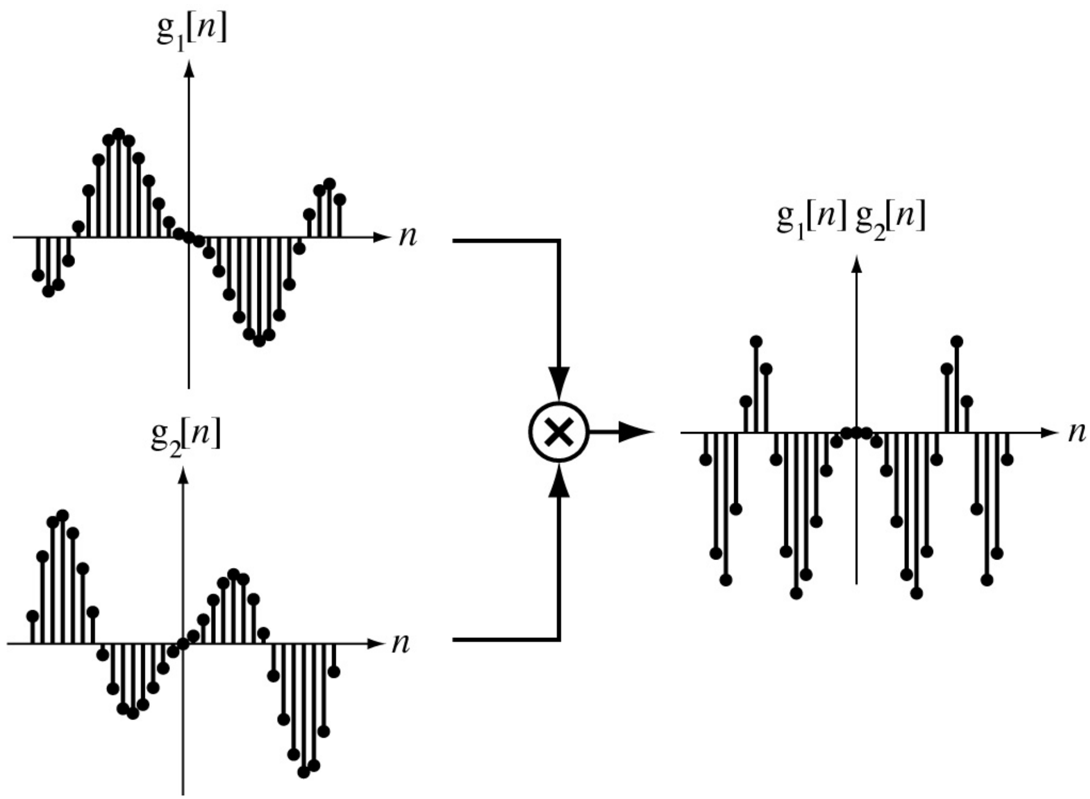
Products of Even and Odd Functions

An Even Function and an Odd Function



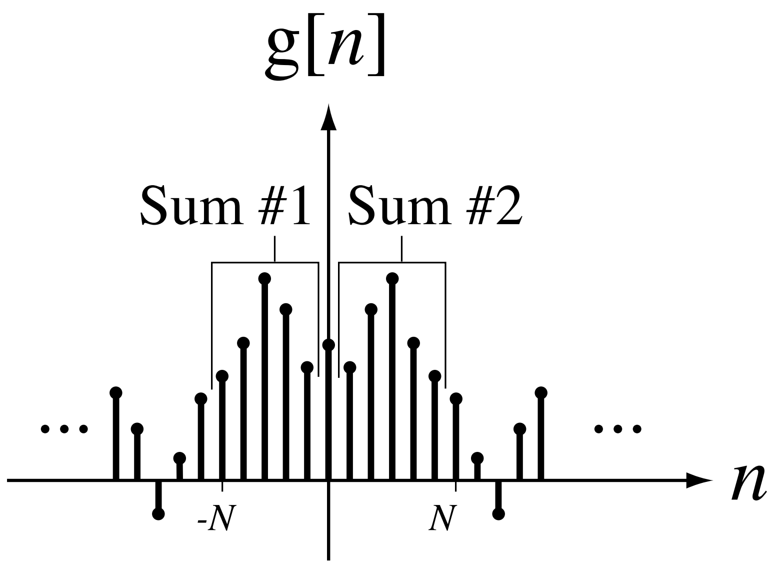
Products of Even and Odd Functions

Two Odd Functions



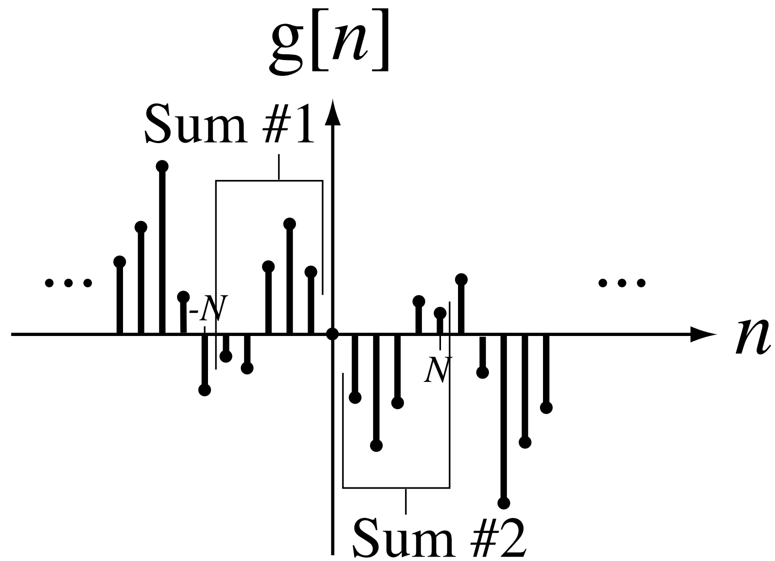
Symmetric Finite Summation

Even Function



Sum #1 = Sum #2

Odd Function



Sum #1 = - Sum #2

$$\sum_{n=-N}^{N} g[n] = g[0] + 2\sum_{n=1}^{N} g[n]$$

$$\sum_{n=-N}^{N} g[n] = 0$$

Periodic Functions

A **periodic** function is one that is invariant to the change of variable $n \to n + mN$ where N is a **period** of the function and m is any integer.

The minimum positive integer value of N for which $\mathrm{g}[n] = \mathrm{g}[n+N]$ is called the **fundamental period** N_0.

Periodic Functions

Find the fundamental period of

$$x[n] = \cos(\pi n / 18) + \sin(10\pi n / 24)$$

$$x[n] = \underbrace{\cos(2\pi n / 36)}_{N_0 = 36} + \underbrace{\sin(2\pi n(5 / 24))}_{N_0 = 24}$$

$$N_0 = \text{LCM}(36, 24) = 72$$

Find the fundamental period of

$$x[n] = \cos(5\pi n / 13) + \sin(8\pi n / 39)$$

$$x[n] = \underbrace{\cos(2\pi n(5 / 26))}_{N_0 = 26} + \underbrace{\sin(2\pi n(4 / 39))}_{N_0 = 39}$$

