Time-Domain Analysis of Systems

Continuous Time

Impulse Response

Continuous-time LTI systems are described by differential equations of the general form,

$$a_n \mathrm{y}^{(n)}(t) + a_{n-1} \mathrm{y}^{(n-1)}(t) + \cdots + a_1 \mathrm{y}'(t) + a_0 \mathrm{y}(t)$$
$$= b_m \mathrm{x}^{(m)}(t) + b_{m-1} \mathrm{x}^{(m-1)}(t) + \cdots + b_1 \mathrm{x}'(t) + b_0 \mathrm{x}(t)$$

For all times, $t < 0$:

If the excitation $\mathrm{x}(t)$ is a unit impulse $\delta(t)$, then for all time $t < 0$ it is zero. The response $\mathrm{y}(t)$ is zero before time $t = 0$ because there has never been an excitation before that time.

Impulse Response

For all time $t > 0$:

The excitation is zero, but there has been a non-zero excitation before $t = 0$, the impulse $\delta(t)$. The impulse puts energy into the system at time $t = 0$ and then goes away. The response is no longer zero. Rather, since the excitation is now zero, it is the homogeneous solution of the differential equation.

Impulse Response

At time $t = 0$:

The excitation is an impulse. The inhomogeneous response in general contains the forcing function (the impulse) and all its unique derivatives. Therefore, it would be possible, in general, for the response to contain an impulse plus all the derivatives of an impulse because these all occur at time $t = 0$ and are zero before and after that time. Whether or not the response actually does contain an impulse or derivatives of an impulse at time $t = 0$ depends on the form of the differential equation

$$a_n \mathrm{y}^{(n)}(t) + a_{n-1} \mathrm{y}^{(n-1)}(t) + \cdots + a_1 \mathrm{y}'(t) + a_0 \mathrm{y}(t)$$
$$= b_m \mathrm{x}^{(m)}(t) + b_{m-1} \mathrm{x}^{(m-1)}(t) + \cdots + b_1 \mathrm{x}'(t) + b_0 \mathrm{x}(t)$$

Impulse Response

$$a_n y^{(n)}(t) + a_{n-1} y^{(n-1)}(t) + \cdots + a_1 y'(t) + a_0 y(t)$$
$$= b_m x^{(m)}(t) + b_{m-1} x^{(m-1)}(t) + \cdots + b_1 x'(t) + b_0 x(t)$$

Case 1: $m < n$

If the response $y(t)$ were to contain an impulse at time $t = 0$ then the nth derivative of the response $y^{(n)}(t)$ would contain the nth derivative of an impulse. Since the highest derivative of the impulse excitation is the mth derivative and $m < n$, the differential equation could not be satisfied at time $t = 0$. Therefore, if $m < n$ the response cannot contain an impulse or any derivatives of an impulse.

Impulse Response

$$a_n\,y^{(n)}(t) + a_{n-1}\,y^{(n-1)}(t) + \cdots + a_1\,y'(t) + a_0\,y(t)$$
$$= b_m\,x^{(m)}(t) + b_{m-1}\,x^{(m-1)}(t) + \cdots + b_1\,x'(t) + b_0\,x(t)$$

Case 2: $m = n$

In this case the highest derivative of the excitation and response are the same and the response will contain an impulse at time $t = 0$ but no derivatives of an impulse.

Case 3: $m > n$

In this case, the response will contain an impulse at time $t = 0$ plus derivatives of an impulse up to the $(m - n)$th derivative. This case is rare in the analysis of practical systems.

Impulse Response

Example

Let a system be described by $\mathrm{y}'(t)+3\mathrm{y}(t)=\mathrm{x}(t)$. If the excitation x is an impulse we have $\mathrm{h}'(t)+3\mathrm{h}(t)=\delta(t)$. We know that $\mathrm{h}(t)=0$ for $t<0$ and that $\mathrm{h}(t)$ is the homogeneous solution for $t>0$ which is $\mathrm{h}(t)=Ke^{-3t}$. There are more derivatives of y than of x in the differential equation. Therefore the impulse response cannot contain an impulse. So the impulse response is of the form

$$\mathrm{h}(t)=Ke^{-3t}\,\mathrm{u}(t).$$

Impulse Response

Example

To find the constant K integrate the differential equation

$\mathrm{h}'(t)+3\,\mathrm{h}(t)=\delta(t)$ over the infinitesimal time range 0^- to 0^+.

$$\int_{0^-}^{0^+}\mathrm{h}'(t)\,dt+3\int_{0^-}^{0^+}\mathrm{h}(t)=\int_{0^-}^{0^+}\delta(t)$$

$$\underbrace{\mathrm{h}(0^+)}_{=K}-\underbrace{\mathrm{h}(0^-)}_{=0}+3\int_{0^-}^{0^+}Ke^{-3t}\,\mathrm{u}(t)\,dt=\underbrace{\mathrm{u}(0^+)}_{=1}-\underbrace{\mathrm{u}(0^-)}_{=0}$$

$$K+3K\left[\frac{e^{-3t}}{-3}\right]_0^{0^+}=K+3K\underbrace{\left[(-1/3)-(-1/3)\right]}_{=0}=1$$

$$K=1\Rightarrow \mathrm{h}(t)=e^{-3t}\,\mathrm{u}(t)$$

Impulse Response

Example

To check the solution, put it into the differential equation to see whether it is satisfied.

$$\frac{d}{dt}\left(e^{-3t}\,\mathrm{u}(t)\right)+3e^{-3t}\,\mathrm{u}(t)=\delta(t)$$

$$e^{-3t}\delta(t)-3e^{-3t}\,\mathrm{u}(t)+3e^{-3t}\,\mathrm{u}(t)=\delta(t)$$

$$\underbrace{e^{-3t}\delta(t)}_{=e^{0}\delta(t)=\delta(t)}=\delta(t)\Rightarrow\delta(t)=\delta(t)\quad\text{Check.}$$

Impulse Response

Example

Let a system be described by $4\,\mathrm{y}'(t)+3\,\mathrm{y}(t)=\mathrm{x}'(t)$. The homogeneous solution is $\mathrm{y}_\mathrm{h}(t)=Ke^{-3t/4}$ and that is the form of the impulse response for $t>0$. The number of y derivatives and the number of x derivatives are the same. Therefore the impulse response has an impulse in it and its form is $\mathrm{h}(t)=Ke^{-3t/4}\,\mathrm{u}(t)+K_\delta\delta(t)$. Integrate between 0^- and 0^+.

$$4\int_{0^-}^{0^+}\mathrm{h}'(t)\,dt+3\int_{0^-}^{0^+}\mathrm{h}(t)\,dt=\int_{0^-}^{0^+}\delta'(t)\,dt$$

Impulse Response

Example

$$4\int_{0^-}^{0^+} \mathrm{h}'(t)\,dt + 3\int_{0^-}^{0^+} \mathrm{h}(t)\,dt = \int_{0^-}^{0^+} \delta'(t)\,dt$$

$$\left\{ \begin{array}{l} 4\left[\underbrace{\mathrm{h}(0^+)}_{=K} - \underbrace{\mathrm{h}(0^-)}_{=0} + K_\delta \left(\underbrace{\delta(0^+)}_{=0} - \underbrace{\delta(0^-)}_{=0} \right) \right] \\ +3\underbrace{\int_{0^-}^{0^+} Ke^{-3t/4}\,\mathrm{u}(t)\,dt}_{=0} + 3K_\delta \left[\underbrace{\mathrm{u}(0^+)}_{=1} - \underbrace{\mathrm{u}(0^-)}_{=0} \right] \end{array} \right\} = \underbrace{\delta(0^+)}_{=0} - \underbrace{\delta(0^-)}_{=0}$$

$$4K + 3K_\delta = 0$$

Impulse Response

Example

Now integrate again over the same infinitesimal interval.

$$4\int_{0^-}^{0^+}\int_{-\infty}^{t} \mathrm{h}'(\lambda)\,d\lambda\,dt + 3\int_{0^-}^{0^+}\int_{-\infty}^{t} Ke^{-3\lambda/4}\,\mathrm{u}(\lambda)\,d\lambda\,dt + 3\int_{0^-}^{0^+}\int_{-\infty}^{t} K_\delta \delta(\lambda)\,d\lambda\,dt = \int_{0^-}^{0^+}\int_{-\infty}^{t} \delta'(\lambda)\,d\lambda\,dt$$

$$\underbrace{4\int_{0^-}^{0^+}\mathrm{h}(t)\,dt}_{=K_\delta} - \underbrace{4K\int_{0^-}^{0^+}\left(1-e^{-3t/4}\right)\mathrm{u}(t)\,dt}_{=0} + 3K_\delta \underbrace{\int_{0^-}^{0^+}\mathrm{u}(t)\,dt}_{=0} = \underbrace{\int_{0^-}^{0^+}\delta(t)\,dt}_{=1}$$

$$4K_\delta = 1 \Rightarrow K_\delta = 1/4 \Rightarrow 4K + 3/4 = 0 \Rightarrow K = -3/16$$

$$\mathrm{h}(t) = (-3/16)e^{-3t/4}\,\mathrm{u}(t) + (1/4)\delta(t)$$

Impulse Response

Example

$$\mathrm{h}(t) = (-3/16)e^{-3t/4}\,\mathrm{u}(t) + (1/4)\delta(t)$$

The original differential equation is $4\,\mathrm{h}'(t) + 3\,\mathrm{h}(t) = \delta'(t)$.

Substituting the solution we get

$$\left\{\begin{array}{l} 4\dfrac{d}{dt}\left[(-3/16)e^{-3t/4}\,\mathrm{u}(t) + (1/4)\delta(t)\right] \\ +3\left[(-3/16)e^{-3t/4}\,\mathrm{u}(t) + (1/4)\delta(t)\right] \end{array}\right\} = \delta'(t)$$

$$\left\{\begin{array}{l} 4\left[(-3/16)e^{-3t/4}\delta(t) + (9/64)e^{-3t/4}\,\mathrm{u}(t) + (1/4)\delta'(t)\right] \\ +3\left[(-3/16)e^{-3t/4}\,\mathrm{u}(t) + (1/4)\delta(t)\right] \end{array}\right\} = \delta'(t)$$

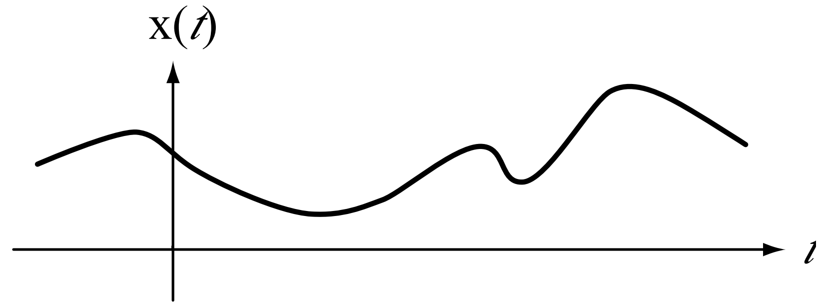
$$-(3/4)e^{-3t/4}\delta(t) + (9/16)e^{-3t/4}\,\mathrm{u}(t) + \delta'(t) - (9/16)e^{-3t/4}\,\mathrm{u}(t) + (3/4)\delta(t) = \delta'(t)$$

$\delta'(t) = \delta'(t)$ Check.

