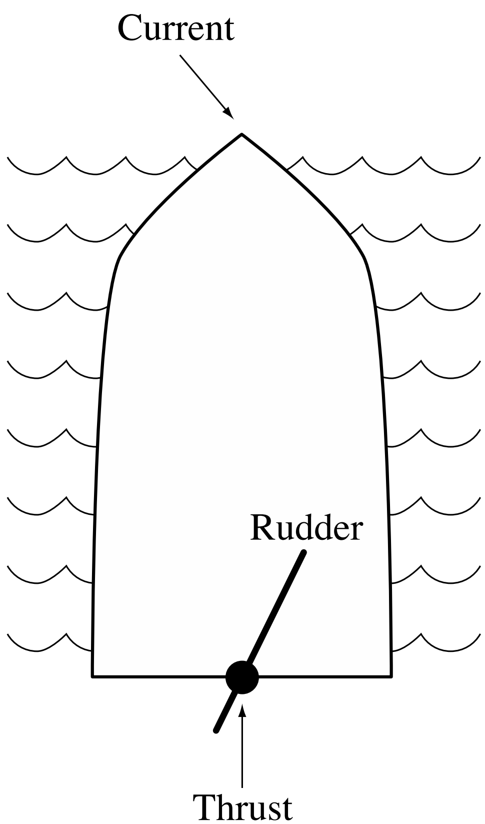
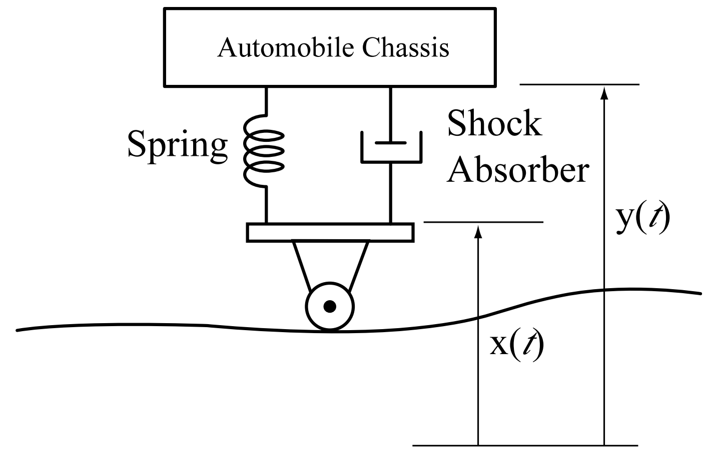
Description of Systems

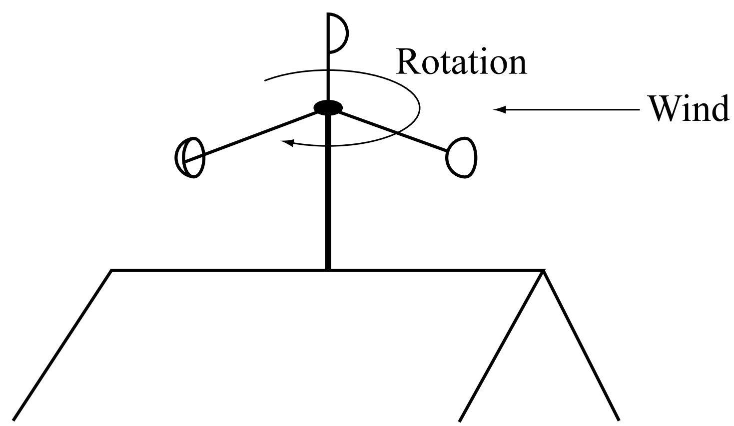
Systems

- Broadly speaking, a system is anything that responds when stimulated or excited
- The systems most commonly analyzed by engineers are artificial systems designed and built by humans
- Engineering system analysis is the application of mathematical methods to the design and analysis of systems

System Examples







Modeling a Mechanical System

A man, 1.8 m tall and weighing 80 kg, bungee jumps off a high bridge over a river. The bridge is 200 m above the water surface and the unstretched bungee cord is 30 m long. The spring constant of the bungee cord is $K_s = 11$ N/m, meaning that, when the cord is stretched, it resists the stretching with a force of 11 newtons per meter of stretch. In the first 30m of free fall,

$$\text{Velocity, } \mathrm{v}(t) = gt \qquad \text{Position, } \mathrm{x}(t) = gt^2/2$$

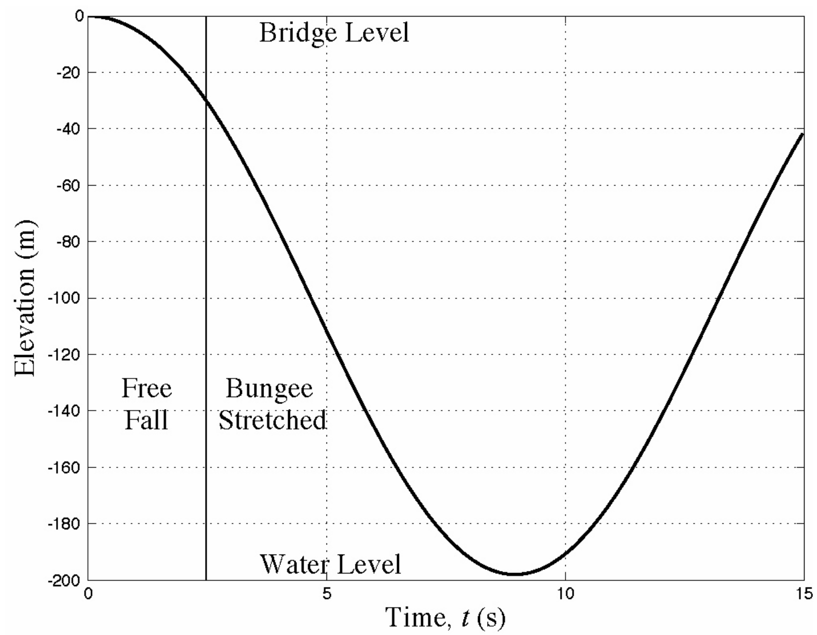
When the bungee cord begins to stretch the man experiences the force of the cord pulling him up (he hopes) and the farther he falls the stronger the upward pull of the cord. Will his hair get wet?

Modeling a Mechanical System

$$\left\{\begin{array}{c}\text{Sum of forces equals}\\ \text{mass times acceleration}\end{array}\right\} \longrightarrow mg - K_s\left(x(t) - 30\right) = m x''(t)$$

$$x(t) = x_h(t) + x_p(t) = K_{h1}\sin\left(\sqrt{K_s/m}\;t\right) + K_{h2}\cos\left(\sqrt{K_s/m}\;t\right) + \underbrace{\frac{mg}{K_s} + 30}_{K_p}$$

$$x(t) = -16.85\sin(0.3708t) - 95.25\cos(0.3708t) + 101.3 \quad , \quad t > 2.47$$



Modeling a Mechanical System

In the **modeling** of this system many physical processes were left out of the model.

1. Air resistance,
2. Energy dissipation in the bungee cord,
3. Horizontal components of the man's velocity,
4. Rotation of the man during the fall,
5. Variation of the acceleration due to gravity as a function of position,

and 6. Variation of the water level in the river .

Leaving out these processes makes the model less accurate but <u>much</u> easier to use and the errors introduced are insignificant.

Modeling a Fluid System

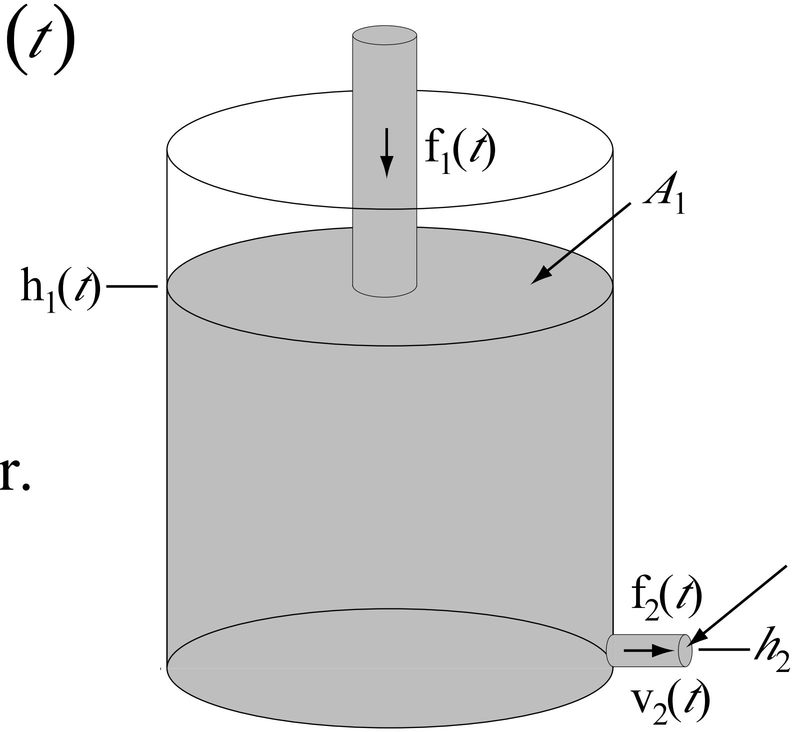
Toricelli's Equation: $\mathrm{v}_2(t) = \sqrt{2g\left[\mathrm{h}_1(t) - h_2\right]}$

where v is water velocity and h is water level

$$\underbrace{A_1 \frac{d}{dt}\left(\mathrm{h}_1(t)\right)}_{\text{Rate of Increase of Water Volume}} + \underbrace{A_2\sqrt{2g\left[\mathrm{h}_1(t) - h_2\right]}}_{\text{Volumetric Outflow Rate}} = \mathrm{f}_1(t)$$

This differential equation is non-linear.

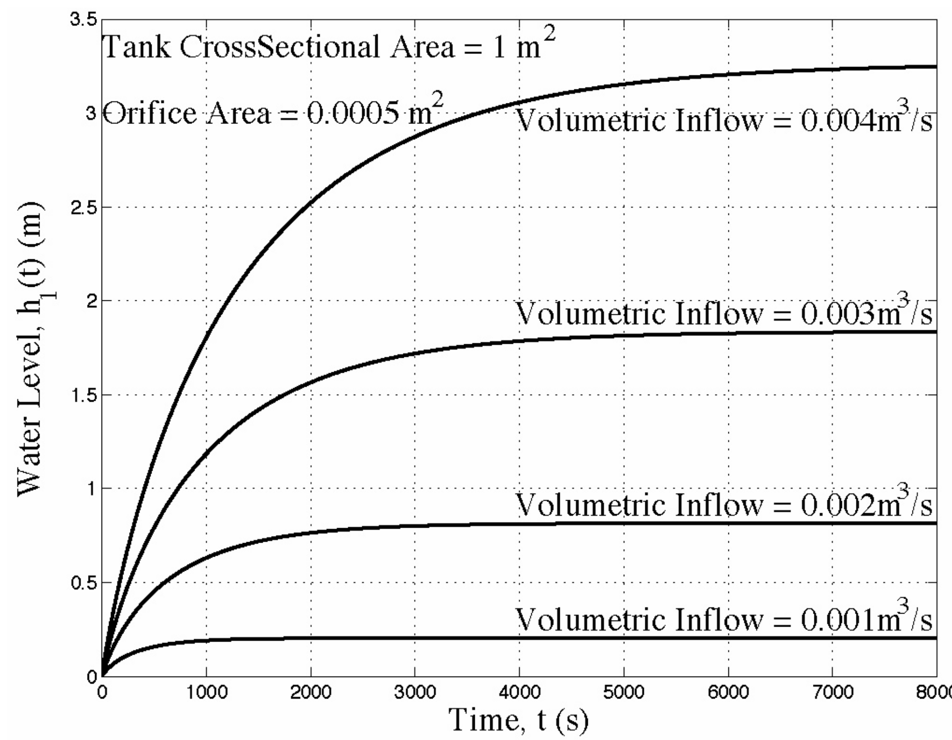
Input Flow



Output Flow

Modeling a Fluid System

This graph shows the water level as a function of time for four different volumetric inflow rates assuming the tank is initially empty. Notice that when the inflow rate is doubled, the final water level is quadrupled. This is a consequence of the **non-linearity** of the **differential equation**. Differential equations that are difficult or impossible to solve analytically can be solved **numerically**.