$$N_0 = \text{LCM}(26, 39) = 78$$

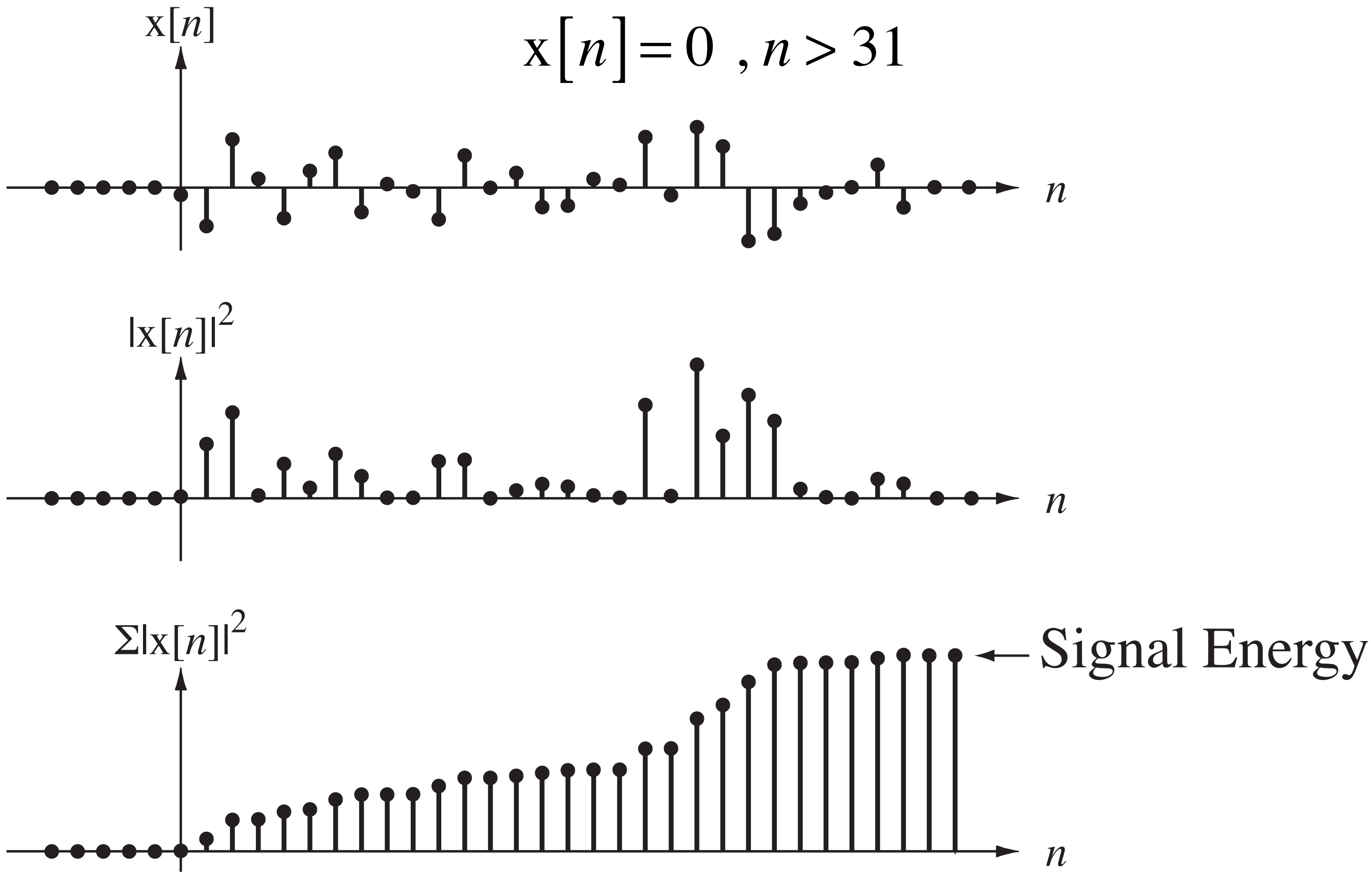
Signal Energy and Power

The signal energy of a signal $\mathrm{x}[n]$ is

$$E_{\mathrm{x}} = \sum_{n=-\infty}^{\infty} \left|\mathrm{x}[n]\right|^2$$

Signal Energy and Power

$$x[n] = 0 \;, n > 31$$



Signal Energy and Power

Find the signal energy of

$$\mathrm{x}[n] = \begin{cases} (5/3)^{2n} & , \ 0 \le n < 8 \\ 0 & , \ \text{otherwise} \end{cases}$$

$$E_{\mathrm{x}} = \sum_{n=-\infty}^{\infty} \left|\mathrm{x}[n]\right|^2 = \sum_{n=0}^{7} \left[(5/3)^{2n}\right]^2 = \sum_{n=0}^{7} \left[(5/3)^4\right]^n$$

Using $\sum_{n=0}^{N-1} r^n = \begin{cases} N & , \ r = 1 \\ \dfrac{1-r^N}{1-r} & , \ r \ne 1 \end{cases}$

$$E_{\mathrm{x}} = \frac{1-\left[(5/3)^4\right]^8}{1-(5/3)^4} \cong 1.871 \times 10^6$$

Signal Energy and Power

Some signals have infinite signal energy. In that case It is usually more convenient to deal with average signal power. The average signal power of a signal $\mathrm{x}[n]$ is

$$P_{\mathrm{x}} = \lim_{N\to\infty} \frac{1}{2N} \sum_{n=-N}^{N-1} \left|\mathrm{x}[n]\right|^2$$

For a periodic signal $\mathrm{x}[n]$ the average signal power is

$$P_{\mathrm{x}} = \frac{1}{N} \sum_{n=\langle N\rangle} \left|\mathrm{x}[n]\right|^2$$

(The notation $\sum_{n=\langle N\rangle}$ means the sum over any set of consecutive n's exactly N in length.)