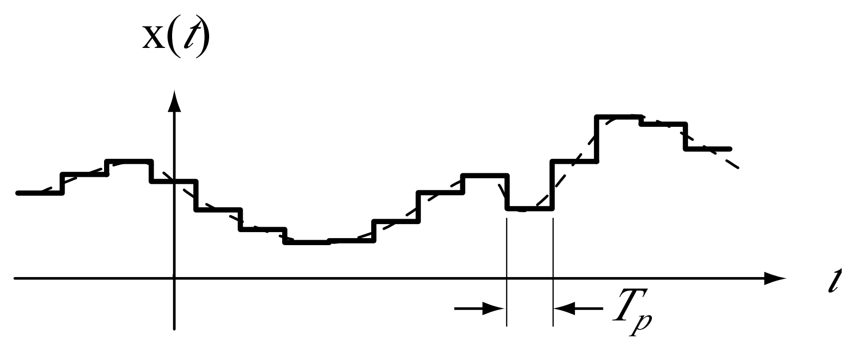
The Convolution Integral

If a continuous-time LTI system is excited by an arbitrary excitation, the response could be found approximately by approximating the excitation as a sequence of contiguous rectangular pulses of width T_p.

Exact Excitation



Approximate Excitation



The Convolution Integral

Approximating the excitation as a pulse train can be expressed mathematically by

$$\mathrm{x}(t) \cong \cdots + \mathrm{x}(-T_p)\,\mathrm{rect}\left(\frac{t+T_p}{T_p}\right) + \mathrm{x}(0)\,\mathrm{rect}\left(\frac{t}{T_p}\right) + \mathrm{x}(T_p)\,\mathrm{rect}\left(\frac{t-T_p}{T_p}\right) + \cdots$$

or

$$\mathrm{x}(t) \cong \sum_{n=-\infty}^{\infty} \mathrm{x}(nT_p)\,\mathrm{rect}\left(\frac{t-nT_p}{T_p}\right)$$

The excitation can be written in terms of pulses of width T_p and unit area

$$\mathrm{x}(t) \cong \sum_{n=-\infty}^{\infty} T_p\,\mathrm{x}(nT_p)\underbrace{\frac{1}{T_p}\,\mathrm{rect}\left(\frac{t-nT_p}{T_p}\right)}_{\text{shifted unit}-\text{area pulse}}$$

The Convolution Integral

Let the response to an unshifted pulse of unit area and width T_p be the "unit pulse response" $\mathrm{h}_p(t)$. Then, invoking linearity, the response to the overall excitation is (approximately) a sum of shifted and scaled unit pulse responses of the form

$$\mathrm{y}(t) \cong \sum_{n=-\infty}^{\infty} T_p \, \mathrm{x}(nT_p) \mathrm{h}_p(t - nT_p)$$

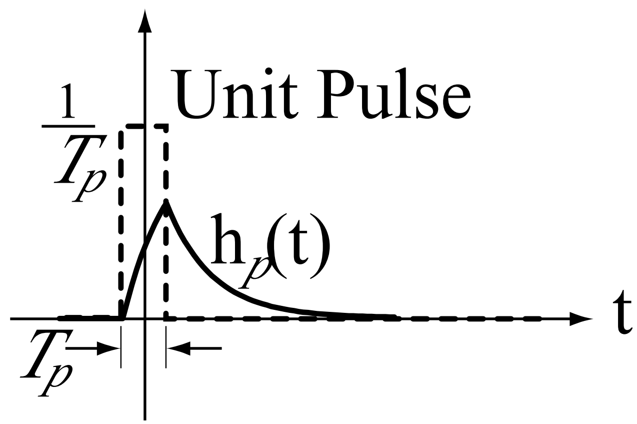
As T_p approaches zero, the unit pulses become unit impulses, the unit pulse response becomes the **unit impulse response** $\mathrm{h}(t)$ and the excitation and response become exact.

The Convolution Integral

Example

Let the unit pulse response be that of the *RC* lowpass filter

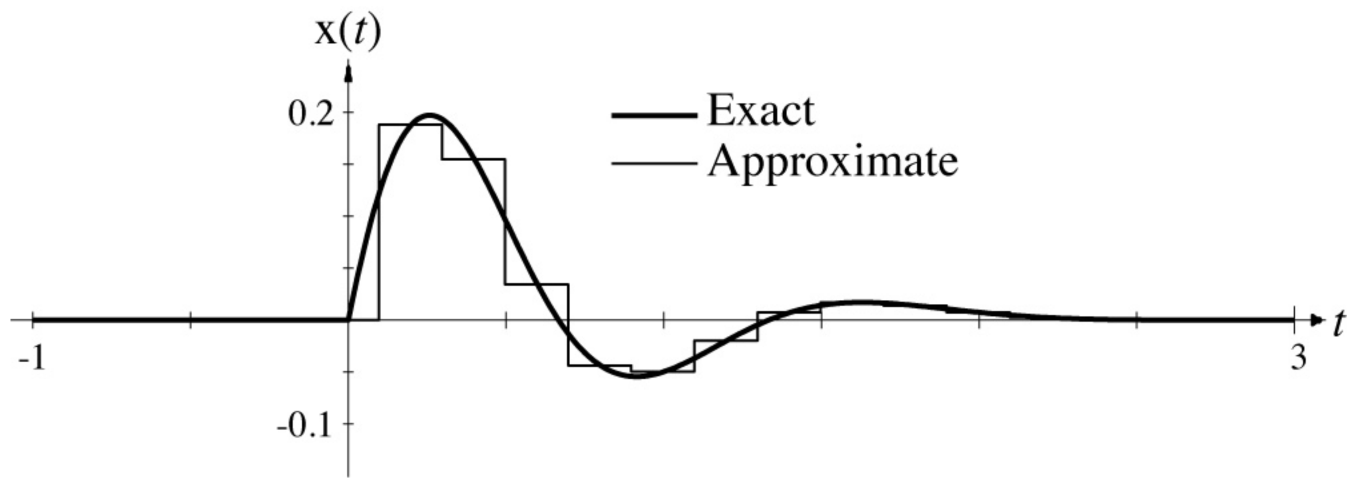
$$\mathrm{h}_p(t) = \left(\frac{1 - e^{-(t+T_p/2)/RC}}{T_p} \right) \mathrm{u}\left(t + \frac{T_p}{2} \right) - \left(\frac{1 - e^{-(t+T_p/2)/RC}}{T_p} \right) \mathrm{u}\left(t - \frac{T_p}{2} \right)$$



The Convolution Integral

Example

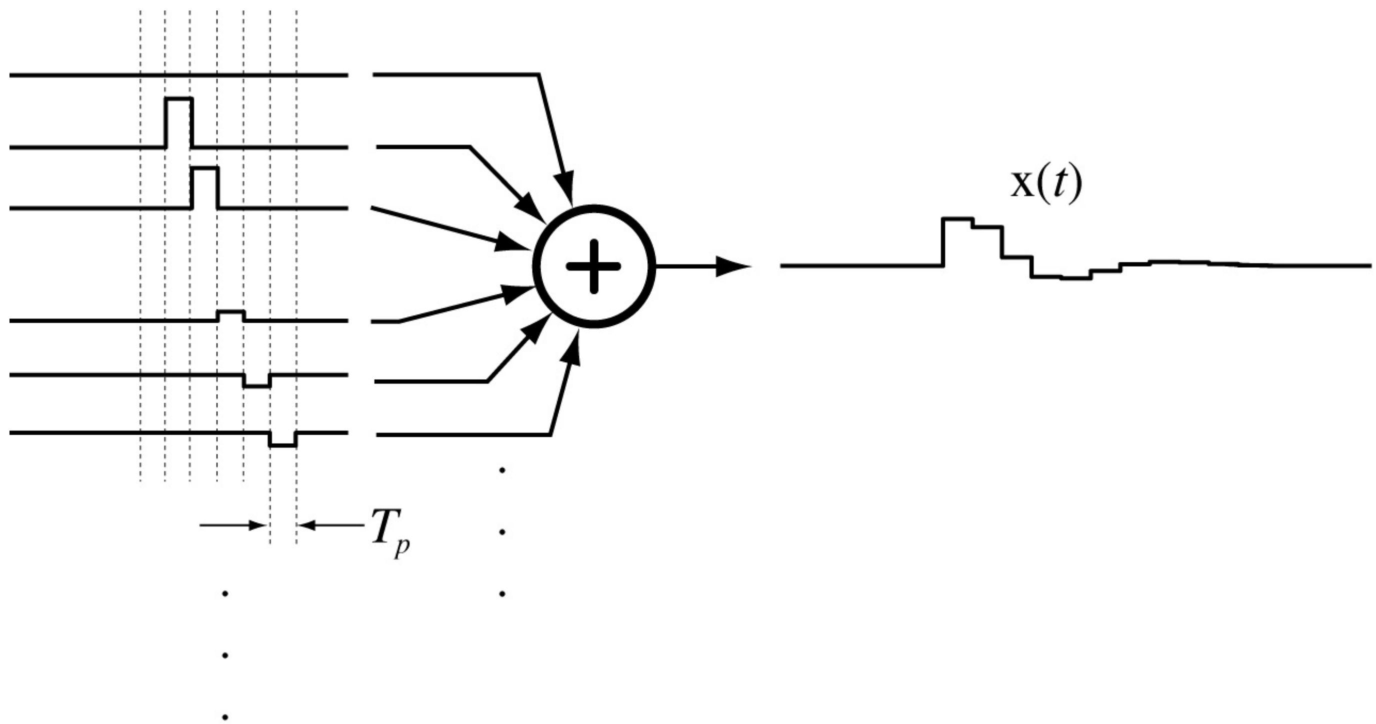
Let $x(t)$ be this smooth waveform and let it be approximated by a sequence of rectangular pulses.



The Convolution Integral

Example

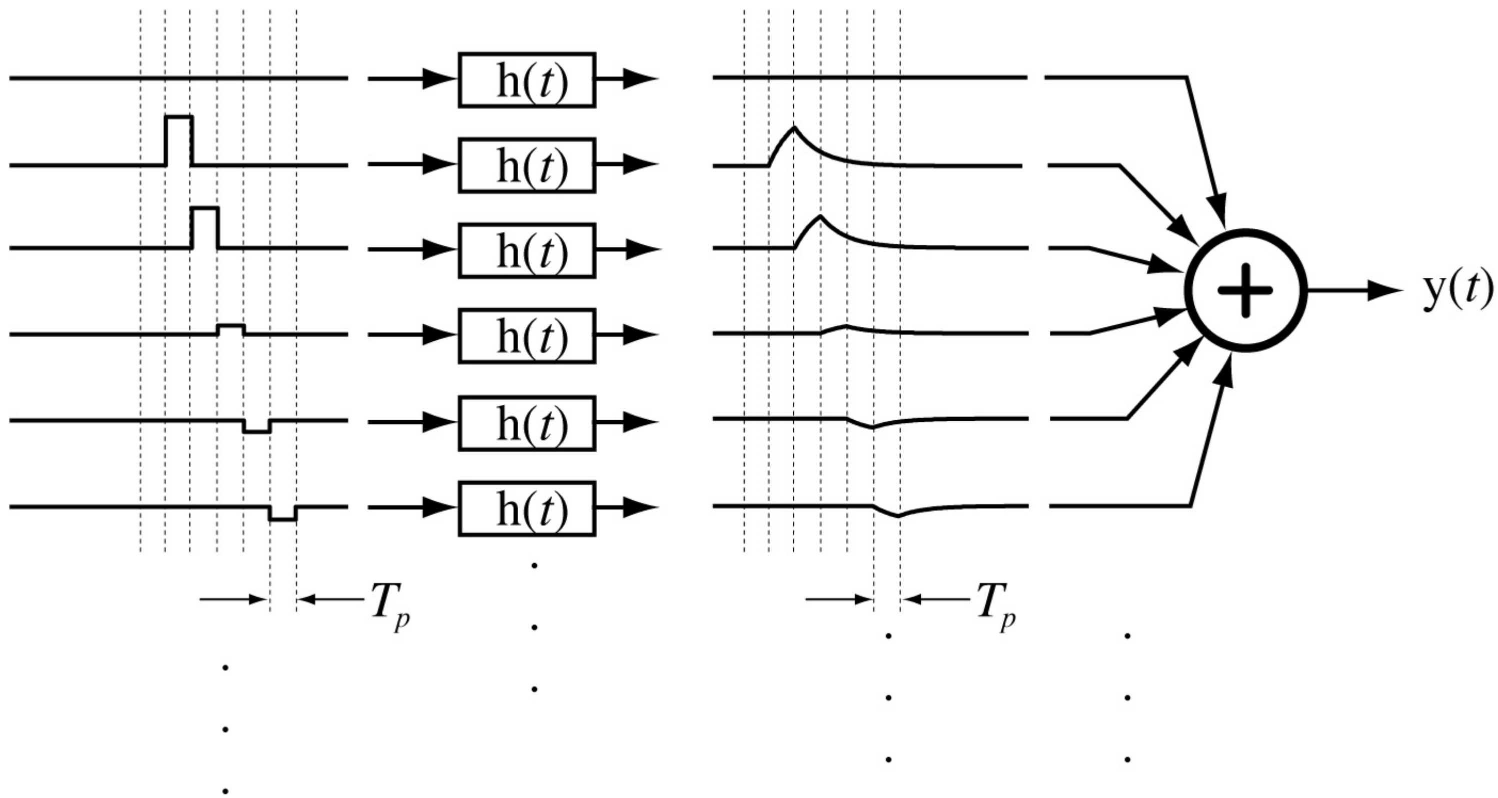
The approximate excitation is a sum of rectangular pulses.