Feedback Systems

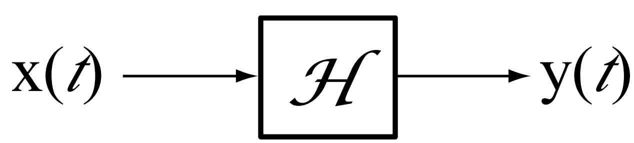
In a **feedback** system the response of the system is "fed back" and combined with the excitation is such a way as to optimize the response in some desired sense. Examples of feedback systems are

1. Temperature control in a house using a thermostat
3. Water level control in the tank of a flush toilet.
5. Pouring a glass of lemonade to the top of the glass without overflowing.
4. A refrigerator ice maker, which keeps the bin full of ice but does not make extra ice.
5. Driving a car.

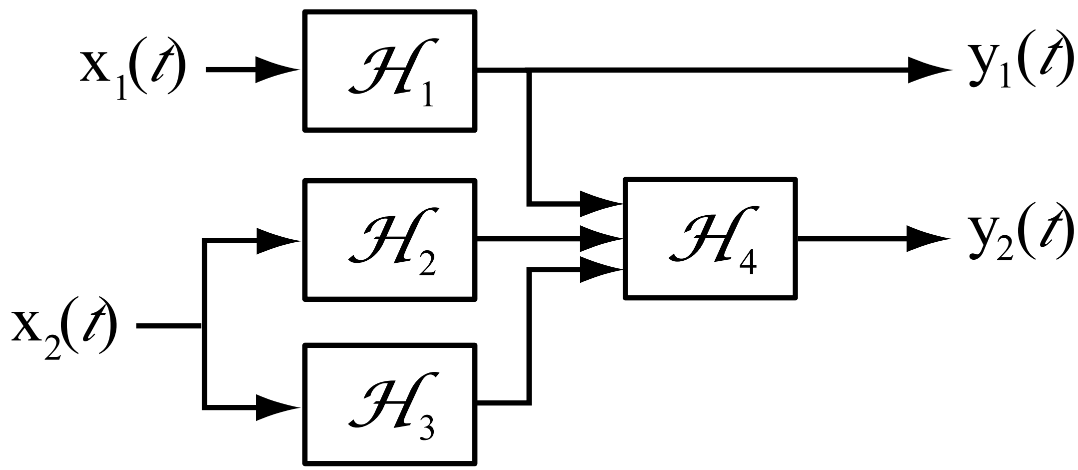
Systems

- Systems have **inputs** and **outputs**
- Systems accept **excitations** or **input signals** at their inputs and produce **responses** or **output signals** at their outputs
- Systems are often usefully represented by **block diagrams**

A single-input, single-output system block diagram

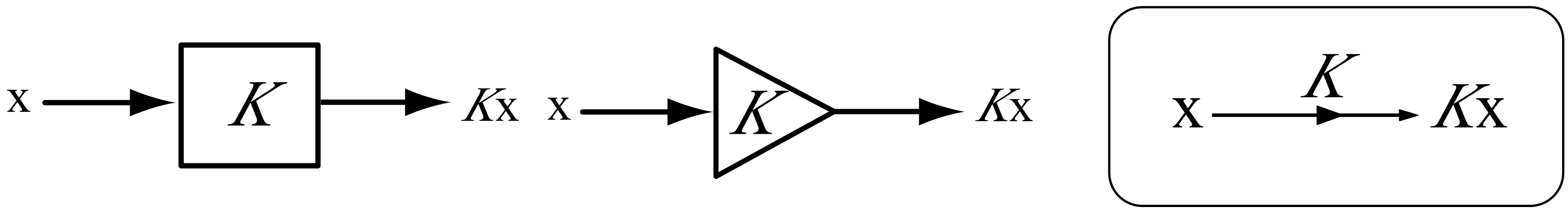


A Multiple-Input, Multiple-Output System Block Diagram

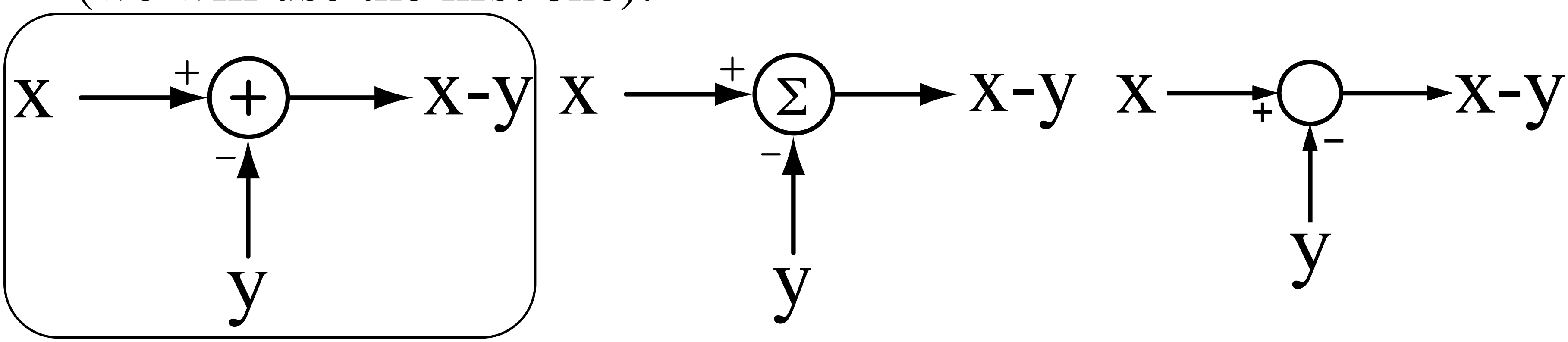


Block Diagram Symbols

Three common block diagram symbols for an **amplifier** (we will use the last one).

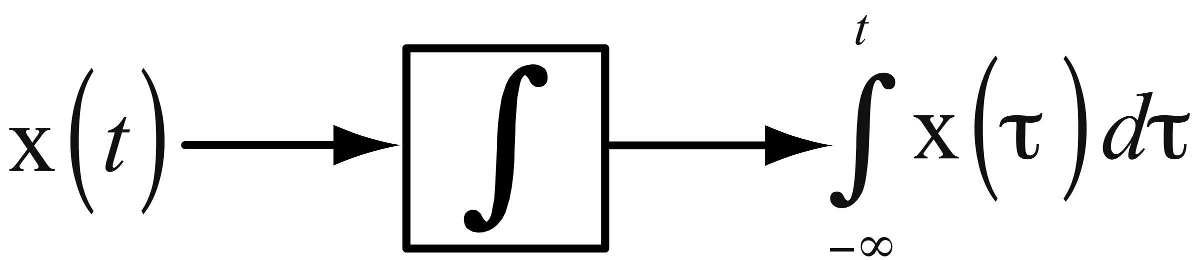


Three common block diagram symbols for a **summing junction** (we will use the first one).



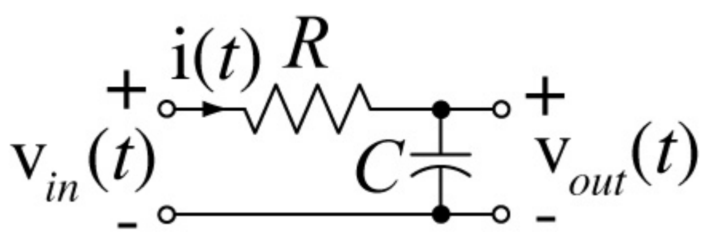
Block Diagram Symbols

Block diagram symbol for an **integrator**



An Electrical Circuit Viewed as a System

An RC **lowpass filter** is a simple electrical system. It is excited by a voltage $v_{in}(t)$ and responds with a voltage $v_{out}(t)$. It can be viewed or modeled as a single-input, single-output system



Zero-State Response of an *RC* Lowpass Filter to a Step Excitation

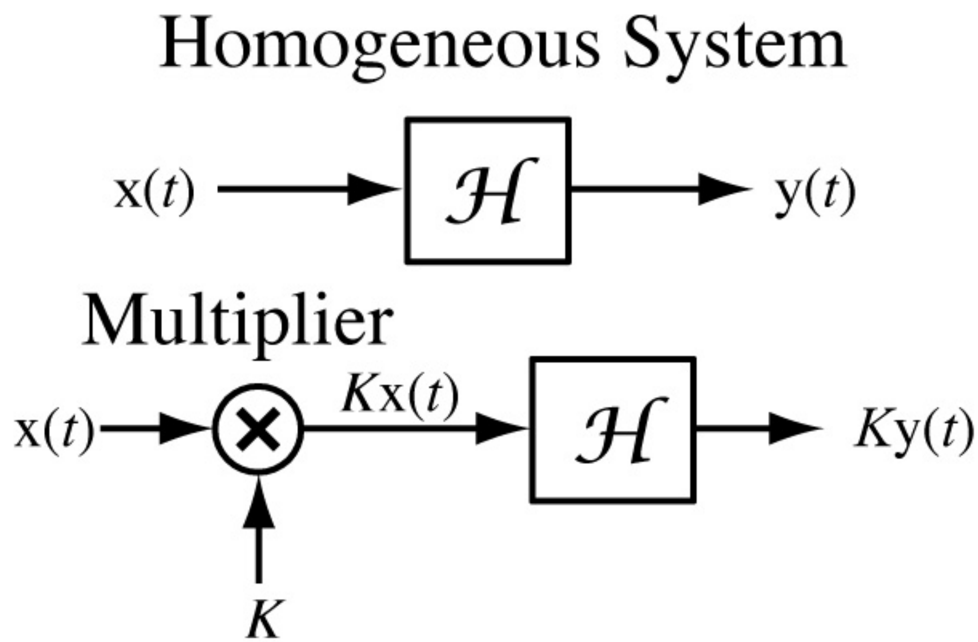
If an *RC* lowpass filter with an initially uncharged capacitor is excited by a step of voltage $v_{in}(t) = A\,u(t)$ its response is $v_{out}(t) = A\left(1 - e^{-t/RC}\right)u(t)$. This response is called the **zero-state** response of this system because there was initially no energy stored in the system. (It was in its zero-energy state.) If the excitation is doubled, the zero-state response also doubles.

Zero-Input Response of an *RC* Lowpass Filter

If an *RC* lowpass filter has an initial charge on the capacitor of V_0 volts and no excitation is applied to the system its **zero-input** response is $\mathrm{v}_{out}(t) = V_0 e^{-t/RC}$, $t > 0$.

Homogeneity

- In a **homogeneous** system, multiplying the excitation by any constant (including complex constants), multiplies the zero-state response by the same constant.



Homogeneity

To test a system for **homogeneity** use this logical process. Apply a signal $\mathrm{g}(t)$ as the excitation $\mathrm{x}_1(t)$ and find the zero-state response $\mathrm{y}_1(t)$. Then apply the signal $K\,\mathrm{g}(t)$ as $\mathrm{x}_2(t)$ where K is a constant and find the zero-state response $\mathrm{y}_2(t)$. If $\mathrm{y}_2(t) = K\,\mathrm{y}_1(t)$ for any arbitrary $\mathrm{g}(t)$ and K, then the system is homogeneous.

If $\mathrm{g}(t) \xrightarrow{\mathcal{H}} \mathrm{y}_1(t)$ and $K\,\mathrm{g}(t) \xrightarrow{\mathcal{H}} K\,\mathrm{y}_1(t)$

$\mathcal{H}$ is Homogeneous

Homogeneity

Let $\mathrm{y}(t) = \exp(\mathrm{x}(t))$. Is this system homogeneous?

Let $\mathrm{x}_1(t) = \mathrm{g}(t)$. Then $\mathrm{y}_1(t) = \exp(\mathrm{g}(t))$.

Let $\mathrm{x}_2(t) = K\mathrm{g}(t)$. Then $\mathrm{y}_2(t) = \exp(K\mathrm{g}(t)) = \left[\exp(\mathrm{g}(t))\right]^K$

$K\mathrm{y}_1(t) = K\exp(\mathrm{g}(t)) \Rightarrow \mathrm{y}_2(t) \neq K\mathrm{y}_1(t)$, Not homogeneous

Let $\mathrm{y}(t) = \mathrm{x}(t) + 2$. Is this system homogeneous?