Signal Energy and Power

Find the average signal power of

$$\mathrm{x}[n] = 2\,\mathrm{sgn}[n] - 4$$

$$P_{\mathrm{x}} = \lim_{N \to \infty} \frac{1}{2N} \sum_{n=-N}^{N-1} \left|\mathrm{x}[n]\right|^2 = \lim_{x \to \infty} \frac{1}{2N} \sum_{n=-N}^{N-1} \left|2\,\mathrm{sgn}[n] - 4\right|^2$$

$$P_{\mathrm{x}} = \lim_{N \to \infty} \frac{1}{2N} \left\{ \underbrace{4 \sum_{n=-N}^{N-1} \underbrace{\mathrm{sgn}^2[n]}_{=1-\delta[n]}}_{=N-1+N=2N-1} + 16 \underbrace{\sum_{n=-N}^{N-1}}_{=2N} 1 - 16 \underbrace{\sum_{n=-N}^{N-1} \mathrm{sgn}[n]}_{=N-1-N=-1} \right\}$$

$$P_{\mathrm{x}} = \lim_{N \to \infty} \frac{1}{2N} \left\{ 4(2N-1) + 32N - 16(-1) \right\} = 20$$

Signal Energy and Power

A signal with finite signal energy is called an **energy signal**.

A signal with infinite signal energy and finite average signal power is called a **power signal**.

Fundamental Period of a Sum of Two Periodic Signals

$$\underbrace{\underbrace{\delta_{14}[n]}_{N_0=14}-6\underbrace{\delta_8[n]}_{N_0=8}}_{N_0=\text{LCM}(14,8)=56}$$

$$-2\cos(3\pi n/12)+11\cos(14\pi n/10)=\underbrace{-2\underbrace{\cos(2\pi n(1/8))}_{N_0=8}+11\underbrace{\cos(2\pi n(7/10))}_{N_0=10}}_{N_0=\text{LCM}(8,10)=40}$$

Impulses and Periodic Impulses

$$\sum_{n=-18}^{33} 38n^2\delta[n+6]=1368 \quad , \quad \sum_{n=-4}^{7} -12(0.4)^n \mathrm{u}[n]\delta_3[n]=-12\left[(0.4)^0+(0.4)^3+(0.4)^6\right]=-12.8172$$

Equivalence Property $27(0.3)^n\,\delta[n-3]=27(0.3)^3\,\delta[n-3]=0.729\delta[n-3]$

Scaling Property $13\delta[3n]=13\delta[n]$, (No scaling property for discrete-time impulses)

$$22\delta_3[4n]=22\sum_{k=-\infty}^{\infty}\delta[4n-3k]=\begin{Bmatrix}22 & , & 4n=3k \\ 0 & , & \text{otherwise}\end{Bmatrix}=\begin{Bmatrix}22 & , & 4n/3=k \\ 0 & , & \text{otherwise}\end{Bmatrix}$$

Since k is an integer, impulses occur only where $4n/3$ is an integer

n	0	1	2	3	4	$\cdots$
$22\delta_3[4n]$	$22(k=0)$	$0(4/3\neq k)$	$0(8/3\neq k)$	$22(k=1)$	$0(16/3\neq k)$	$\cdots$

Signal Energy and Signal Power

$$\mathrm{x}[n] = n(-1.3)^n \left(\mathrm{u}[n] - \mathrm{u}[n-4]\right) \Rightarrow E_{\mathrm{x}} = \sum_{n=-\infty}^{\infty} \left| n(-1.3)^n \left(\mathrm{u}[n] - \mathrm{u}[n-4]\right)\right|^2$$

$$E_{\mathrm{x}} = \sum_{n=0}^{3} n^2 (1.3)^{2n} = 0 + 1.3^2 + 4 \times 1.3^4 + 9 \times 1.3^6 = 56.5557$$

$\mathrm{x}[n]$ is periodic and one period of $\mathrm{x}[n]$ is described by

$$\mathrm{x}[n] = n(1-n) \;,\; 3 \le n < 6$$

$$P_{\mathrm{x}} = \frac{1}{N} \sum_{n=\langle N \rangle} \left|\mathrm{x}[n]\right|^2 \;,$$

n	3	4	5
$\mathrm{x}[n]$	-6	-12	-20

$$P_{\mathrm{x}} = \frac{1}{3}[36 + 144 + 400] = \frac{580}{3} = 193.333\cdots$$