The Convolution Integral

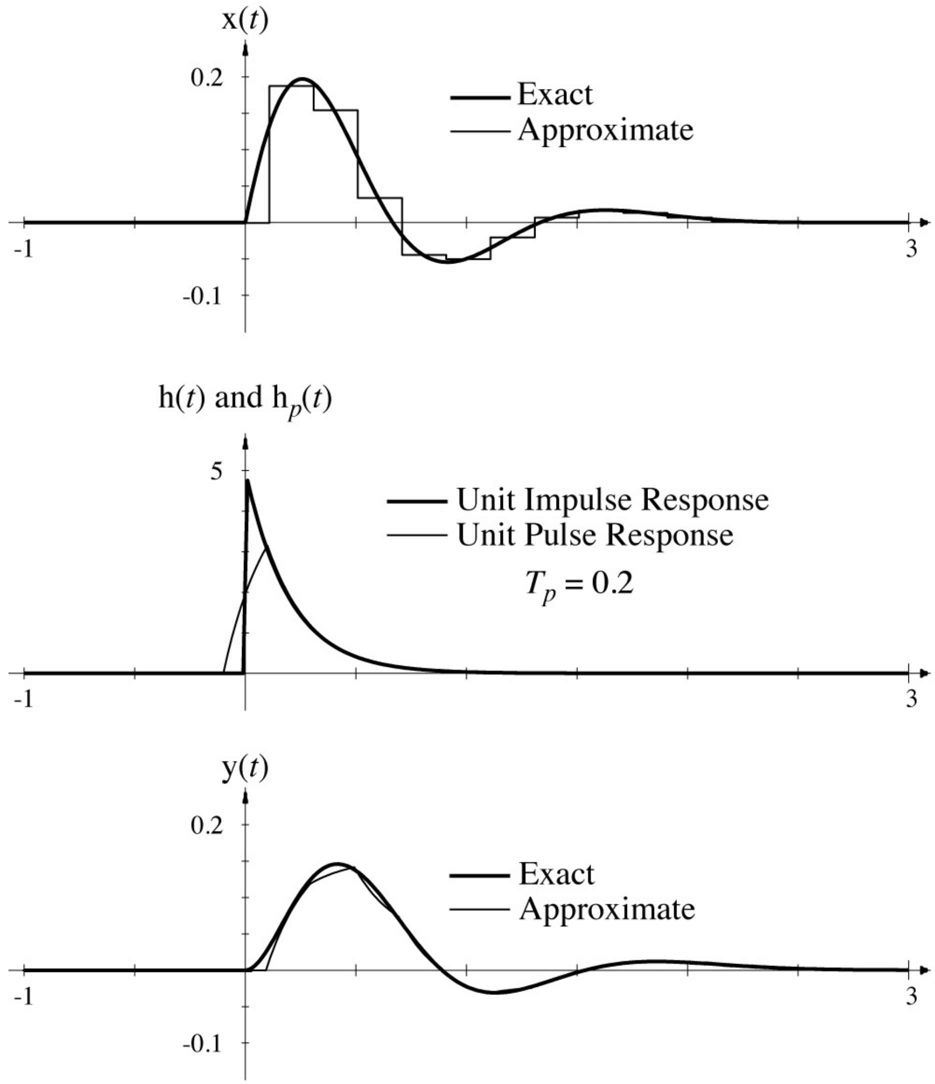
Example

The approximate response is a sum of pulse responses.



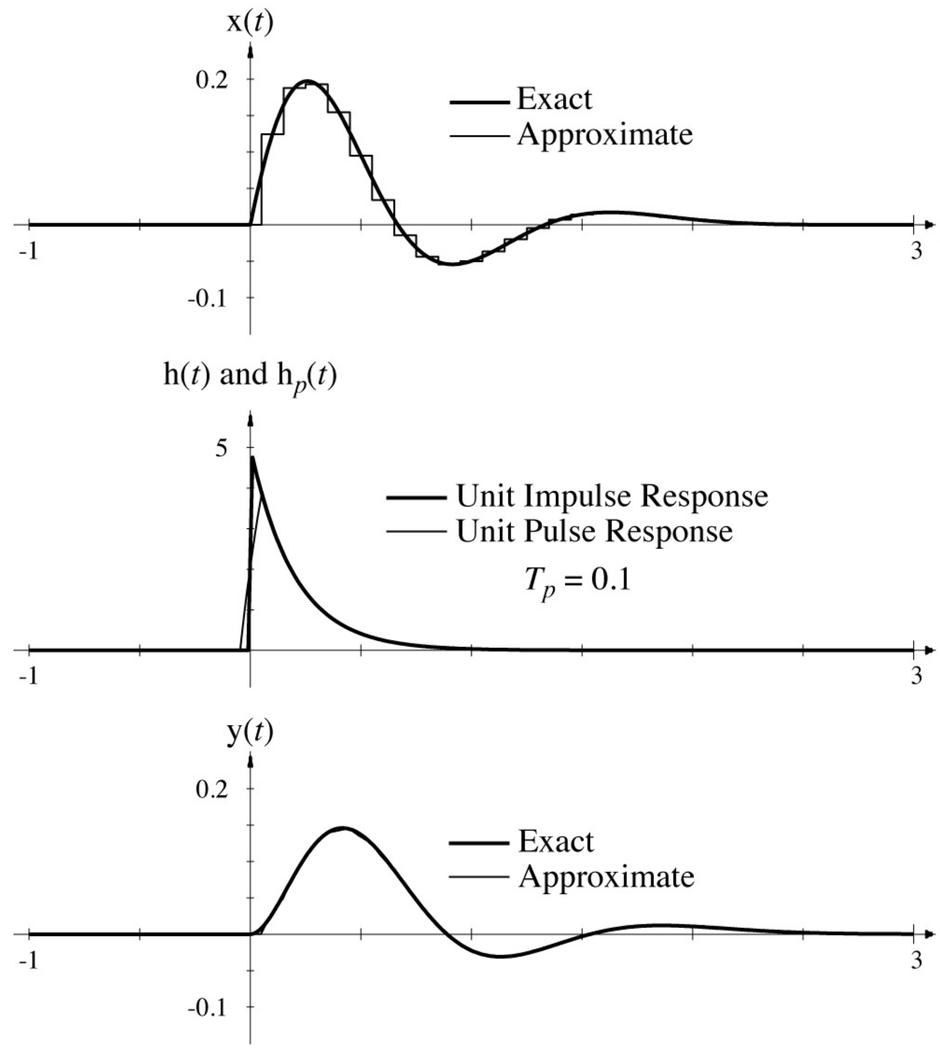
The Convolution Integral

$T_p = 0.1$ s



The Convolution Integral

$T_p = 0.05\ \text{s}$



The Convolution Integral

As T_p approaches zero, the expressions for the approximate excitation and response approach the limiting exact forms

Superposition Integral

$$\mathrm{x}(t) = \int_{-\infty}^{\infty} \mathrm{x}(\tau)\delta(t-\tau)\,d\tau$$

Convolution Integral

$$\mathrm{y}(t) = \int_{-\infty}^{\infty} \mathrm{x}(\tau)\mathrm{h}(t-\tau)\,d\tau$$

The Convolution Integral

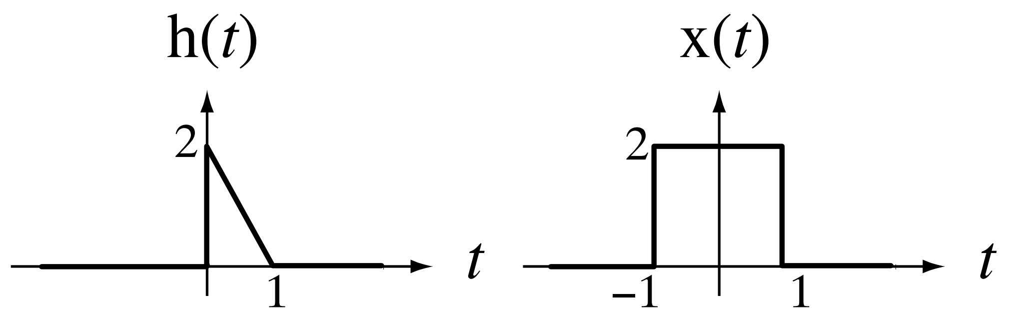
Another (quicker) way to develop the convolution integral is to start with $x(t) = \int_{-\infty}^{\infty} x(\tau)\delta(t-\tau)\,d\tau$ which follows directly from the sampling property of the impulse. If $h(t)$ is the impulse response of the system, and if the system is LTI, then the response to $x(\tau)\delta(t-\tau)$ must be $x(\tau)h(t-\tau)$. Then, invoking additivity, if $x(t) = \int_{-\infty}^{\infty} x(\tau)\delta(t-\tau)\,d\tau$, then $y(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)\,d\tau$.

A Graphical Illustration of the Convolution Integral

The convolution integral is defined by

$$\mathrm{x}(t)*\mathrm{h}(t)=\int_{-\infty}^{\infty}\mathrm{x}(\tau)\mathrm{h}(t-\tau)\,d\tau$$

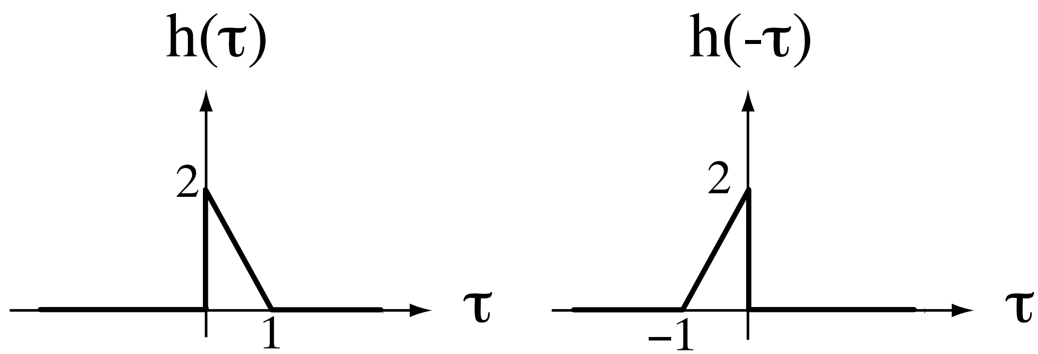
For illustration purposes let the excitation $\mathrm{x}(t)$ and the impulse response $\mathrm{h}(t)$ be the two functions below.