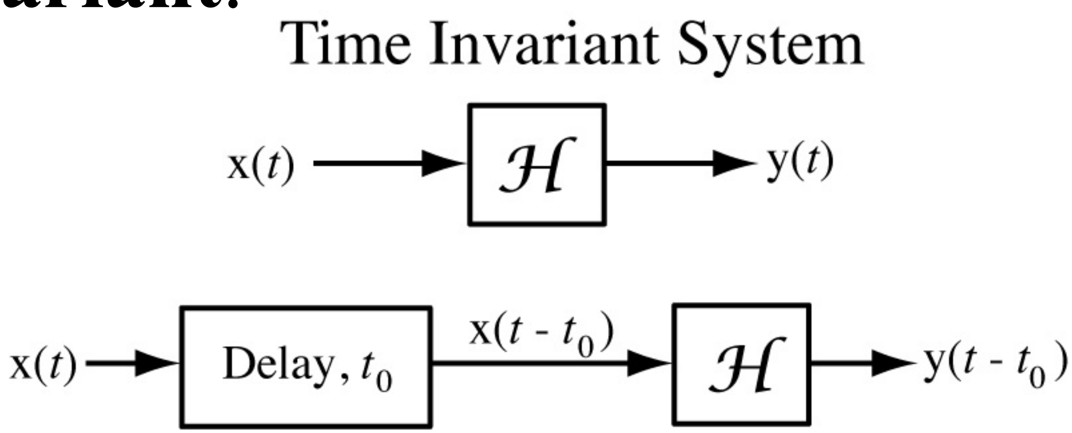
Let $\mathrm{x}_1(t) = \mathrm{g}(t)$. Then $\mathrm{y}_1(t) = \mathrm{g}(t) + 2$.

Let $\mathrm{x}_2(t) = K\mathrm{g}(t)$. Then $\mathrm{y}_2(t) = K\mathrm{g}(t) + 2$

$K\mathrm{y}_1(t) = K\mathrm{g}(t) + 2K \Rightarrow \mathrm{y}_2(t) \neq K\mathrm{y}_1(t)$, Not homogeneous

Time Invariance

- If an excitation causes a zero-state response and delaying the excitation simply delays the zero-state response by the same amount of time, regardless of the amount of delay, the system is **time invariant**.



If $\mathrm{g}(t) \xrightarrow{\mathcal{H}} \mathrm{y}_1(t)$ and $\mathrm{g}(t-t_0) \xrightarrow{\mathcal{H}} \mathrm{y}_1(t-t_0) \Rightarrow \mathcal{H}$ is Time Invariant

This test must succeed for any g and any t_0.

Time Invariance

Let $y(t) = \exp(x(t))$. Is this system time invariant?

Let $x_1(t) = g(t)$. Then $y_1(t) = \exp(g(t))$.

Let $x_2(t) = g(t - t_0)$. Then $y_2(t) = \exp(g(t - t_0))$

$y_1(t - t_0) = \exp(g(t - t_0)) \Rightarrow y_2(t) = y_1(t - t_0)$, Time Invariant

Let $y(t) = x(t/2)$. Is this system time invariant?

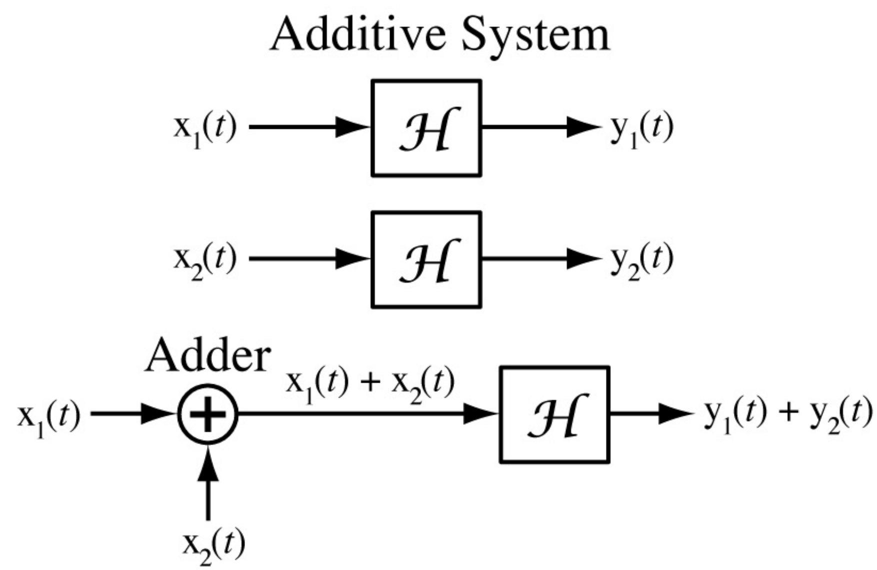
Let $x_1(t) = g(t)$. Then $y_1(t) = g(t/2)$.

Let $x_2(t) = g(t - t_0)$. Then $y_2(t) = g(t/2 - t_0)$

$y_1(t - t_0) = g((t - t_0)/2) \Rightarrow y_2(t) \neq y_1(t - t_0)$, Time **Variant**

Additivity

If one excitation causes a zero-state response and another excitation causes another zero-state response and if, for any arbitrary excitations, the sum of the two excitations causes a zero-state response that is the sum of the two zero-state responses, the system is said to be **additive**.



$$\text{If } g(t) \xrightarrow{\mathcal{H}} y_1(t) \text{ and } h(t) \xrightarrow{\mathcal{H}} y_2(t)$$

$$\text{and } g(t)+h(t) \xrightarrow{\mathcal{H}} y_1(t)+y_2(t) \Rightarrow \mathcal{H} \text{ is Additive}$$

Additivity

Let $\mathrm{y}(t)=\mathrm{u}(\mathrm{x}(t))$. Is this system additive?

Let $\mathrm{x}_1(t)=\mathrm{g}(t)$. Then $\mathrm{y}_1(t)=\mathrm{u}(\mathrm{g}(t))$.

Let $\mathrm{x}_2(t)=\mathrm{h}(t)$. Then $\mathrm{y}_2(t)=\mathrm{u}(\mathrm{h}(t))$.

Let $\mathrm{x}_3(t)=\mathrm{g}(t)+\mathrm{h}(t)$. Then $\mathrm{y}_3(t)=\mathrm{u}(\mathrm{g}(t)+\mathrm{h}(t))$

$\mathrm{y}_1(t)+\mathrm{y}_2(t)=\mathrm{u}(\mathrm{g}(t))+\mathrm{u}(\mathrm{h}(t))\neq\mathrm{u}(\mathrm{g}(t)+\mathrm{h}(t))$. Not additive.

(For example, at time $t=3$, if $\mathrm{g}(3)=4$ and $\mathrm{h}(3)=2$,

$\mathrm{y}_1(3)+\mathrm{y}_2(3)=\mathrm{u}(4)+\mathrm{u}(2)=1+1=2$. But $\mathrm{y}_3(3)=\mathrm{u}(4+6)=1$.)

Linearity and LTI Systems

- If a system is both homogeneous and additive it is **linear**.
- If a system is both linear and time-invariant it is called an **LTI** system
- Some systems that are non-linear can be accurately approximated for analytical purposes by a linear system for small excitations

Linearity and LTI Systems

In an LTI system, the analysis of the effect of an excitation on a system can be found by expressing the excitation as the sum of simpler signals, finding the responses to those signals individually and then adding those responses. Let $\mathrm{x}(t) = \mathrm{rect}(t/4)$. We could express $\mathrm{x}(t)$ as $\mathrm{u}(t+2) - \mathrm{u}(t-2)$ or we could express it as $0.75\,\mathrm{rect}(t/4) + 0.25\,\mathrm{rect}(t/4)$ or any other convenient sum of functions that equals $\mathrm{x}(t)$.

Stability

- Any system for which the response is bounded for any arbitrary bounded excitation, is called a **bounded-input-bounded-output (BIBO)** stable system
- A continuous-time LTI system described by a differential equation is stable if the **eigenvalues** of the solution of the equation all have negative real parts

Causality

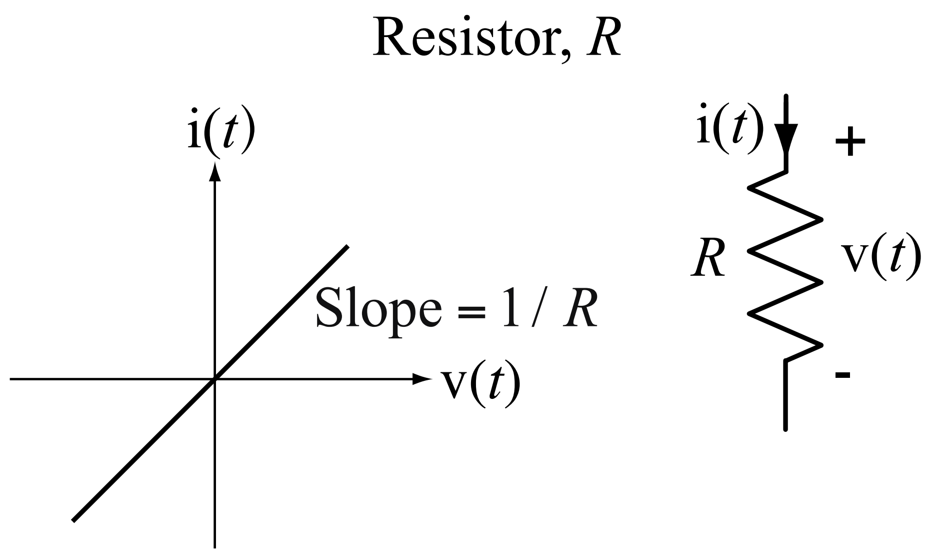
- Any system for which the zero-state response occurs only during or after the time in which the excitation is applied is called a **causal** system.
- Strictly speaking, all real physical systems are causal

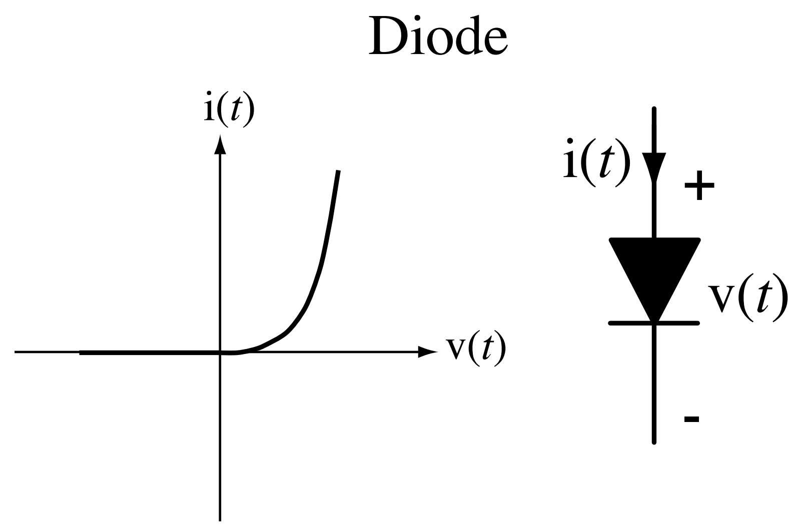
Memory

- If a system's zero-state response at any arbitrary time depends only on the excitation at that same time and not on the excitation or response at any other time it is called a **static** system and is said to have no **memory**. All static systems are causal.
- A system whose zero-state response at some arbitrary time depends on anything other than the excitation at that same time is called a **dynamic** system and is said to have memory
- Any system containing an integrator has memory

Static Non-Linearity

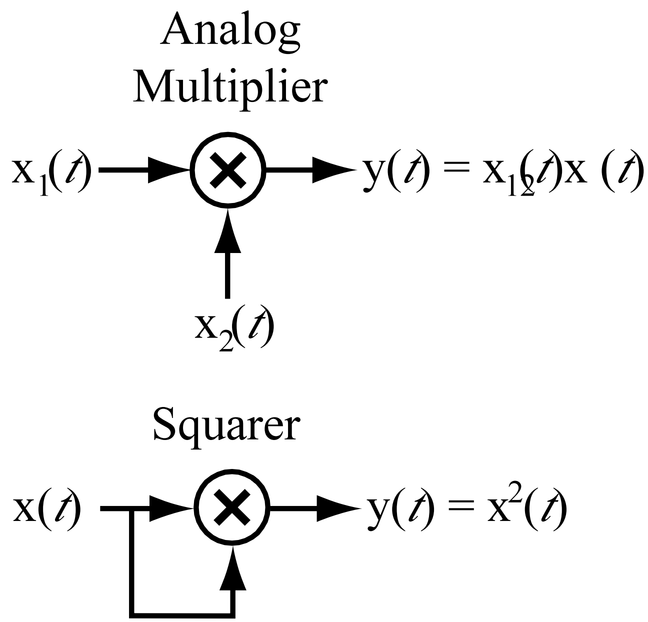
- Many real systems are non-linear because the relationship between excitation amplitude and response amplitude is non-linear