A Graphical Illustration of the Convolution Integral

In the convolution integral there is a factor $\mathrm{h}(t-\tau)$.

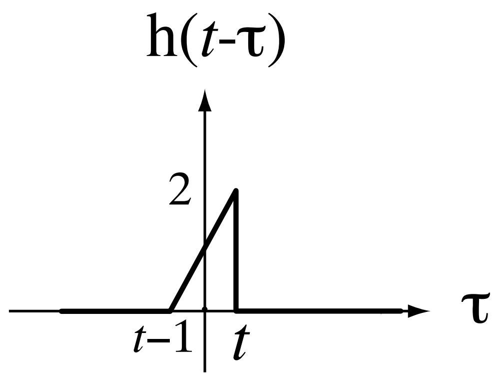
We can begin to visualize this quantity in the graphs below.



A Graphical Illustration of the Convolution Integral

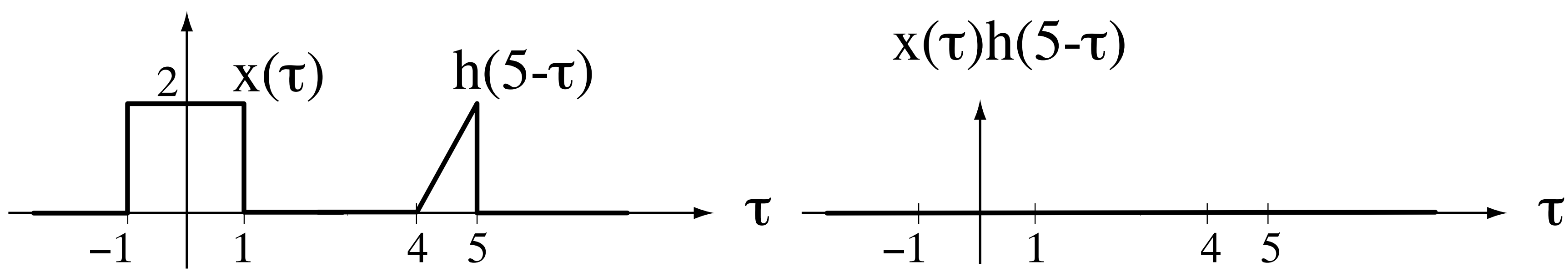
The functional transformation in going from $\mathrm{h}(t)$ to $\mathrm{h}(t-\tau)$ is

$$\mathrm{h}(\tau) \xrightarrow{\tau \to -\tau} \mathrm{h}(-\tau) \xrightarrow{\tau \to \tau - t} \mathrm{h}(-(\tau - t)) = \mathrm{h}(t-\tau)$$



A Graphical Illustration of the Convolution Integral

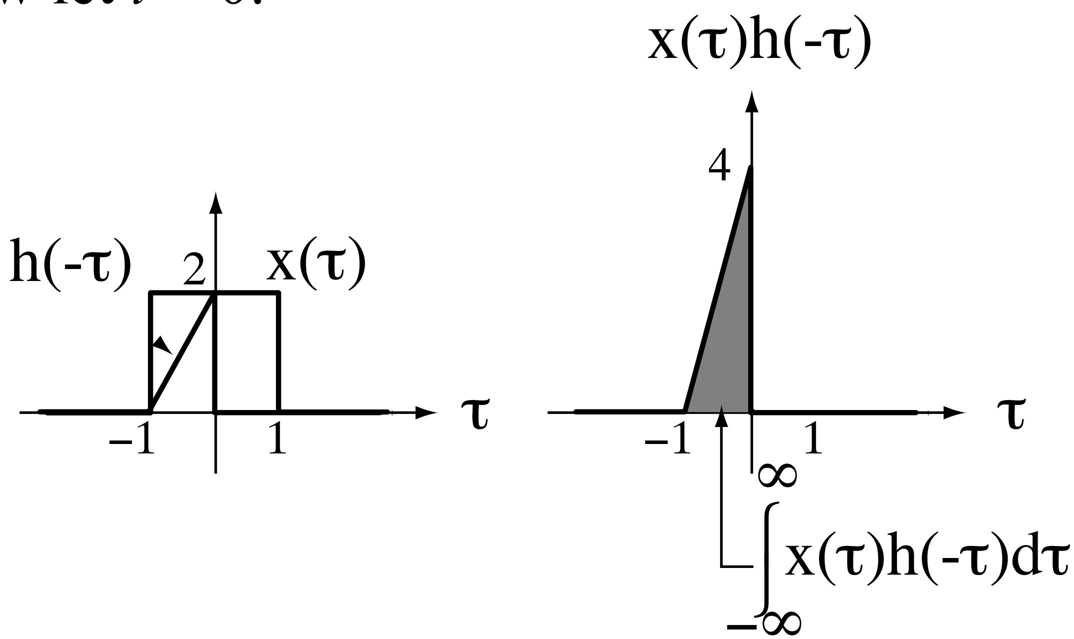
The convolution value is the area under the product of $\mathrm{x}(t)$ and $\mathrm{h}(t-\tau)$. This area depends on what t is. First, as an example, let $t = 5$.



For this choice of t the area under the product is zero.
Therefore if $\mathrm{y}(t) = \mathrm{x}(t) * \mathrm{h}(t)$ then $\mathrm{y}(5) = 0$.

A Graphical Illustration of the Convolution Integral

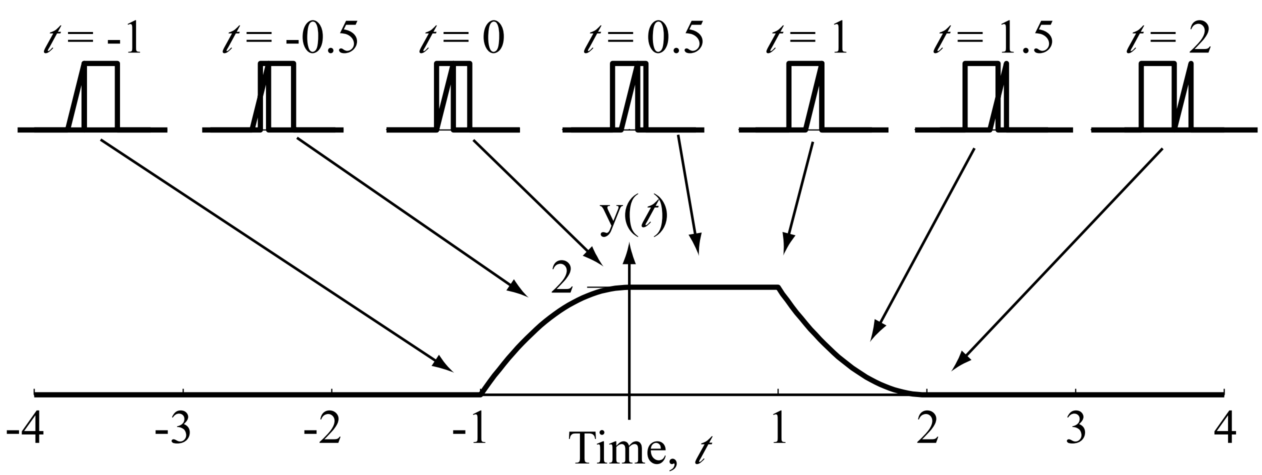
Now let $t = 0$.



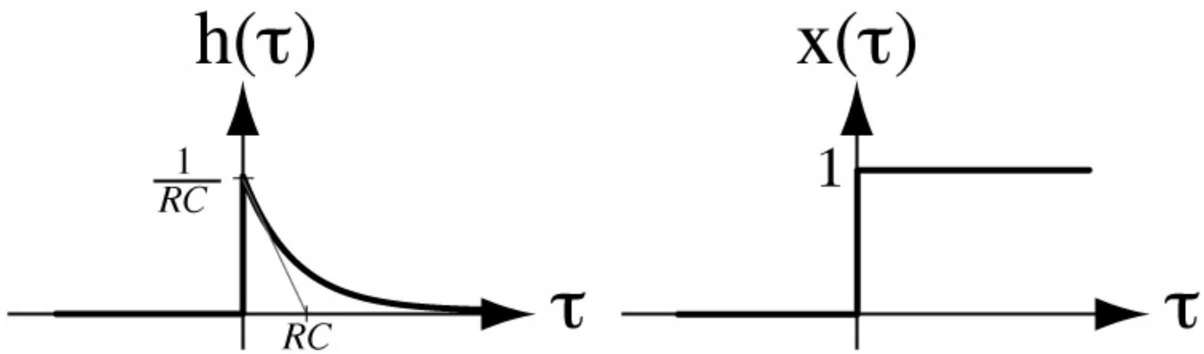
Therefore $y(0) = 2$, the area under the product.

A Graphical Illustration of the Convolution Integral

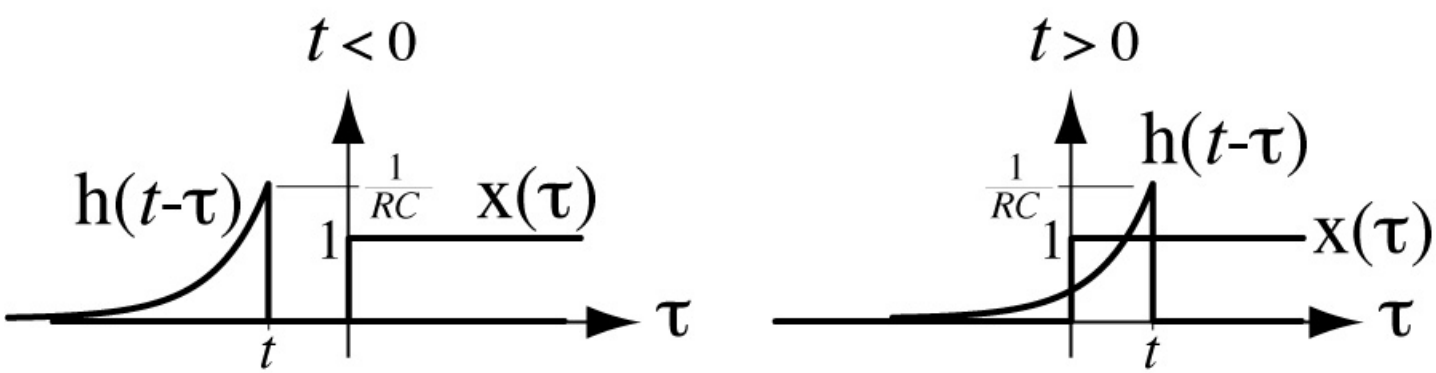
The process of convolving to find y(t) is illustrated below.