Static Non-Linearity

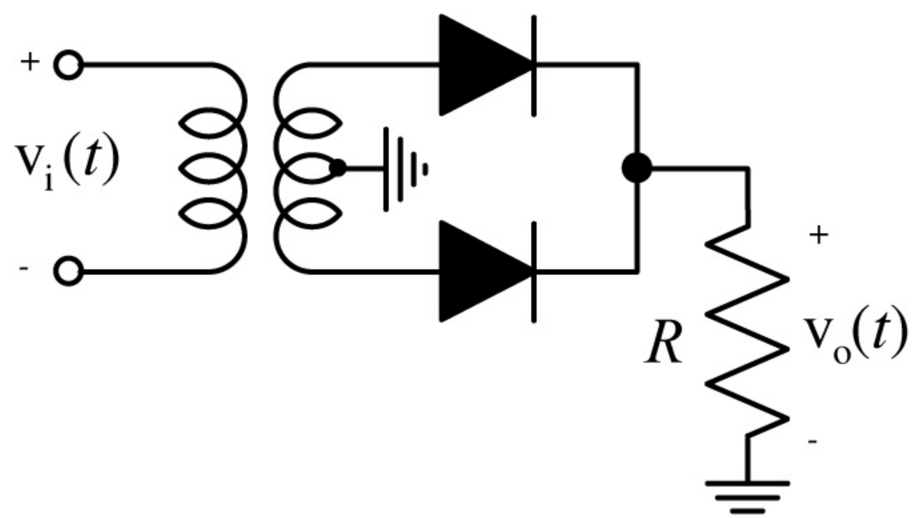
- For an analog multiplier, if the two excitations are the same single excitation signal, the response signal is the square of that single excitation signal and doubling the excitation would cause the response to increase by a factor of 4
- Such a system is not homogeneous and therefore not linear



Invertibility

A system is said to be **invertible** if unique excitations produce unique zero-state responses. In other words, if a system is invertible, knowledge of the zero-state response is sufficient to determine the excitation

This full-wave rectifier is a non-invertible system



Dynamics of Second-Order Systems

The eigenfunction of an LTI system is the complex exponential. The eigenvalues of a second-order system are either both real or occur in a complex-conjugate pair. The general solution form is a sum of two complex exponentials and a constant. For example, the capacitor voltage in a series *RLC* circuit excited by a voltage step of height *A* is

$$\mathrm{v}_{out}(t) = K_1 e^{\left(-\alpha + \sqrt{\alpha^2 - \omega_n^2}\right)t} + K_2 e^{\left(-\alpha - \sqrt{\alpha^2 - \omega_n^2}\right)t} + A$$

where $\alpha = R/2L$ and $\omega_n^2 = 1/LC$. α is the **damping factor** and ω_n is the **natural radian frequency**.

Dynamics of Second-Order Systems

The solution form $K_1 e^{\left(-\alpha+\sqrt{\alpha^2-\omega_n^2}\right)t} + K_2 e^{\left(-\alpha-\sqrt{\alpha^2-\omega_n^2}\right)t}$ applies to all second-order LTI systems. It can also be written as $K_1 e^{(-\alpha+j\omega_c)t} + K_2 e^{(-\alpha-j\omega_c)t}$ where $\omega_c = \omega_n\sqrt{1-\zeta^2}$ and ω_c is the **critical radian frequency** and $\zeta = \alpha/\omega_n$ is the **damping ratio**.

Complex Sinusoid Excitation

Any LTI system excited by a complex sinusoid responds with another complex sinusoid of the same frequency but generally a different magnitude and phase. In the case of the *RLC* circuit if the excitation is $\mathrm{v}_{in}(t) = Ae^{j2\pi f_0 t}$ the response is $\mathrm{v}_{out}(t) = Be^{j2\pi f_0 t}$ where *A* and *B* are, in general, complex. *B* can be found by substituting the solution form into the differential equation and finding the particular solution. In the *RLC* circuit

$$B = \frac{A}{(j2\pi f_0)^2 LC + j2\pi f_0 RC + 1}$$

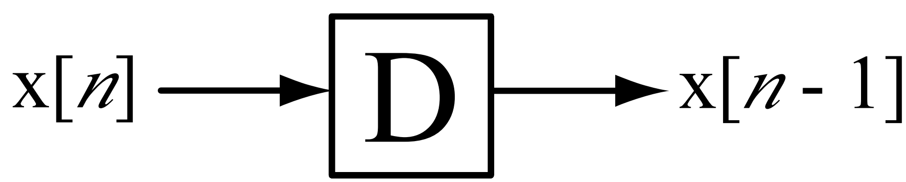
Discrete-Time Systems

- With the increase in speed and decrease in cost of digital system components, **discrete-time systems** have experienced, and are still experiencing, rapid growth in modern engineering system design
- Discrete-time systems are usually described by **difference equations**

Block Diagram Symbols

The block diagram symbols for a summing junction and an amplifier are the same for discrete-time systems as they are for continuous-time systems.

Block diagram symbol for a **delay**



Discrete-Time Systems

In a discrete-time system events occur at points in time but not between those points. The most important example is a **digital computer**. Significant events occur at the end of each clock cycle and nothing of significance (to the computer user) happens between those points in time.

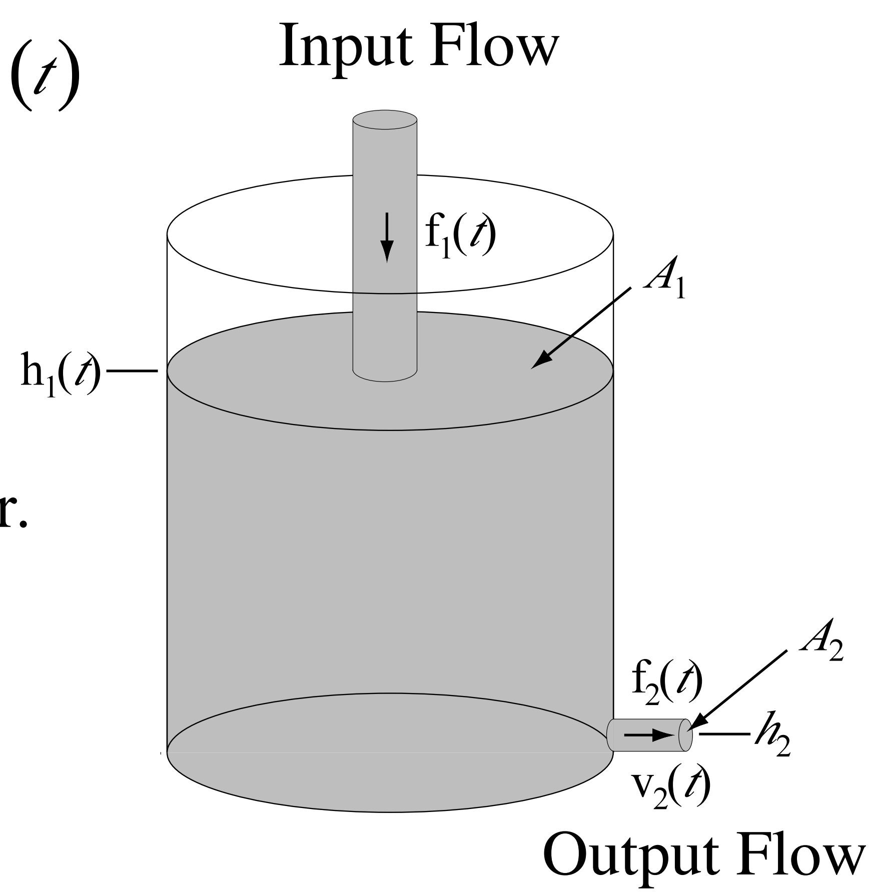
Discrete-time systems can be described by **difference** (not differential) **equations**. Let a discrete-time system generate an excitation signal $y[n]$ where n is the number of discrete-time intervals that have elapsed since some beginning time $n = 0$. Then, for example a simple discrete-time system might be described by

$$y[n] = 1.97\,y[n-1] - y[n-2]$$

Modeling a Fluid System

Toricelli's Equation: $\mathrm{v}_2(t) = \sqrt{2g\left[\mathrm{h}_1(t) - h_2\right]}$

$$\underbrace{A_1 \frac{d}{dt}\left(\mathrm{h}_1(t)\right)}_{\text{Rate of Increase of Water Volume}} + \underbrace{A_2\sqrt{2g\left[\mathrm{h}_1(t) - h_2\right]}}_{\text{Volumetric Outflow Rate}} = \mathrm{f}_1(t)$$



This differential equation is non-linear.

Solving a Differential Equation Numerically

The differential equation that models the fluid system

$$A_1 \frac{d}{dt}\left(\mathrm{h}_1(t)\right) + A_2\sqrt{2g\left[\mathrm{h}_1(t) - h_2\right]} = \mathrm{f}_1(t)$$

can be approximated by the difference equation

$$A_1 \frac{\mathrm{h}_1\left((n+1)T_s\right) - \mathrm{h}_1\left(nT_s\right)}{T_s} + A_2\sqrt{2g\left[\mathrm{h}_1\left(nT_s\right) - h_2\right]} \cong \mathrm{f}_1\left(nT_s\right)$$

by replacing all derivatives by finite differences of the form

$$\frac{d}{dt}\left(\mathrm{h}_1(t)\right) \cong \frac{\mathrm{h}_1\left((n+1)T_s\right) - \mathrm{h}_1\left(nT_s\right)}{T_s}$$

Solving a Differential Equation Numerically

The difference equation

$$A_1 \frac{\mathrm{h}_1\left(\left(n+1\right)T_s\right)-\mathrm{h}_1\left(nT_s\right)}{T_s}+A_2\sqrt{2g\left[\mathrm{h}_1\left(nT_s\right)-h_2\right]}\cong \mathrm{f}_1\left(nT_s\right)$$

can be written in the new discrete-time notation as

$$\mathrm{h}_1[n]\cong \frac{1}{A_1}\left\{T_s\,\mathrm{f}_1[n-1]+A_1\,\mathrm{h}_1[n-1]-A_2T_s\sqrt{2g\left(\mathrm{h}_1[n-1]-h_2\right)}\right]$$

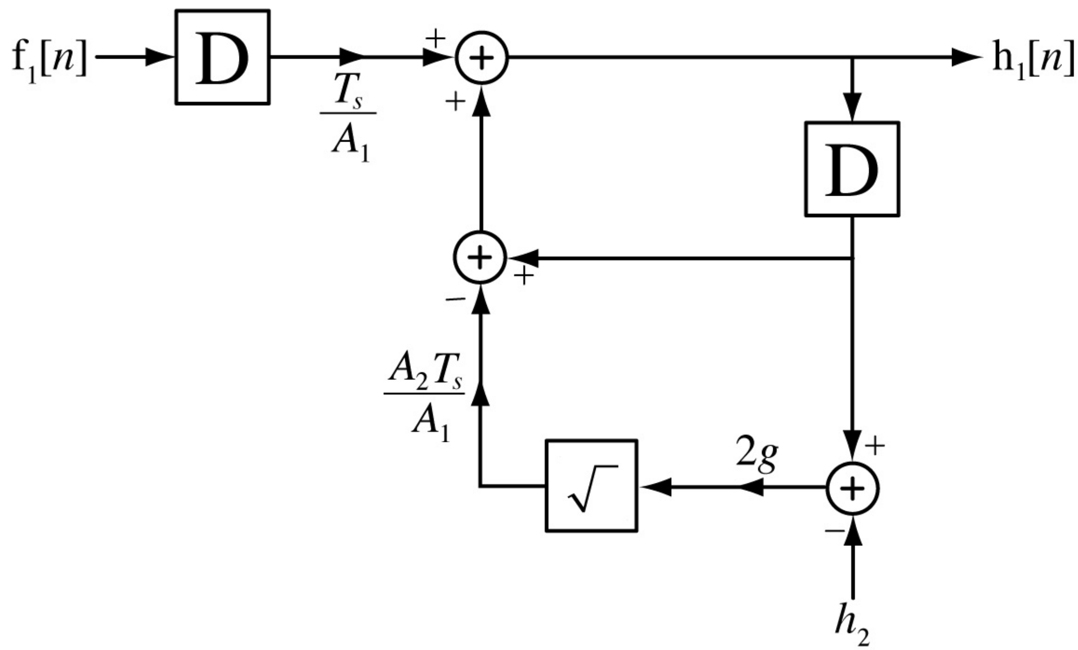
which expresses the present value of the water level in terms of the immediate past value of the water level.

Solving a Differential Equation Numerically

The difference equation

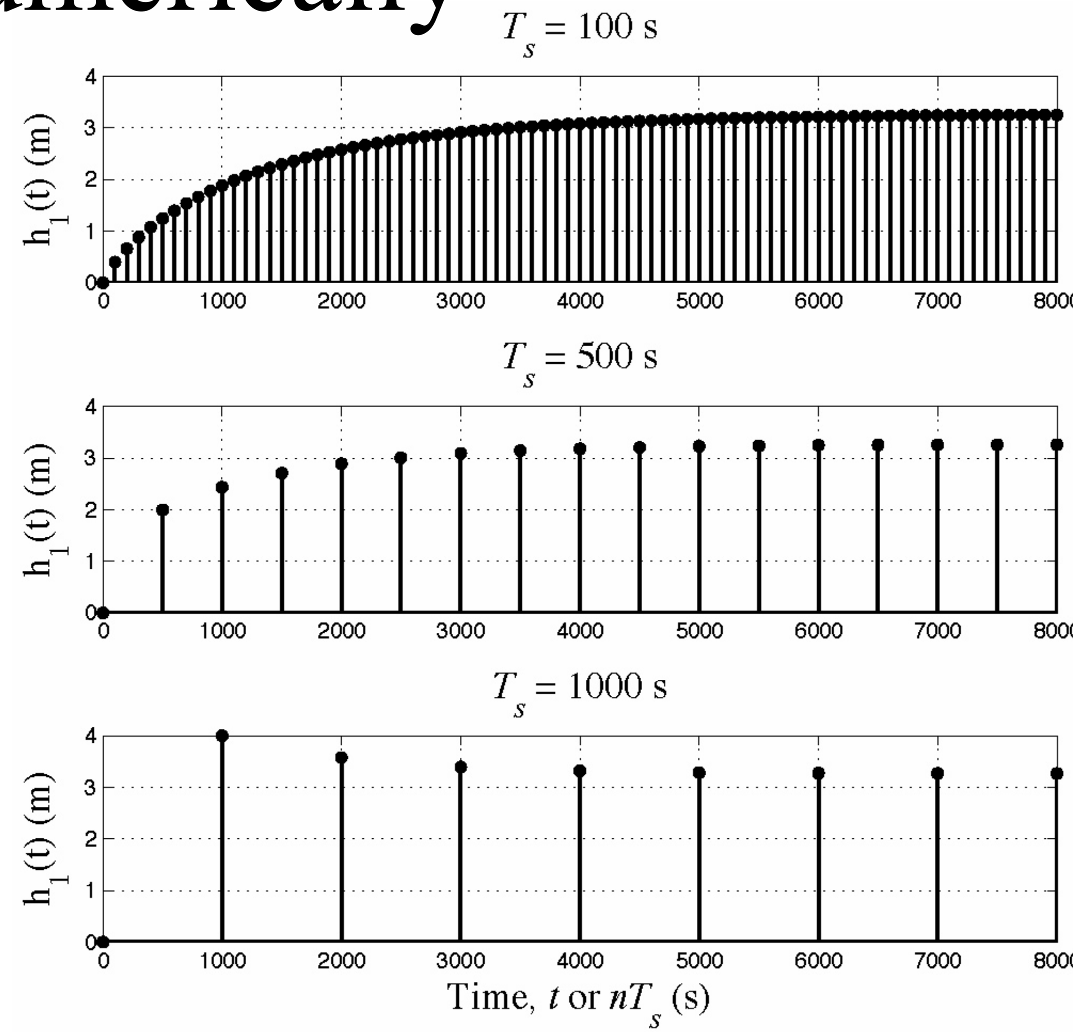
$$h_1[n] \cong \frac{1}{A_1}\left\{T_s f_1[n-1] + A_1 h_1[n-1] - A_2 T_s \sqrt{2g\left(h_1[n-1] - h_2\right)}\right]$$

describes a system that can also be described by a block diagram.



Solving a Differential Equation Numerically

The accuracy of the approximate numerical solution depends on the time step T_s used. Smaller time steps usually result in more accurate solutions.



Discrete-Time Systems

The equation

$$y[n] = 1.97\,y[n-1] - y[n-2]$$

says in words

"The signal value at any time n is 1.97 times the signal value at the previous time $[n-1]$ minus the signal value at the time before that $[n-2]$."

If we know the signal value at any two times, we can compute its value at all other (discrete) times. This is quite similar to a second-order differential equation for which knowledge of two independent initial conditions allows us to find the solution for all time and the solution methods are very similar.

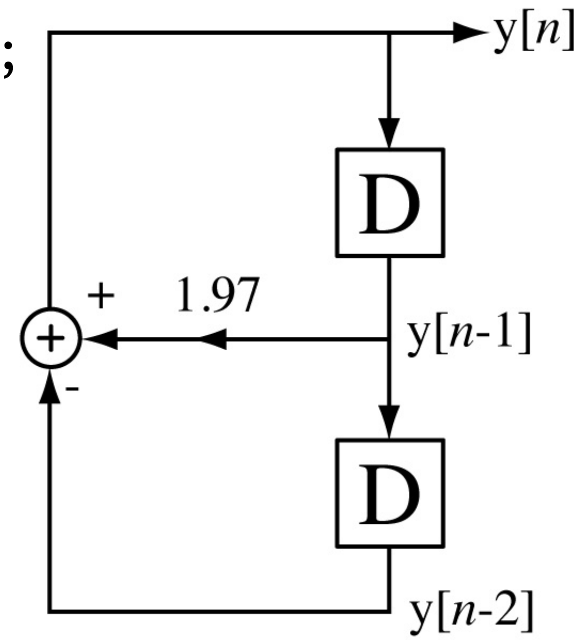
Discrete-Time Systems

$$y[n] = 1.97\,y[n-1] - y[n-2]$$

We could solve this equation by iteration using a computer.

```
yn = 1 ; yn1 = 0 ;      ⟵ Initial Conditions
while 1,
    yn2 = yn1 ; yn1 = yn ; yn = 1.97*yn1 - yn2 ;
end
```

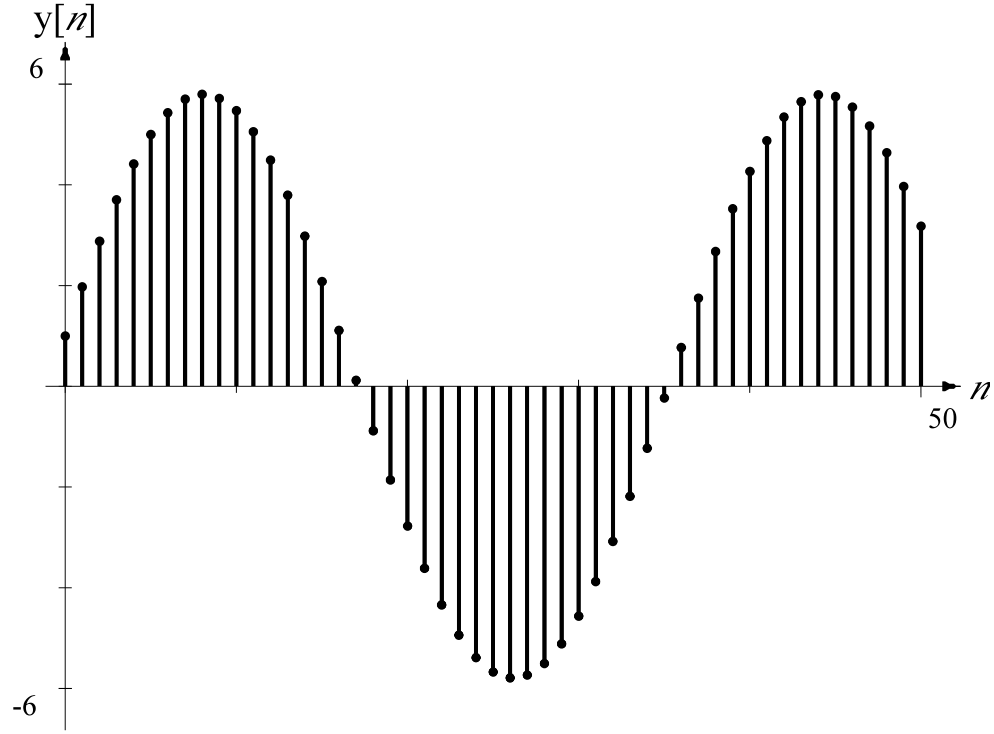
We could also describe the system with a block diagram.



Discrete-Time Systems

$$y[n] = 1.97\,y[n-1] - y[n-2]$$

With the initial conditions $y[1] = 1$ and $y[0] = 0$ the (zero-input) response is $y[n]$.



Solving Difference Equations

On the previous two slides we found the solution to

$$\mathrm{y}[n] = 1.97\,\mathrm{y}[n-1] - \mathrm{y}[n-2]$$

by iteration as a sequence of numbers for $\mathrm{y}[n]$. We can also solve linear, constant-coefficient ordinary difference equations with techniques that are very similar to those used to solve linear, constant-coefficient ordinary differential equations. The eigenfunction of this type of equation is the complex exponential z^n. As a first example let the equation be $2\,\mathrm{y}[n] - \mathrm{y}[n-1] = 0$. The homogeneous solution of this equation is then $\mathrm{y}_h[n] = Kz^n$. Substituting that into the equation we get $2Kz^n - Kz^{n-1} = 0$. This is the characteristic equation. Dividing through by Kz^{n-1} we get $2z - 1 = 0$ and the solution is $z = 1/2$.

Solving Difference Equations

The eigenvalue for the equation $2\,\mathrm{y}[n]-\mathrm{y}[n-1]=0$ is then $z=1/2$ and the homogeneous solution is $\mathrm{y}_h[n]=K(1/2)^n$. Since the equation is homogeneous, the homogeneous solution is also the total solution. To find K we need an initial condition. Let it be $\mathrm{y}[0]=3$. Then $\mathrm{y}[0]=K(1/2)^0=K=3$ and $\mathrm{y}[n]=3(1/2)^n$.

Solving Difference Equations

The solution of inhomogeneous equations is also similar to differential equation techniques. Let $5\mathrm{y}[n]-3\mathrm{y}[n-1]=(1/3)^n$ with an initial condition of $\mathrm{y}[0]=-1$. The characteristic equation is $5z-3=0$. The eigenvalue is $3/5$ and the homogeneous solution is $\mathrm{y}_h[n]=K(3/5)^n$. The particular solution is a linear combination of the forcing function $(1/3)^n$ and all its unique differences. The first backward difference of $(1/3)^n$ is $(1/3)^n-(1/3)^{n-1}$, which can be written as $-2(1/3)^n$. This is just the same function but with a different multiplying constant. So the first difference of an exponential is also an exponential. Therefore the only functional form we need for the particular solution is $K_p(1/3)^n$.

Solving Difference Equations

Substituting the particular solution form into the difference equation we get $5K_p(1/3)^n - 3K_p(1/3)^{n-1} = (1/3)^n$. Solving,

$$K_p = \frac{(1/3)^n}{5(1/3)^n - 3(1/3)^{n-1}} = \frac{1/3}{5(1/3) - 3} = -1/4.$$ Then the total solution is $\mathrm{y}[n] = K(3/5)^n - (1/4)(1/3)^n$. Applying initial conditions $\mathrm{y}[0] = K(3/5)^0 - (1/4)(1/3)^0 = K - 1/4 = -1$

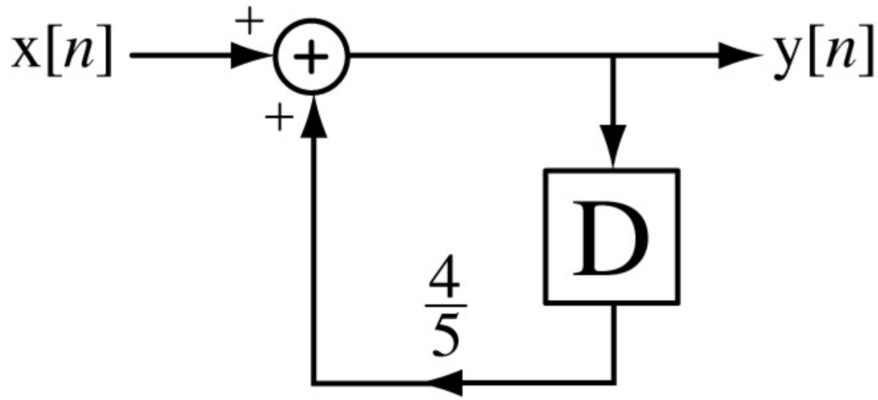
Therefore $K = -3/4$ and the total solution is

$$\mathrm{y}[n] = (-3/4)(3/5)^n - (1/4)(1/3)^n.$$

Stability

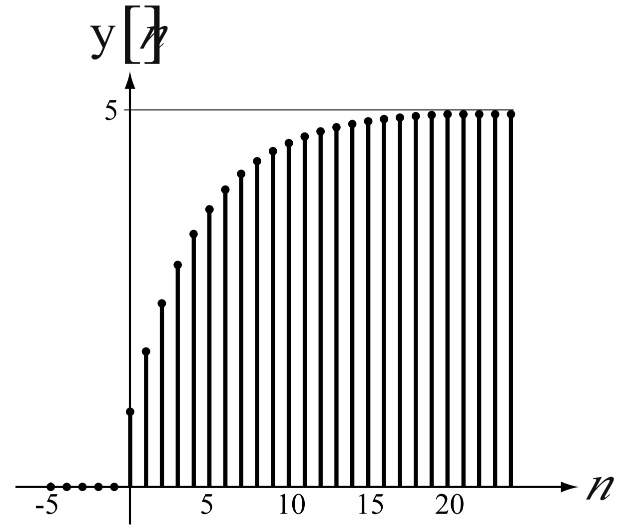
- Any system for which the response is bounded for any arbitrary bounded excitation, is called a **bounded-input-bounded-output (BIBO)** stable system
- A discrete-time LTI system described by a difference equation is stable if the **eigenvalues** of the solution of the equation <u>all</u> have magnitudes less than one