A Graphical Illustration of the Convolution Integral



$$v_{out}(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau = \int_{-\infty}^{\infty} u(\tau)\frac{e^{-(t-\tau)/RC}}{RC}u(t-\tau)d\tau$$



A Graphical Illustration of the Convolution Integral

$t < 0: \quad \mathrm{v}_{out}(t) = 0$

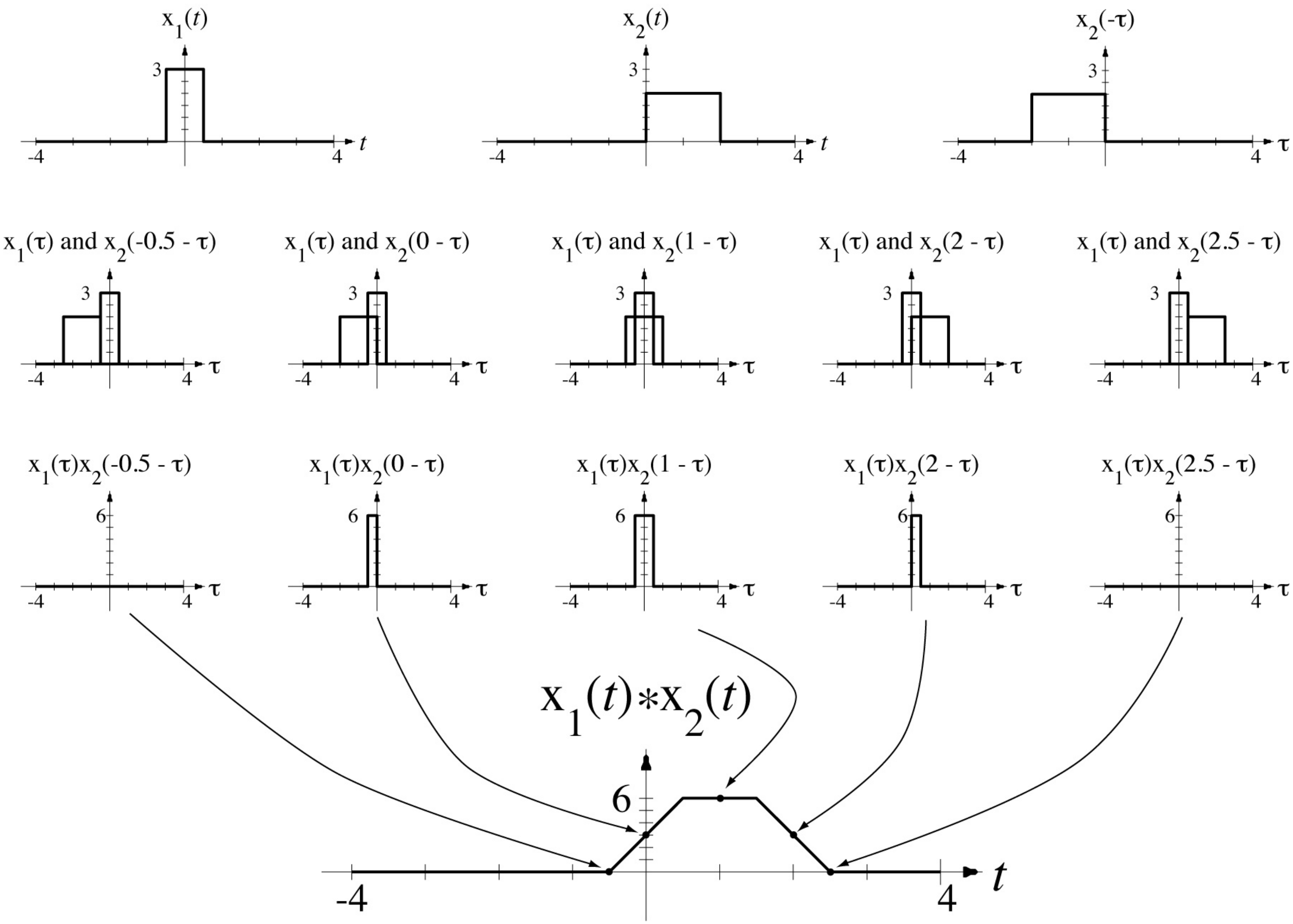
$t > 0: \quad \mathrm{v}_{out}(t) = \int_{-\infty}^{\infty} \mathrm{u}(\tau) \frac{e^{-(t-\tau)/RC}}{RC} \mathrm{u}(t-\tau)\, d\tau$

$$\mathrm{v}_{out}(t) = \frac{1}{RC}\int_{0}^{t} e^{-(t-\tau)/RC}\, d\tau = \frac{1}{RC}\left[\frac{e^{-(t-\tau)/RC}}{-1/RC}\right]_0^t = \left[-e^{-(t-\tau)/RC}\right]_0^t = 1 - e^{-t/RC}$$

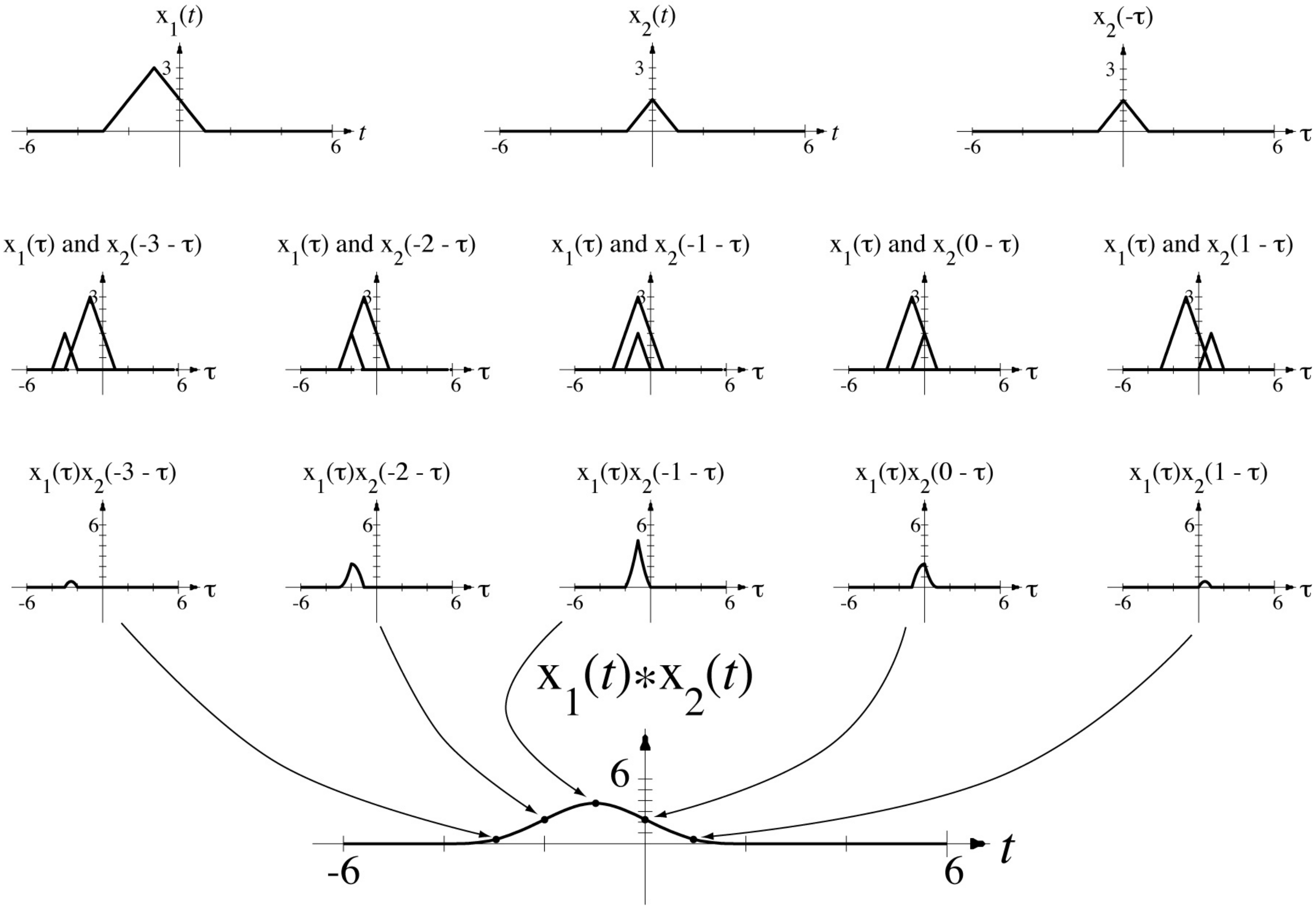
For all time, t:

$\mathrm{v}_{out}(t) = \left(1 - e^{-t/RC}\right)\mathrm{u}(t)$

Convolution Example



Convolution Example



Convolution Integral Properties

$$\mathrm{x}(t) * A\delta(t-t_0) = A\mathrm{x}(t-t_0)$$

If $\mathrm{g}(t) = \mathrm{g}_0(t) * \delta(t)$ then $\mathrm{g}(t-t_0) = \mathrm{g}_0(t-t_0) * \delta(t) = \mathrm{g}_0(t) * \delta(t-t_0)$

If $\mathrm{y}(t) = \mathrm{x}(t) * \mathrm{h}(t)$ then $\mathrm{y}'(t) = \mathrm{x}'(t) * \mathrm{h}(t) = \mathrm{x}(t) * \mathrm{h}'(t)$

and $\mathrm{y}(at) = |a|\mathrm{x}(at) * \mathrm{h}(at)$

Commutativity

$$\mathrm{x}(t) * \mathrm{y}(t) = \mathrm{y}(t) * \mathrm{x}(t)$$

Associativity

$$\left[\mathrm{x}(t) * \mathrm{y}(t)\right] * \mathrm{z}(t) = \mathrm{x}(t) * \left[\mathrm{y}(t) * \mathrm{z}(t)\right]$$

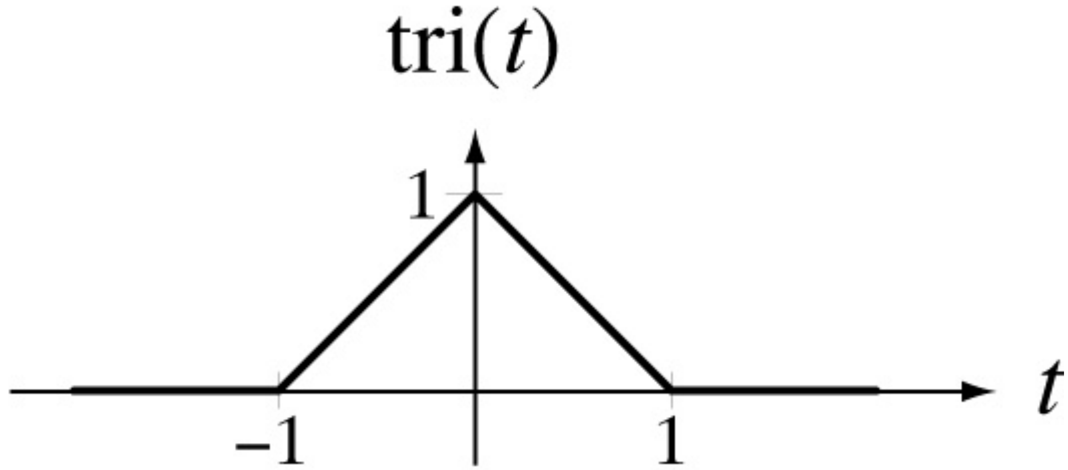
Distributivity

$$\left[\mathrm{x}(t) + \mathrm{y}(t)\right] * \mathrm{z}(t) = \mathrm{x}(t) * \mathrm{z}(t) + \mathrm{y}(t) * \mathrm{z}(t)$$