A System

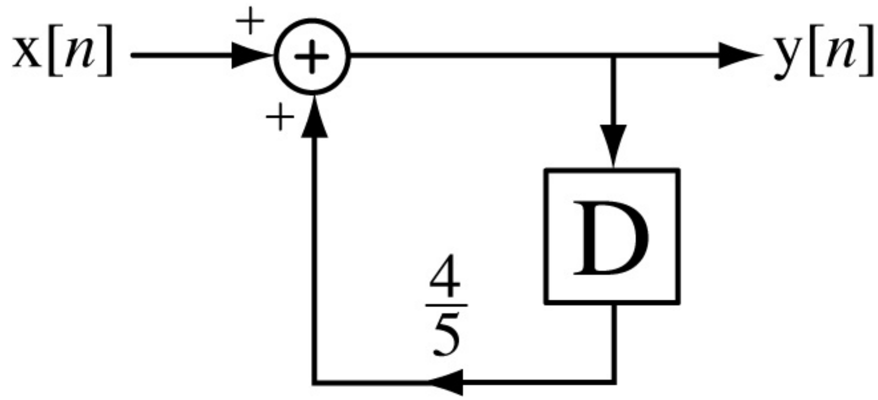


If the excitation x[n] is the unit sequence, the zero-state response is

$$y[n]=\left[5-4(4/5)^n\right]u[n]$$



A System



If the excitation is doubled, the zero-state response doubles. If two signals are added to form the excitation, the zero-state response is the sum of the zero-state responses to those two signals. If the excitation is delayed by some time, the zero-state response is delayed by the same time. This system is linear and time invariant.

System Properties

- The properties of discrete-time systems have the same meaning as they do in continuous-time systems

Eigenfunctions of LTI Systems

- The **eigenfunction** of an LTI system is the complex exponential
- The **eigenvalues** are either real or, if complex, occur in complex conjugate pairs
- Any LTI system excited by a complex sinusoid responds with another complex sinusoid of the same frequency, but generally a different amplitude and phase
- All these statements are true of both continuous-time and discrete-time systems

$\mathrm{y}(t) = \mathrm{x}(\sin(t)) \Rightarrow$ Causal?, Linear?

$\mathrm{y}(t) = \mathrm{x}(\sin(t))$ For any time $t < 0$, $\mathrm{y}(t)$ depends on a value of x in the future, because for $t < 0$, $\sin(t) > t$. Non-causal.

Let $\mathrm{x}_1(t) = \mathrm{g}(t)$. Then $\mathrm{y}_1(t) = \mathrm{g}(\sin(t)) \to K\mathrm{y}_1(t) = K\mathrm{g}(\sin(t))$.

Let $\mathrm{x}_2(t) = K\mathrm{g}(t)$. Then $\mathrm{y}_2(t) = K\mathrm{g}(\sin(t)) = K\mathrm{y}_1(t)$ Homogeneous

Let $\mathrm{x}_2(t) = \mathrm{h}(t)$. Then $\mathrm{y}_2(t) = \mathrm{h}(\sin(t))$.

Let $\mathrm{x}_3(t) = \mathrm{g}(t) + \mathrm{h}(t)$.

Then $\mathrm{y}_3(t) = K\mathrm{g}(\sin(t)) + K\mathrm{h}(\sin(t)) = K[\mathrm{y}_1(t) + \mathrm{y}_2(t)]$

Additive $\to$ Linear

$$\mathrm{y}[n] = \sum_{m=n-k}^{n+k} \mathrm{x}[m] \ , \ k \text{ a finite positive integer}$$

Linear?, Time-Invariant?, Stable?

Let $\mathrm{x}_1[n] = \mathrm{g}[n]$. Then $\mathrm{y}_1[n] = \sum_{m=n-k}^{n+k} \mathrm{g}[m]$.

Let $\mathrm{x}_2[n] = K\mathrm{g}[n]$. Then $\mathrm{y}_2[n] = K \sum_{m=n-k}^{n+k} \mathrm{g}[m] = K\mathrm{y}_1[n]$

Homogeneous

Let $\mathrm{x}_2[n] = \mathrm{h}[n]$. Then $\mathrm{y}_2[n] = \sum_{m=n-k}^{n+k} \mathrm{h}[m]$

Let $\mathrm{x}_3[n] = \mathrm{g}[n] + \mathrm{h}[n]$. Then $\mathrm{y}_3[n] = \sum_{m=n-k}^{n+k} \left(\mathrm{g}[m] + \mathrm{h}[m]\right) = \mathrm{y}_1[n] + \mathrm{y}_2[n]$

Additive $\rightarrow$ Linear

$$\mathrm{y}[n]=\sum_{m=n-k}^{n+k}\mathrm{x}[m]\ ,\ k \text{ a finite positive integer}$$

$$\text{Let } \mathrm{x}_1[n]=\mathrm{g}[n].\ \text{Then } \mathrm{y}_1[n]=\sum_{m=n-k}^{n+k}\mathrm{g}[m].$$

$$\text{Let } \mathrm{x}_2[n]=\mathrm{g}[n-n_0].\ \text{Then } \mathrm{y}_2[n]=\sum_{m=n-k}^{n+k}\mathrm{g}[m-n_0]$$

$$\text{Let } q=m-n_0 \Rightarrow m=q+n_0.\ \text{Then } \mathrm{y}_2[n]=\sum_{q=n-k-n_0}^{n+k-n_0}\mathrm{g}[q].$$

$$\mathrm{y}_1[n-n_0]=\sum_{m=n-k-n_0}^{n+k-n_0}\mathrm{g}[m]=\mathrm{y}_2[n]\rightarrow \text{Time-Invariant}$$

If the upper bound on $\mathrm{x}[n]$ is B, what is the upper bound on $\mathrm{y}[n]$?

$$\text{If } \mathrm{x}[n]\le B,\ \text{then } \mathrm{y}[n]\le\sum_{k=n-k}^{n+k}B=(2k+1)B.$$

Upper bound on y is $(2k+1)B \rightarrow$ BIBO Stable

$y(t) = t^2 x(t-1)$, Linear?, Time-Invariant?

Let $x_1(t) = g(t)$. Then $y_1(t) = t^2 g(t-1)$.

Let $x_2(t) = K g(t)$. Then $y_2(t) = t^2 K g(t-1) = K y_1(t) \rightarrow$ Homogeneous

Let $x_2(t) = h(t)$. Then $y_2(t) = t^2 h(t-1)$.

Let $x_3(t) = g(t) + h(t)$.

Then $y_3(t) = t^2 \left[g(t-1) + h(t-1) \right] = y_1(t) + y_2(t) \rightarrow$ Additive $\rightarrow$ Linear

Let $x_1(t) = g(t)$. Then $y_1(t) = t^2 g(t-1)$ and $y_1.(t-t_0) = (t-t_0)^2 g(t-t_0-1)$

Let $x_2(t) = g(t-t_0)$. Then $y_2(t) = t^2 g(t-1-t_0) \neq y_1(t-t_0)$

Time <u>Variant</u>

$y[n] = x^2[n-2]$, Linear?, Time-Invariant?

Let $x_1[n] = g[n]$. Then $y_1[n] = g^2[n-2]$.

Let $x_2[n] = K g[n]$. Then $y_2[n] = (K g[n-2])^2 \neq K y_1[n]$

Inhomogenous $\rightarrow$ Non-Linear

Let $x_2[n] = g[n-n_0]$. Then $y_2[n] = g^2[n-n_0-2] = y_1[n-n_0]$

Time-Invariant

$$y[n] = x[n+1] - x[n-1] \text{ , Linear?, Time-Invariant?}$$

$$\text{Let } x_1[n] = g[n]. \text{ Then } y_1[n] = g[n+1] - g[n-1].$$

$$\text{Let } x_2[n] = K g[n]. \text{ Then } y_2[n] = K g[n+1] - K g[n-1].$$

$$y_2[n] = K y_1[n] \rightarrow \text{Homogeneous}$$

$$\text{Let } x_2[n] = h[n]. \text{ Then } y_2[n] = h[n+1] - h[n-1].$$

$$\text{Let } x_3[n] = g[n] + h[n].$$

$$\text{Then } y_3[n] = g[n+1] + h[n+1] - \left(g[n-1] + h[n-1]\right).$$

$$y_3[n] = y_1[n] + y_2[n] \rightarrow \text{Additive} \rightarrow \text{Linear}$$

$$\text{Let } x_2[n] = g[n - n_0].$$

$$\text{Then } y_2[n] = g[n - n_0 + 1] - g[n - n_0 - 1].$$

$$y_2[n] = y_1[n - n_0] \rightarrow \text{Time-Invariant}$$

$y(t) = x(t-2) + x(2-t)$, Linear?, Time-Invariant?, Dynamic?, Causal?, Stable?

Let $x_1(t) = g(t)$. Then $y_1(t) = g(t-2) + g(2-t)$.

Let $x_2(t) = K g(t)$. Then $y_2(t) = K\left[g(t-2) + g(2-t)\right] = K y_1(t)$

Homogeneous

Let $x_2(t) = h(t)$. Then $y_2(t) = h(t-2) + h(2-t)$.

Let $x_3(t) = g(t) + h(t)$. Then $y_3(t) = \left[g(t-2) + h(t-2)\right] + \left[g(2-t) + h(2-t)\right]$.

$y_3(t) = y_1(t) + y_2(t)$. $\rightarrow$ Additive $\rightarrow$ Linear

Let $x_2(t) = g(t - t_0)$. Then $y_2(t) = g(t - 2 - t_0) + g(2 - t - t_0)$.

$y_2(t) \neq y_1(t - t_0) \rightarrow$ Time Variant

$y(t)$ depends on x at other times $\rightarrow$ Dynamic

For $t < 1$, $y(t)$ depends on $x(t)$ at future times. $\rightarrow$ Non-Causal

Since y is a simple linear combination of shifted versions of x,

if x is bounded, so is y. $\rightarrow$ BIBO Stable

$y(t)=x(t)\cos(3t)$, Linear?, Time-Invariant?, Dynamic?, Causal?, Stable?

Let $x_1(t)=g(t)$. Then $y_1(t)=g(t)\cos(3t)$.

Let $x_2(t)=Kg(t)$. Then $y_2(t)=Kg(t)\cos(3t)=Ky_1(t)$

Homogeneous

Let $x_2(t)=h(t)$. Then $y_2(t)=h(t)\cos(3t)$.

Let $x_3(t)=g(t)+h(t)$. Then $y_3(t)=\left[g(t)+h(t)\right]\cos(3t)=y_1(t)+y_2(t)$

Additive $\rightarrow$ Linear

Let $x_2(t)=g(t-t_0)$. Then $y_2(t)=g(t-t_0)\cos(3t)$.

$y_1(t-t_0)=g(t-t_0)\cos(3(t-t_0))\neq y_2(t)$

Time Variant

$y(t)$ depends on x only at time $t \rightarrow$ Static

Static $\rightarrow$ Causal

If $x(t)$ is bounded, then $x(t)\cos(3t)$ is also. $\rightarrow$ Stable

$$y(t) = \int_{-\infty}^{2t} x(\tau)\,d\tau,$$ Linear?, Time-Invariant?, Dynamic?, Causal?, Stable?

Let $x_1(t) = g(t)$. Then $y_1(t) = \int_{-\infty}^{2t} g(\tau)\,d\tau$.