The Unit Triangle Function

$$\text{tri}(t) = \begin{cases} 1-|t| & , \ |t|<1 \\ 0 & , \ |t| \geq 1 \end{cases} = \text{ramp}(t+1) - 2\,\text{ramp}(t) + \text{ramp}(t-1)$$

$$= \text{rect}(t) * \text{rect}(t)$$

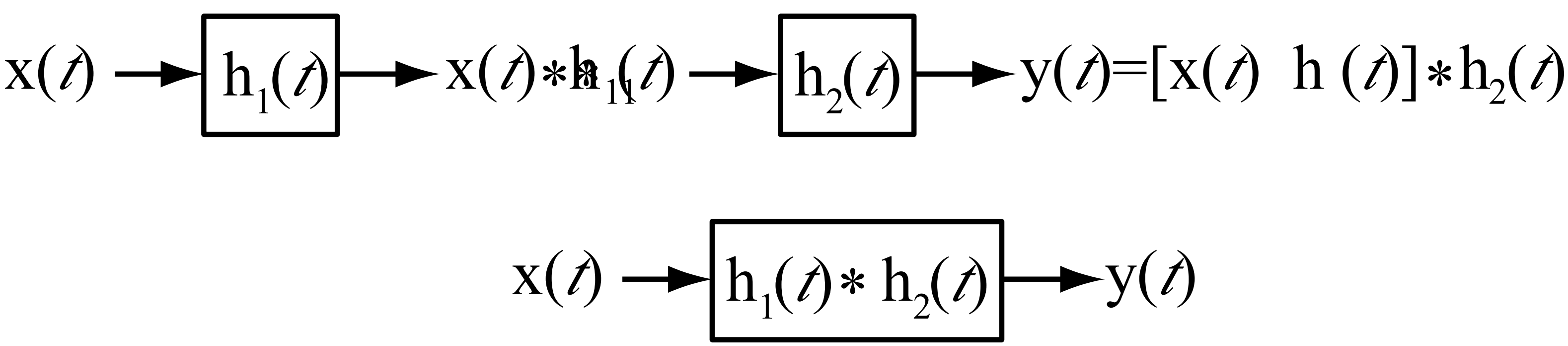


The unit triangle, is the convolution of a unit rectangle with Itself.

System Interconnections

If the output signal from a system is the input signal to a second system the systems are said to be **cascade** connected.

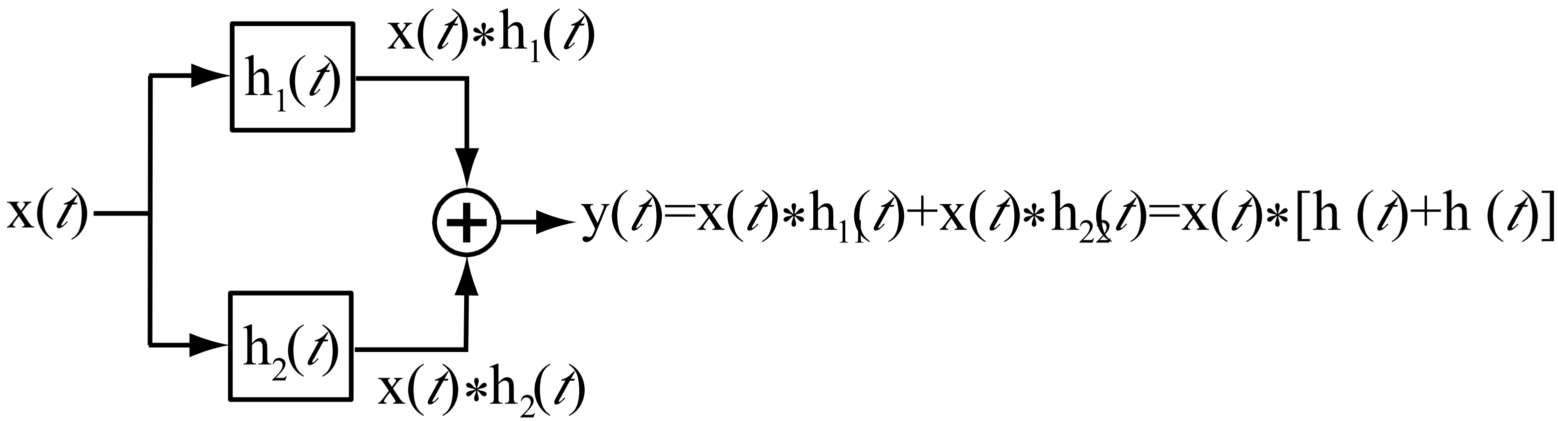
It follows from the associative property of convolution that the impulse response of a cascade connection of LTI systems is the convolution of the individual impulse responses of those systems.

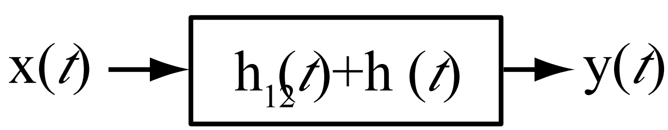


System Interconnections

If two systems are excited by the same signal and their responses are added they are said to be **parallel** connected.

It follows from the distributive property of convolution that the impulse response of a parallel connection of LTI systems is the sum of the individual impulse responses.





Unit Impulse Response and Unit Step Response

In any LTI system let an excitation $\mathrm{x}(t)$ produce the response $\mathrm{y}(t)$. Then the excitation $\frac{d}{dt}(\mathrm{x}(t))$ will produce the response $\frac{d}{dt}(\mathrm{y}(t))$. It follows then that the unit impulse response $\mathrm{h}(t)$ is the first derivative of the unit step response $\mathrm{h}_{-1}(t)$ and, conversely that the unit step response $\mathrm{h}_{-1}(t)$ is the integral of the unit impulse response $\mathrm{h}(t)$.

Stability and Impulse Response

A system is BIBO stable if its impulse response is **absolutely integrable**. That is if

$$\int_{-\infty}^{\infty} \left|\mathrm{h}(t)\right| dt \text{ is finite.}$$

Systems Described by Differential Equations

The most general form of a differential equation describing an LTI system is $\sum_{k=0}^{N} a_k \mathrm{y}^{(k)}(t) = \sum_{k=0}^{M} b_k \mathrm{x}^{(k)}(t)$. Let $\mathrm{x}(t) = Xe^{st}$ and let $\mathrm{y}(t) = Ye^{st}$. Then $\mathrm{x}^{(k)}(t) = s^k Xe^{st}$ and $\mathrm{y}^{(k)}(t) = s^k Ye^{st}$ and

$$\sum_{k=0}^{N} a_k s^k Ye^{st} = \sum_{k=0}^{M} b_k s^k Xe^{st}.$$

The differential equation has become an algebraic equation.

$$Ye^{st}\sum_{k=0}^{N} a_k s^k = Xe^{st}\sum_{k=0}^{M} b_k s^k \Rightarrow \frac{Y}{X} = \frac{\sum_{k=0}^{M} b_k s^k}{\sum_{k=0}^{N} a_k s^k}$$

Systems Described by Differential Equations

The transfer function for systems of this type is

$$\mathrm{H}(s) = \frac{\sum_{k=0}^{M} b_k s^k}{\sum_{k=0}^{N} a_k s^k} = \frac{b_M s^M + b_{M-1} s^{M-1} + \cdots + b_2 s^2 + b_1 s + b_0}{a_N s^N + a_{N-1} s^{N-1} + \cdots + a_2 s^2 + a_1 s + a_0}$$

This type of function is called a **rational function** because it is a ratio of polynomials in s. The transfer function encapsulates all the system characteristics and is of great importance in signal and system analysis.

Systems Described by Differential Equations

Now let $\mathrm{x}(t) = Xe^{j\omega t}$ and let $\mathrm{y}(t) = Ye^{j\omega t}$.

This change of variable $s \to j\omega$ changes the transfer function to the **frequency response**.

$$\mathrm{H}(j\omega) = \frac{b_M (j\omega)^M + b_{M-1}(j\omega)^{M-1} + \cdots + b_2 (j\omega)^2 + b_1 (j\omega) + b_0}{a_N (j\omega)^N + a_{N-1}(j\omega)^{N-1} + \cdots + a_2 (j\omega)^2 + a_1 (j\omega) + a_0}$$

Frequency response describes how a system responds to a sinusoidal excitation, as a function of the frequency of that excitation.

Systems Described by Differential Equations

It is shown in the text that if an LTI system is excited by a sinusoid $\mathrm{x}(t) = A_{\mathrm{x}} \cos(\omega_0 t + \theta_{\mathrm{x}})$ that the response is $\mathrm{y}(t) = A_{\mathrm{y}} \cos(\omega_0 t + \theta_{\mathrm{y}})$ where $A_{\mathrm{y}} = |\mathrm{H}(j\omega_0)| A_{\mathrm{x}}$ and $\theta_{\mathrm{y}} = \measuredangle \mathrm{H}(j\omega_0) + \theta_{\mathrm{x}}$.

MATLAB System Objects

A MATLAB system object is a special kind of variable in MATLAB that contains all the information about an LTI system. It can be created with the `tf` (transfer function) command whose syntax is

```
sys = tf(num,den)
```

where `num` is a vector of numerator coefficients of powers of s, `den` is a vector of denominator coefficients of powers of s, both in descending order and `sys` is the system object.

MATLAB System Objects

Example

The transfer function

$$H_1(s) = \frac{s^2 + 4}{s^5 + 4s^4 + 7s^3 + 15s^2 + 31s + 75}$$

can be created by the commands

```
»num = [1 0 4] ;  den = [1 4 7 15 31 75] ;

»H1 = tf(num,den) ;

»H1

Transfer function:

                  s^2 + 4

-------------------------------------

  s^5 + 4 s^4 + 7 s^3 + 15 s^2 + 31 s + 75
```

Discrete Time

Impulse Response

Discrete-time LTI systems are described mathematically by difference equations of the form

$$a_0\,\mathrm{y}[n]+a_1\,\mathrm{y}[n-1]+\cdots+a_N\,\mathrm{y}[n-N]$$
$$=b_0\,\mathrm{x}[n]+b_1\,\mathrm{x}[n-1]+\cdots+b_M\,\mathrm{x}[n-M]$$

For any excitation $\mathrm{x}[n]$ the response $\mathrm{y}[n]$ can be found by finding the response to $\mathrm{x}[n]$ as the only forcing function on the right-hand side and then adding scaled and time-shifted versions of that response to form $\mathrm{y}[n]$.

If $\mathrm{x}[n]$ is a unit impulse, the response to it as the only forcing function is simply the homogeneous solution of the difference equation with initial conditions applied. The impulse response is conventionally designated by the symbol $\mathrm{h}[n]$.

Impulse Response

Since the impulse is applied to the system at time $n = 0$, that is the only excitation of the system and the system is causal, the impulse response is zero before time $n = 0$.

$$\mathrm{h}[n] = 0 \ , \ n < 0$$

After time $n = 0$, the impulse has come and gone and the excitation is again zero. Therefore for $n > 0$, the solution of the difference equation describing the system is the homogeneous solution.

$$\mathrm{h}[n] = \mathrm{y}_h[n] \ , \ n > 0$$

Therefore, the impulse response is of the form,

$$\mathrm{h}[n] = \mathrm{y}_h[n]\mathrm{u}[n]$$

Impulse Response Example

Example

Let a system be described by $4\,\mathrm{y}[n]-3\,\mathrm{y}[n-1]=\mathrm{x}[n]$. Then, if the excitation is a unit impulse, $4\,\mathrm{h}[n]-3\,\mathrm{h}[n-1]=\delta[n]$.