Let $x_2(t) = K g(t)$. Then $y_2(t) = K \int_{-\infty}^{2t} g(\tau)\,d\tau = K y_1(t)$.

Homogeneous

Let $x_2(t) = h(t)$. Then $y_2(t) = \int_{-\infty}^{2t} h(\tau)\,d\tau$

Let $x_3(t) = g(t) + h(t)$. Then $y_3(t) = \int_{-\infty}^{2t} \left[g(\tau) + h(\tau) \right] d\tau = y_1(t) + y_2(t)$

Additive $\rightarrow$ Linear

Let $x_2(t) = g(t - t_0)$. Then $y_2(t) = \int_{-\infty}^{2t} g(\tau - t_0)\,d\tau$. Let $\lambda = \tau - t_0$.

Then $y_2(t) = \int_{-\infty}^{2t - t_0} g(\lambda)\,d\lambda$. $y_1(t - t_0) = \int_{-\infty}^{2(t - t_0)} g(\tau)\,d\tau \rightarrow$ Time <u>Variant</u>

$\mathrm{y}(t)=\int_{-\infty}^{2t}\mathrm{x}(\tau)d\tau$, Linear?, Time-Invariant?, Dynamic?, Causal?, Stable?

$\mathrm{y}(t)$ depends on values of $\mathrm{x}(t)$ at other times $\rightarrow$ Dynamic

For any $t>0$, $\mathrm{y}(t)$ depends on all values of $\mathrm{x}(t)$ up to

time $2t. \rightarrow$ Non-Causal

If $\mathrm{x}(t)$ is a constant, there is no upper bound on $\mathrm{y}(t)$.

Unstable

Also, by Leibniz's rule for differentiating an integral

$\mathrm{y}'(t)=\mathrm{x}(2t) \Rightarrow$ Eigenvalue is 0. Real part is not negative.

Unstable

$$\mathrm{y}(t)=\begin{cases}0, & t<0\\ \mathrm{x}(t)+\mathrm{x}(t-2), & t\geq 0\end{cases}$$ Linear?, Time-Invariant?, Dynamic?, Causal?, Stable?

Let $\mathrm{x}_1(t)=\mathrm{g}(t)$. Then $\mathrm{y}_1(t)=\begin{cases}0, & t<0\\ \mathrm{g}(t)+\mathrm{g}(t-2), & t\geq 0\end{cases}$.

Let $\mathrm{x}_2(t)=K\mathrm{g}(t)$. Then $\mathrm{y}_2(t)=\begin{cases}0, & t<0\\ K\mathrm{g}(t)+K\mathrm{g}(t-2), & t\geq 0\end{cases}=K\mathrm{y}_1(t)$

Homogeneous

Let $\mathrm{x}_2(t)=\mathrm{h}(t)$. Then $\mathrm{y}_2(t)=\begin{cases}0, & t<0\\ \mathrm{h}(t)+\mathrm{h}(t-2), & t\geq 0\end{cases}$

Let $\mathrm{x}_3(t)=\mathrm{g}(t)+\mathrm{h}(t)$. Then $\mathrm{y}_3(t)=\begin{cases}0, & t<0\\ \mathrm{g}(t)+\mathrm{h}(t)+\mathrm{g}(t-2)+\mathrm{h}(t-2), & t\geq 0\end{cases}$

$\mathrm{y}_3(t)=\mathrm{y}_1(t)+\mathrm{y}_2(t)\rightarrow$ Additive $\rightarrow$ Linear

Let $\mathrm{x}_2(t)=\mathrm{g}(t-t_0)$. Then $\mathrm{y}_2(t)=\begin{cases}0, & t<0\\ \mathrm{g}(t-t_0)+\mathrm{g}(t-t_0-2), & t\geq 0\end{cases}$

$$\mathrm{y}_1(t-t_0)=\begin{cases}0, & t-t_0<0\\ \mathrm{g}(t-t_0)+\mathrm{g}(t-t_0-2), & t-t_0\geq 0\end{cases}\Rightarrow \mathrm{y}_2(t)\neq \mathrm{y}_1(t-t_0)$$

Time <u>Variant</u>

$$y(t)=\begin{cases}0, & t<0\\ x(t)+x(t-2), & t\geq 0\end{cases}$$

Linear?, Time-Invariant?, Dynamic?, Causal?, Stable?

$y(t)$ depends on $x(t-2) \rightarrow$ Dynamic

$y(t)$ depends only on x at the same or earlier times $\rightarrow$ Causal

If $x(t)$ is bounded, $x(t-2)$ is bounded and $y(t)$ is bounded. $\rightarrow$ Stable

$$y(t) = \begin{cases} 0, & x(t) < 0 \\ x(t) + x(t-2), & x(t) \geq 0 \end{cases}$$

Linear?, Time-Invariant?, Dynamic?, Causal?, Stable?

Let $x_1(t) = g(t)$. Then $y_1(t) = \begin{cases} 0, & g(t) < 0 \\ g(t) + g(t-2), & g(t) \geq 0 \end{cases}$.

Let $x_2(t) = Kg(t)$. Then $y_2(t) = \begin{cases} 0, & Kg(t) < 0 \\ Kg(t) + Kg(t-2), & Kg(t) \geq 0 \end{cases}$.

$$Ky_1(t) = \begin{cases} 0, & g(t) < 0 \\ Kg(t) + Kg(t-2), & g(t) \geq 0 \end{cases} \neq y_2(t)$$

For example, let $g(t) = 1$. Then $y_1(t) = 2$.

Let $K = -1$. Then $y_2(t) = 0 \neq -2$.

Inhomogeneous $\rightarrow$ Non-Linear

$$\mathrm{y}(t)=\begin{cases}0, & \mathrm{x}(t)<0 \\ \mathrm{x}(t)+\mathrm{x}(t-2), & \mathrm{x}(t)\geq 0\end{cases}$$

Linear?, Time-Invariant?, Dynamic?, Causal?, Stable?

Let $\mathrm{x}_1(t)=\mathrm{g}(t)$. Then $\mathrm{y}_1(t)=\begin{cases}0, & \mathrm{g}(t)<0 \\ \mathrm{g}(t)+\mathrm{g}(t-2), & \mathrm{g}(t)\geq 0\end{cases}$.

Let $\mathrm{x}_2(t)=\mathrm{g}(t-t_0)$. Then $\mathrm{y}_2(t)=\begin{cases}0, & \mathrm{g}(t-t_0)<0 \\ \mathrm{g}(t-t_0)+\mathrm{g}(t-t_0-2), & \mathrm{g}(t-t_0)\geq 0\end{cases}$.

$$\mathrm{y}_1(t-t_0)=\begin{cases}0, & \mathrm{g}(t-t_0)<0 \\ K\mathrm{g}(t-t_0)+K\mathrm{g}(t-t_0-2), & \mathrm{g}(t-t_0)\geq 0\end{cases}=\mathrm{y}_2(t)$$

Time Invariant

$\mathrm{y}(t)$ depends on $\mathrm{x}(t-2) \rightarrow$ Dynamic

$\mathrm{y}(t)$ depends only on x at the same or earlier times $\rightarrow$ Causal

If $\mathrm{x}(t)$ is bounded, $\mathrm{x}(t-2)$ is bounded and $\mathrm{y}(t)$ is bounded. $\rightarrow$ Stable

$y[n] = x[-n]$ Linear?, Time-Invariant?, Dynamic?, Causal?, Stable?

Let $x_1[n] = g[n]$. Then $y_1[n] = g[-n]$.

Let $x_2[n] = K g[n]$. Then $y_2[n] = K g[-n] = K y_1[n]. \rightarrow$ Homogeneous

Let $x_2[n] = h[n]$. Then $y_2[n] = h[-n]$.

Let $x_3[n] = g[n] + h[n]$. Then $y_3[n] = g[-n] + h[-n]$.

$y_3[n] = y_1[n] + y_2[n] \rightarrow$ Additive $\rightarrow$ Linear

$y_1[n - n_0] = g[-(n - n_0)]$

Let $x_2[n] = g[n - n_0]$. Then $y_2[n] = g[-n - n_0] \neq y_1[n - n_0]. \rightarrow$ Time <u>Variant</u>

y at any time n depends on x at time $-n \rightarrow$ Dynamic

y at any time n depends on x at time $-n$. For negative n, $-n$ is the future.

Non-Causal

If x is bounded, y is bounded. $\rightarrow$ BIBO Stable

$y[n] = x[n-2] - 2x[n-8]$ Linear?, Time-Invariant?, Dynamic?, Causal?, Stable?

Let $x_1[n] = g[n]$. Then $y_1[n] = g[n-2] - 2g[n-8]$.

Let $x_2[n] = K g[n]$. Then $y_2[n] = K(g[n-2] - 2g[n-8]) = K y_1[n]$.

Homogeneous

Let $x_2[n] = h[n]$. Then $y_2[n] = h[n-2] - 2h[n-8]$.

Let $x_3[n] = g[n] + h[n]$. Then $y_3[n] = g[n-2] + h[n-2] - 2(g[n-8] + h[n-8])$.

$y_3[n] = y_1[n] + y_2[n] \rightarrow$ Additive $\rightarrow$ Linear

Let $x_2[n] = g[n-n_0]$. Then $y_2[n] = g[n-n_0-2] - 2g[n-n_0-8] = y_1[n-n_0]$.

Time Invariant

y at any time n depends on x at other times $\rightarrow$ Dynamic

y at any time n depends only on x at earlier times. $\rightarrow$ Causal

If x is bounded, y is bounded. $\rightarrow$ BIBO Stable

$$\mathrm{y}[n]=\begin{cases}\mathrm{x}[n] & , \ n \geq 1 \\ 0 & , \ n=0 \\ \mathrm{x}[n+1] & , \ n \leq -1\end{cases}$$

Linear?, Time-Invariant?, Dynamic?, Causal?, Stable?

Let $\mathrm{x}_1[n]=\mathrm{g}[n]$. Then $\mathrm{y}_1[n]=\begin{cases}\mathrm{g}[n] & , \ n \geq 1 \\ 0 & , \ n=0 \\ \mathrm{g}[n+1] & , \ n \leq -1\end{cases}$.

Let $\mathrm{x}_2[n]=K\mathrm{g}[n]$. Then $\mathrm{y}_2[n]=\begin{cases}K\mathrm{g}[n] & , \ n \geq 1 \\ 0 & , \ n=0 \\ K\mathrm{g}[n+1] & , \ n \leq -1\end{cases} = K\mathrm{y}_1[n]$.

Homogeneous

$$y[n] = \begin{cases} x[n] & , \ n \geq 1 \\ 0 & , \ n = 0 \\ x[n+1] & , \ n \leq -1 \end{cases}$$

Linear?, Time-Invariant?, Dynamic?, Causal?, Stable?

Let $x_1[n] = g[n]$. Then $y_1[n] = \begin{cases} g[n] & , \ n \geq 1 \\ 0 & , \ n = 0 \\ g[n+1] & , \ n \leq -1 \end{cases}$.

Let $x_2[n] = h[n]$. Then $y_2[n] = \begin{cases} h[n] & , \ n \geq 1 \\ 0 & , \ n = 0 \\ h[n+1] & , \ n \leq -1 \end{cases}$.

Let $x_3[n] = g[n] + h[n]$. Then $y_3[n] = \begin{cases} g[n] + h[n] & , \ n \geq 1 \\ 0 & , \ n = 0 \\ g[n+1] + h[n+1] & , \ n \leq -1 \end{cases}$.

$y_3[n] = y_1[n] + y_2[n] \rightarrow$ Additive $\rightarrow$ Linear

$$\text{Let } x_1[n] = g[n]. \text{ Then } y_1[n] = \begin{cases} g[n] & , \ n \geq 1 \\ 0 & , \ n = 0 \\ g[n+1] & , \ n \leq -1 \end{cases}.$$

$$\text{Let } x_2[n] = g[n-n_0]. \text{ Then } y_2[n] = \left\{ \begin{matrix} g[n-n_0] & , \ n \geq 1 \\ 0 & , \ n = 0 \\ g[n-n_0+1] & , \ n \leq -1 \end{matrix} \right\} \neq y_1[n-n_0].$$

Time <u>Variant</u>

y at any time n depends on x at other times $\rightarrow$ Dynamic

y at negative times n depends on x at time $n+1$. $\rightarrow$ Non-Causal

If x is bounded, y is bounded. $\rightarrow$ BIBO Stable

$y[n] = x[4n+1]$ Linear?, Time-Invariant?, Dynamic?, Causal?, Stable?

Let $x_1[n] = g[n]$. Then $y_1[n] = g[4n+1]$.