The **eigenfunction** is the complex exponential z^n. Substituting into the homogeneous difference equation, $4z^n-3z^{n-1}=0$. Dividing through by z^{n-1}, $4z-3=0$. Solving, $z=3/4$. The homogeneous solution is then of the form $\mathrm{h}[n]=K(3/4)^n$.

Impulse Response Example

Example

The constant K in the homogeneous solution can be found by applying initial conditions. For the case of unit impulse excitation at time $n = 0$,

$$4\,\mathrm{h}[0] - 3\underbrace{\mathrm{h}[0-1]}_{=0} = \mathrm{x}[0] = 1 \Rightarrow \mathrm{h}[0] = 1/4$$

$$\mathrm{h}[0] = K(3/4)^0 = K = 1/4$$

$$\mathrm{h}[n] = \begin{cases} 0 & , \; n < 0 \\ (1/4)(3/4)^n, & n \geq 0 \end{cases}$$

$$\mathrm{h}[n] = (1/4)(3/4)^n \,\mathrm{u}[n]$$

Impulse Response Example

Example

Let a system be described by $3\,\mathrm{y}[n]+2\,\mathrm{y}[n-1]+\mathrm{y}[n-2]=\mathrm{x}[n]$.

Then, if the excitation is a unit impulse,

$$3\,\mathrm{h}[n]+2\,\mathrm{h}[n-1]+\mathrm{h}[n-2]=\delta[n]$$

The eigenfunction is the complex exponential z^n. Substituting into the homogeneous difference equation,

$$3z^n+2z^{n-1}+z^{n-2}=0.$$

Dividing through by z^{n-2}, $3z^2+2z+1=0$.

Solving, $z=-0.333\pm j0.4714$. The homogeneous solution is then of the form

$$\mathrm{h}[n]=K_1(-0.333+j0.4714)^n+K_2(-0.333-j0.4714)^n$$

Impulse Response Example

Example

The constants in the homogeneous solution can be found by applying initial conditions. For the case of unit impulse excitation at time $n = 0$,

$$3\mathrm{h}[0] + 2\underbrace{\mathrm{h}[0-1]}_{=0} + \underbrace{\mathrm{h}[0-2]}_{=0} = \mathrm{x}[0] = 1 \Rightarrow \mathrm{h}[0] = 1/3$$

$$3\mathrm{h}[1] + 2\underbrace{\mathrm{h}[1-1]}_{=1/3} + \underbrace{\mathrm{h}[1-2]}_{=0} = \mathrm{x}[1] = 0 \Rightarrow \mathrm{h}[1] = -2/9$$

$$\mathrm{h}[0] = K_1(-0.333 + j0.4714)^0 + K_2(-0.333 - j0.4714)^0 = K_1 + K_2 = 1/3$$

$$\mathrm{h}[1] = K_1(-0.333 + j0.4714) + K_2(-0.333 - j0.4714) = -2/9$$

$$K_1 = 0.1665 + j0.1181 \;,\; K_2 = 0.1665 - j0.1181$$

Impulse Response Example

Example

The impulse response is then

$$\mathrm{h}[n]=\begin{bmatrix}(0.1665+j0.1181)(-0.333+j0.4714)^n \\ +(0.1665-j0.1181)(-0.333-j0.4714)^n\end{bmatrix}\mathrm{u}[n]$$

which can also be written in the forms,

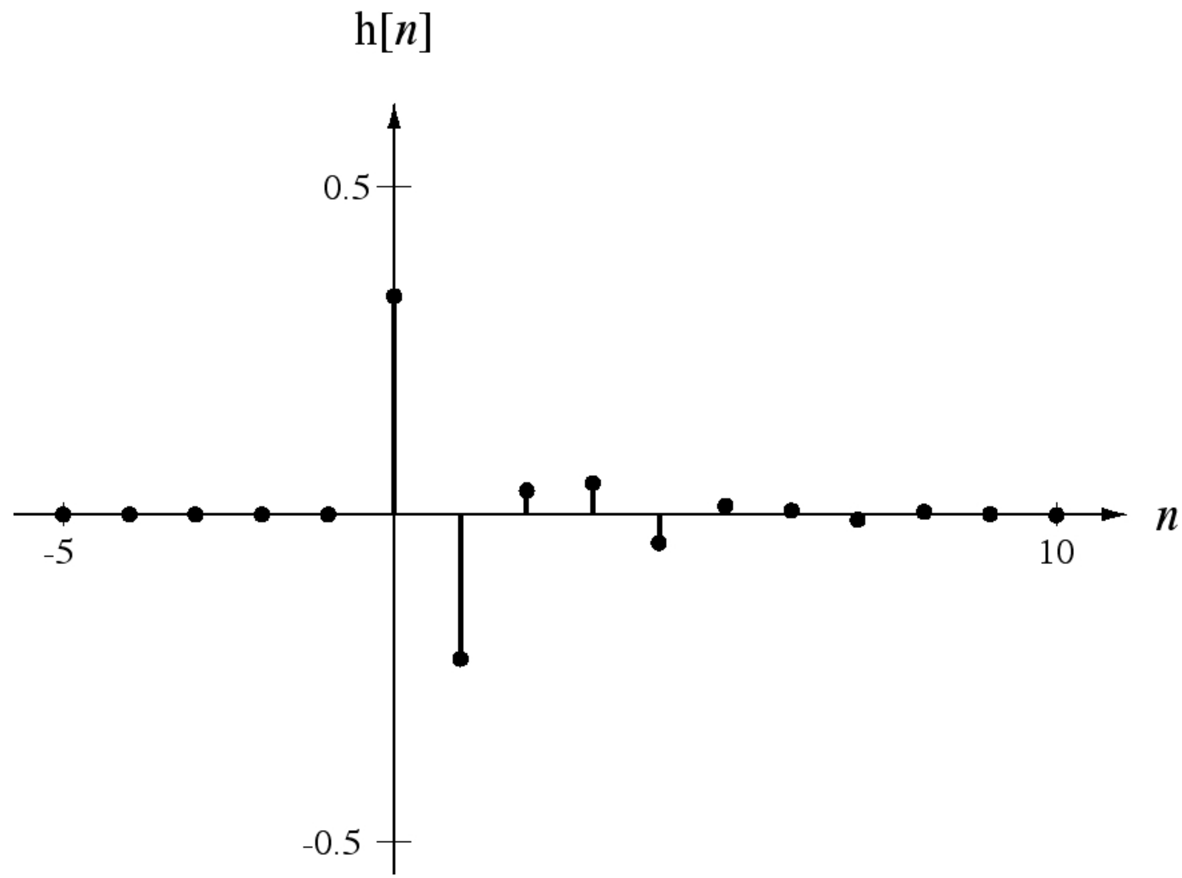
$$\mathrm{h}[n]=(0.5722)^n\begin{bmatrix}(0.1665+j0.1181)e^{j2.1858n} \\ +(0.1665-j0.1181)e^{-j2.1858n}\end{bmatrix}\mathrm{u}[n]$$

$$\mathrm{h}[n]=(0.5722)^n\begin{bmatrix}0.1665\left(e^{j2.1858n}+e^{-j2.1858n}\right) \\ +j0.1181\left(e^{j2.1858n}-e^{-j2.1858n}\right)\end{bmatrix}\mathrm{u}[n]$$

$$\mathrm{h}[n]=(0.5722)^n\left[0.333\cos(2.1858n)-0.2362\sin(2.1858n)\right]\mathrm{u}[n]$$

$$\mathrm{h}[n]=0.4083(0.5722)^n\cos(2.1858n+0.6169)$$

Impulse Response Example

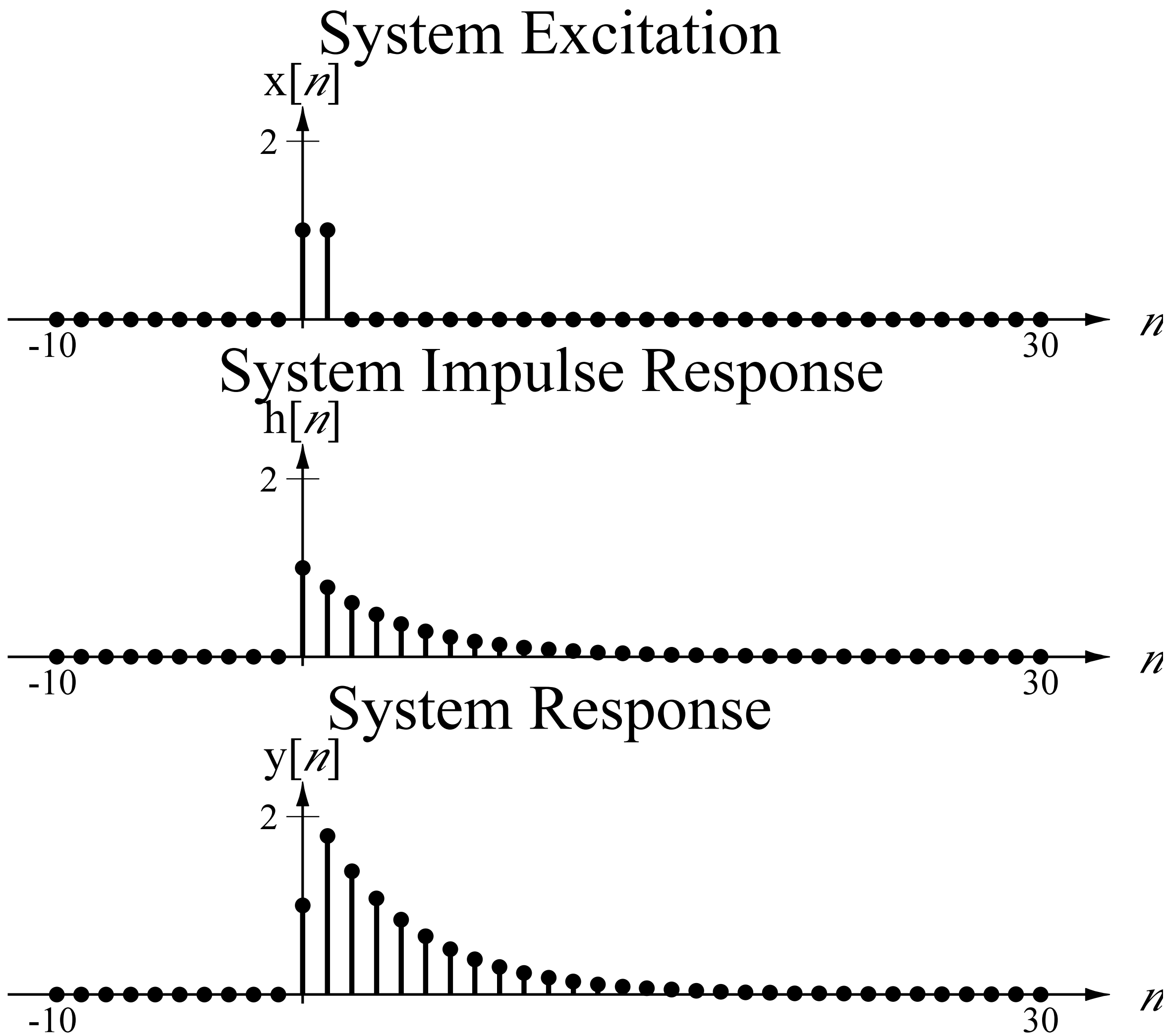


$$h[n] = 0.4083(0.5722)^n \cos(2.1858n + 0.6169)$$

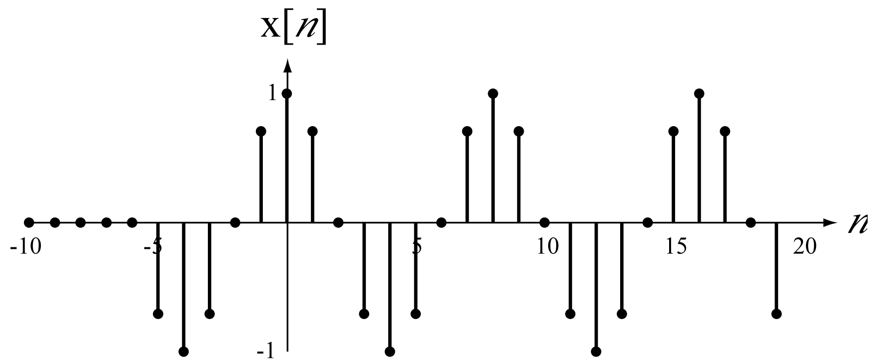
System Response

- Once the response to a unit impulse is known, the response of any LTI system to any arbitrary excitation can be found
- Any arbitrary excitation is simply a sequence of amplitude-scaled and time-shifted impulses
- Therefore the response is simply a sequence of amplitude-scaled and time-shifted impulse <u>responses</u>