Let $x_2[n] = K g[n]$. Then $y_2[n] = K g[4n+1] = K y_1[n]$.

Homogeneous

Let $x_2[n] = h[n]$. Then $y_2[n] = h[4n+1]$.

Let $x_3[n] = g[n] + h[n]$. Then $y_3[n] = g[4n+1] + h[4n+1]$.

$y_3[n] = y_1[n] + y_2[n] \rightarrow$ Additive $\rightarrow$ Linear

Let $x_2[n] = g[n - n_0]$. Then $y_2[n] = g[4n+1-n_0] \neq y_1[n-n_0] = g[4(n-n_0)+1]$.

Time <u>Variant</u>

y at any time n depends on x at other times $\rightarrow$ Dynamic

y at any time n depends on x at time $4n+1$.

For $n > 0$ that is in the future. $\rightarrow$ Non-Causal

If x is bounded, y is bounded. $\rightarrow$ BIBO Stable

$$\mathrm{y}(t)=\frac{1}{\mathrm{x}(t)}\left(\frac{d\mathrm{x}(t)}{dt}\right)^2$$, Homogeneous?, Additive?

Let $\mathrm{x}_1(t)=\mathrm{g}(t)$. Then $\mathrm{y}_1(t)=\frac{1}{\mathrm{g}(t)}\left(\frac{d\mathrm{g}(t)}{dt}\right)^2 \rightarrow K\mathrm{y}_1(t)=\frac{K}{\mathrm{g}(t)}\left(\frac{d\mathrm{g}(t)}{dt}\right)^2$

Let $\mathrm{x}_2(t)=K\mathrm{g}(t)$. Then $\mathrm{y}_2(t)=\frac{1}{K\mathrm{g}(t)}\left(\frac{d(K\mathrm{g}(t))}{dt}\right)^2=\frac{K}{\mathrm{g}(t)}\left(\frac{d\mathrm{g}(t)}{dt}\right)^2=K\mathrm{y}_1(t)$

Homogeneous

Let $\mathrm{x}_2(t)=\mathrm{h}(t)$. Then $\mathrm{y}_2(t)=\frac{1}{\mathrm{h}(t)}\left(\frac{d\mathrm{h}(t)}{dt}\right)^2$

Let $\mathrm{x}_3(t)=\mathrm{g}(t)+\mathrm{h}(t)$. Then $\mathrm{y}_3(t)=\frac{1}{\mathrm{g}(t)+\mathrm{h}(t)}\left[\frac{d(\mathrm{g}(t)+\mathrm{h}(t))}{dt}\right]^2$

$$\mathrm{y}_3(t)=\frac{1}{\mathrm{g}(t)+\mathrm{h}(t)}\left[\frac{d\mathrm{g}(t)}{dt}+\frac{d\mathrm{h}(t)}{dt}\right]^2=\frac{1}{\mathrm{g}(t)+\mathrm{h}(t)}\left[\left(\frac{d\mathrm{g}(t)}{dt}\right)^2+\left(\frac{d\mathrm{h}(t)}{dt}\right)^2+2\frac{d\mathrm{g}(t)}{dt}\frac{d\mathrm{h}(t)}{dt}\right]$$

$\mathrm{y}_3(t)\neq\mathrm{y}_1(t)+\mathrm{y}_2(t)\rightarrow$ Not Additive $\rightarrow$ Non-Linear

For example, let $\mathrm{g}(t)=t^2$ and let $\mathrm{h}(t)=3t+2$. Then $\frac{d\mathrm{g}(t)}{dt}=2t$ and $\frac{d\mathrm{h}(t)}{dt}=3$.

Then $\mathrm{y}_1(t)=\frac{4t^2}{t^2}=4$ and $\mathrm{y}_2(t)=\frac{9}{3t+2}$ and $\mathrm{y}_1(t)+\mathrm{y}_2(t)=\frac{12t+8+9}{3t+2}=4\frac{t+17/12}{t+2/3}$.

$$\mathrm{y}_3(t)=\frac{4t^2+9+2\times 2t\times 3}{t^2+3t+2}=\frac{4t^2+12t+9}{t^2+3t+2}=4\frac{t^2+3t+9/4}{t^2+3t+2}\neq 4\frac{t+17/12}{t+2/3}.$$

$$\mathrm{y}[n]=\frac{\mathrm{x}[n]\mathrm{x}[n-2]}{\mathrm{x}[n-1]}, \quad \text{Homogeneous?, Additive?}$$

Let $\mathrm{x}_1[n]=\mathrm{g}[n]$. Then $\mathrm{y}_1[n]=\frac{\mathrm{g}[n]\mathrm{g}[n-2]}{\mathrm{g}[n-1]} \to K\mathrm{y}_1[n]=K\frac{\mathrm{g}[n]\mathrm{g}[n-2]}{\mathrm{g}[n-1]}$

Let $\mathrm{x}_2[n]=K\mathrm{g}[n]$. Then $\mathrm{y}_2[n]=\frac{K\mathrm{g}[n]K\mathrm{g}[n-2]}{K\mathrm{g}[n-1]}=K\frac{\mathrm{g}[n]\mathrm{g}[n-2]}{\mathrm{g}[n-1]}=K\mathrm{y}_1[n]$

Homogeneous

Let $\mathrm{x}_2[n]=\mathrm{h}[n]$. Then $\mathrm{y}_2[n]=\frac{\mathrm{h}[n]\mathrm{h}[n-2]}{\mathrm{h}[n-1]}$.

Let $\mathrm{x}_3[n]=\mathrm{g}[n]+\mathrm{h}[n]$. Then $\mathrm{y}_3[n]=\frac{(\mathrm{g}[n]+\mathrm{h}[n])(\mathrm{g}[n-2]+\mathrm{h}[n-2])}{\mathrm{g}[n-1]+\mathrm{h}[n-1]}$.

$\mathrm{y}_3[n]\neq \mathrm{y}_1[n]+\mathrm{y}_2[n] \to$ Not Additive $\to$ Non-Linear

$y(t) = x(t-4)$, Invertible?

Let $t \to t+4$. Then $y(t+4) = x(t) \to$ Invertible

$y(t) = \cos(x(t))$, Invertible?

$x(t) = \cos^{-1}(y(t))$ The $\cos^{-1}$ function is multiple-valued.

Not Invertible.

$y[n] = n x[n]$, Invertible?

$x[n] = y[n]/n$. When $n = 0$, $x[n]$ is undefined. $\to$ Not Invertible.

If $y[n]$ is zero, that can be because $x[n] = 0$ or because $n = 0$ (or both).

So when $y[n] = 0$ we cannot determine $x[n]$ from $y[n]$.

$$\mathrm{y}[n]=\begin{cases}\mathrm{x}[n-1] & , \; n\geq 1 \\ 0 & , \; n=0 \\ \mathrm{x}[n] & , \; n\leq -1\end{cases} \quad , \quad \text{Invertible?}$$

When $n=0$, $\mathrm{y}[n]=0$ regardless of the value of $\mathrm{x}[n]$. So, when $n=0$, $\mathrm{x}[n]$ cannot be determined by knowledge of $\mathrm{y}[n]$. $\rightarrow$ Not Invertible

$\mathrm{y}[n]=\mathrm{x}[n]\mathrm{x}[n-1]$, Invertible?

$\mathrm{y}[n]=\mathrm{x}[n]\mathrm{x}[n-1] \rightarrow \mathrm{x}[n]=\mathrm{y}[n]/\mathrm{x}[n-1]$

$\mathrm{y}[n]=\mathrm{x}[n]\mathrm{x}[n-1] \rightarrow \mathrm{x}[n-1]=\mathrm{y}[n]/\mathrm{x}[n] \rightarrow \mathrm{x}[n]=\mathrm{y}[n+1]/\mathrm{x}[n+1]$

So we cannot determine $\mathrm{x}[n]$ without first determining either $\mathrm{x}[n-1]$ or $\mathrm{x}[n+1]$. $\rightarrow$ Not Invertible

Example: Let $\mathrm{x}[n]=(-1)^n$. Then $\mathrm{y}[n]=(-1)^n(-1)^{n-1}=-1$.

$\mathrm{y}[n]$ always equals -1 and that can occur with $\mathrm{x}[n]=1$ and $\mathrm{x}[n-1]=-1$ or with $\mathrm{x}[n]=-1$ and $\mathrm{x}[-n]=1$.

$$\mathrm{y}[n] = \mathrm{x}[1-n], \quad \text{Invertible?}$$

$$\mathrm{y}[n] = \mathrm{x}[1-n] \rightarrow \mathrm{y}[n-1] = \mathrm{x}[-n] \rightarrow \mathrm{x}[n] = \mathrm{y}[-(n-1)] = \mathrm{y}[1-n]$$

Invertible

$$\mathrm{y}[n] = \sum_{m=-\infty}^{n} (1/2)^{n-m}\,\mathrm{x}[m], \quad \text{Invertible?}$$

$$\mathrm{y}[n] - \mathrm{y}[n-1] = \sum_{m=-\infty}^{n} (1/2)^{n-m}\,\mathrm{x}[m] - \sum_{m=-\infty}^{n-1} (1/2)^{n-1-m}\,\mathrm{x}[m]$$

$$\mathrm{y}[n] - \mathrm{y}[n-1] = \underbrace{(1/2)^{n-n}}_{=1}\mathrm{x}[n] + \underbrace{\sum_{m=-\infty}^{n-1} (1/2)^{n-1-m}\,\mathrm{x}[m] - \sum_{m=-\infty}^{n-1} (1/2)^{n-1-m}\,\mathrm{x}[m]}_{=0}$$

$$\mathrm{x}[n] = \mathrm{y}[n] - \mathrm{y}[n-1] \rightarrow \text{Invertible}$$

$$\mathrm{y}(t)=\int_{-\infty}^{t} e^{-(t-\tau)}\mathrm{x}(\tau)d\tau, \quad \text{Invertible?}$$

Leibniz's rule for differentiating an integral is $\dfrac{d}{dx}\displaystyle\int_{-\infty}^{x}\mathrm{g}(\lambda)d\lambda=\mathrm{g}(x)$.

Applying it to this case, $\mathrm{y}'(t)=e^{-(t-t)}\mathrm{x}(t)=\mathrm{x}(t)\rightarrow$ Invertible

$$\mathrm{y}(t)=\mathrm{x}'(t) \quad , \quad \text{Invertible?}$$

If $\mathrm{x}(t)=\displaystyle\int_{-\infty}^{t}\mathrm{y}(\tau)d\tau+K$, then $\mathrm{x}'(t)=\mathrm{y}(t)$.

Therefore, $\mathrm{y}(t)=\mathrm{x}'(t)\Rightarrow\mathrm{x}(t)=\displaystyle\int_{-\infty}^{t}\mathrm{y}(\tau)d\tau+K$

*W*e can determine $\mathrm{x}(t)$ to within an additive constant K,
but not exactly. $\rightarrow$ Not Invertible

$$y(t)=x(2t) \quad , \text{ Invertible?}$$

$$y(t/2)=x(t)\rightarrow \text{ Invertible}$$

$$y[n]=\begin{cases} x[n/2], & n \text{ even} \\ 0 & , n \text{ odd} \end{cases} \quad , \text{ Invertible?}$$

$$y[2n]=\begin{cases} x[n], & 2n \text{ even} \\ 0 & , 2n \text{ odd} \leftarrow 2n \text{ can never be odd.} \end{cases}$$

Therefore

$$x[n]=y[2n]\rightarrow \text{Invertible}$$

If $\mathrm{x}[n]=(0.9)^n\,\mathrm{u}[n]$ and $\mathrm{h}[n]=\mathrm{u}[n-4]$ and $\mathrm{y}[n]=\mathrm{x}[n]*\mathrm{h}[n]$ find $\mathrm{y}[n]$.

$$\mathrm{y}[n]=\sum_{m=-\infty}^{\infty}\mathrm{x}[m]\mathrm{h}[n-m]=\sum_{m=-\infty}^{\infty}(0.9)^m\,\mathrm{u}[m]\mathrm{u}[n-m-4]$$

$$\mathrm{y}[n]=\left\{\begin{array}{ll}\sum_{m=0}^{n-4}(0.9)^m & ,\ n\geq 4\\ 0 & ,\ n<4\end{array}\right\}=\frac{1-(0.9)^{n-3}}{1-0.9}\mathrm{u}[n-4]$$

$$=10\left[1-(0.9)^{n-3}\right]\mathrm{u}[n-4]$$

This starts with value 1 at time $n=4$ and approaches, in a decaying exponential form, a final value of 10 as $n\to\infty$.