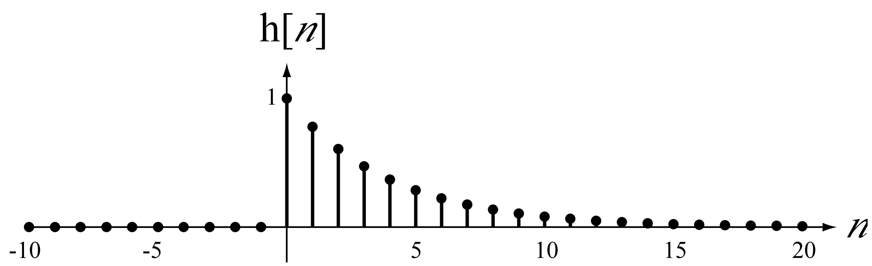
Simple System Response Example



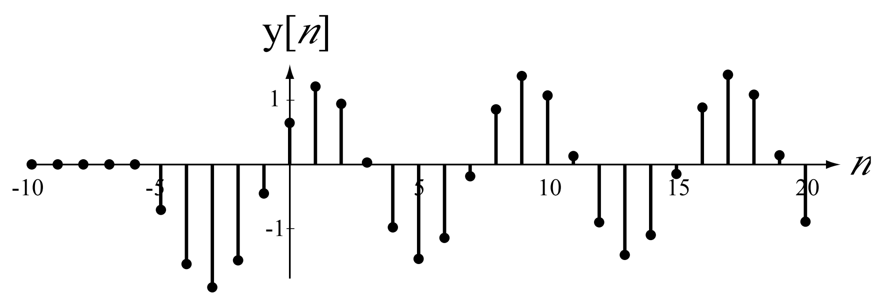
More Complicated System Response Example



System Excitation



System Impulse Response



System Response

The Convolution Sum

The response $\mathrm{y}[n]$ to an arbitrary excitation $\mathrm{x}[n]$ is of the form

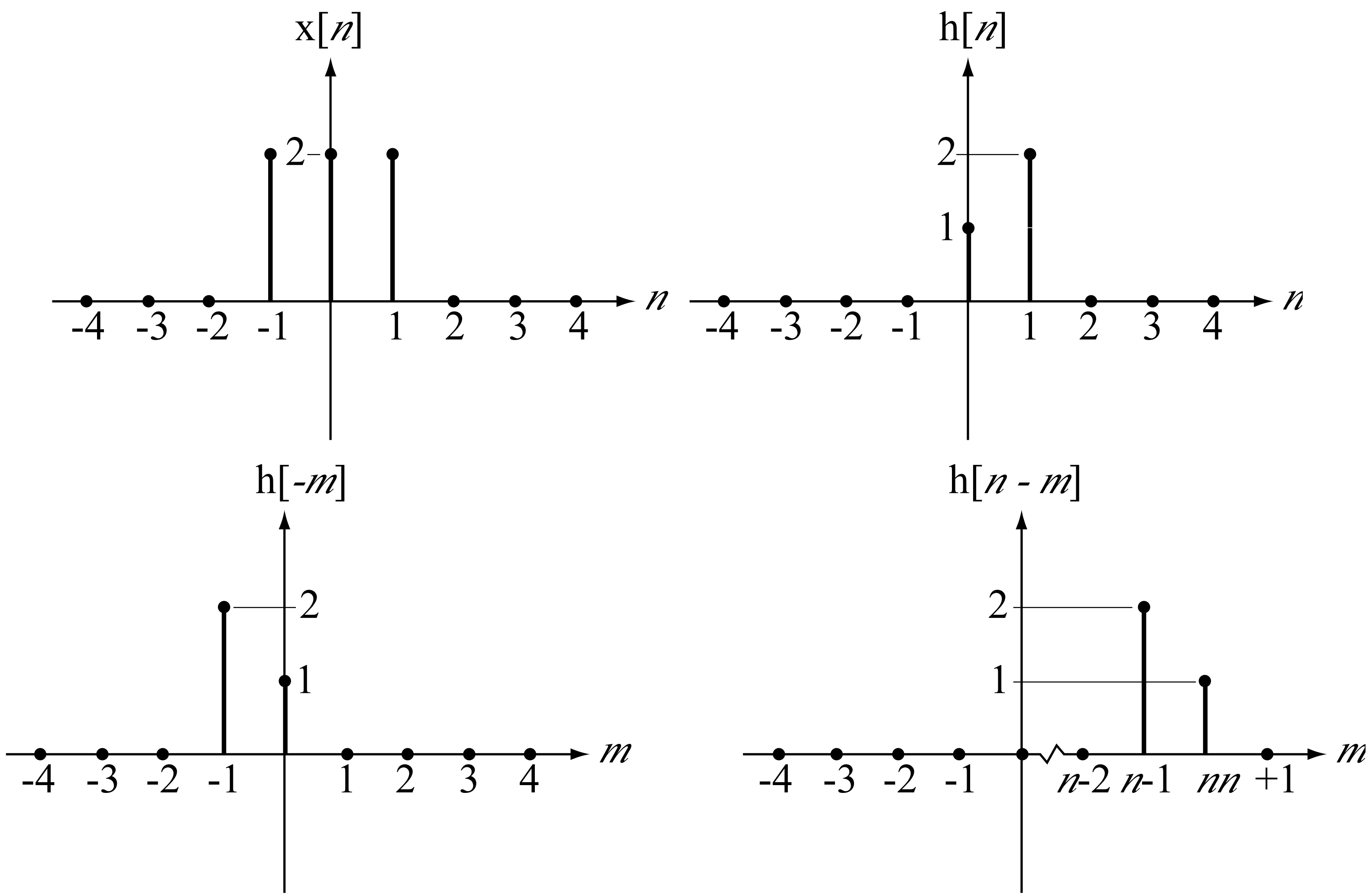
$$\mathrm{y}[n] = \cdots \mathrm{x}[-1]\mathrm{h}[n+1] + \mathrm{x}[0]\mathrm{h}[n] + \mathrm{x}[1]\mathrm{h}[n-1] + \cdots$$

where $\mathrm{h}[n]$ is the impulse response. This can be written in a more compact form

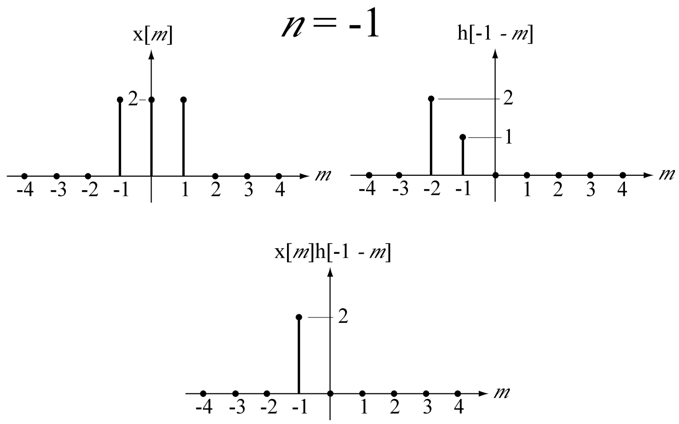
$$\mathrm{y}[n] = \sum_{m=-\infty}^{\infty} \mathrm{x}[m]\mathrm{h}[n-m]$$

called the **convolution sum**.

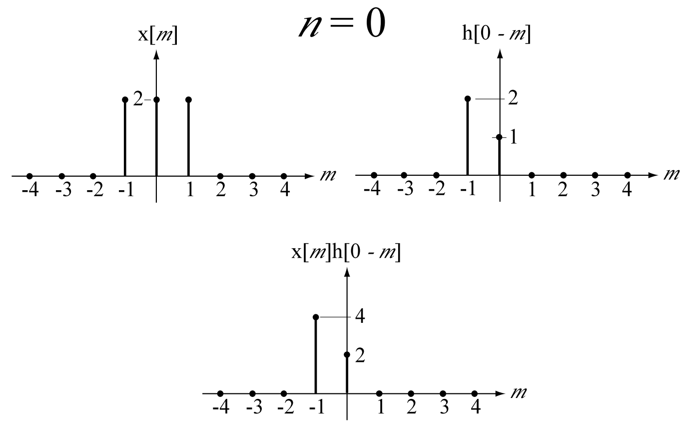
A Convolution Sum Example



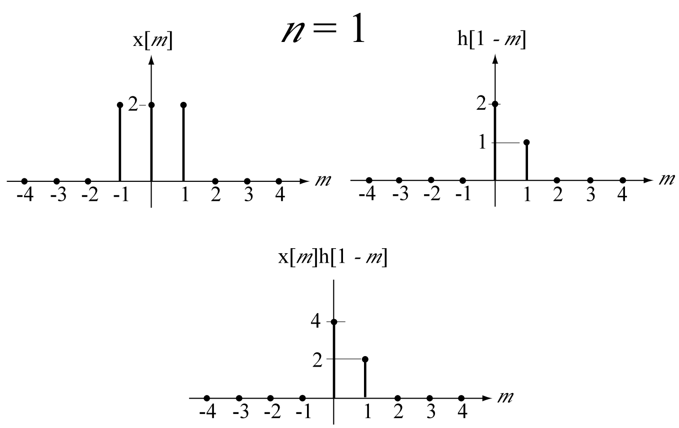
A Convolution Sum Example



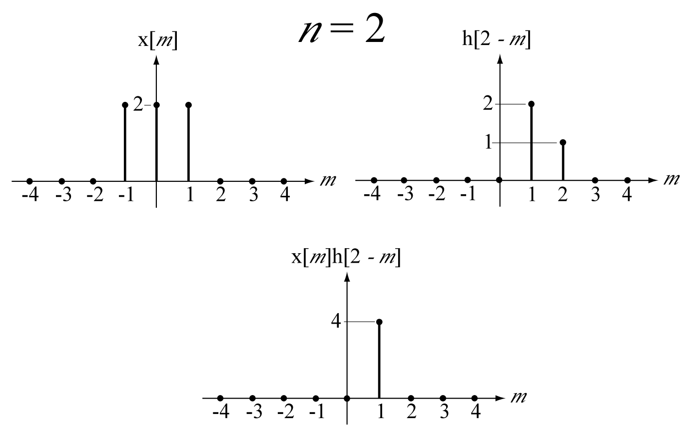
y[-1] = 2



y[0] = 6

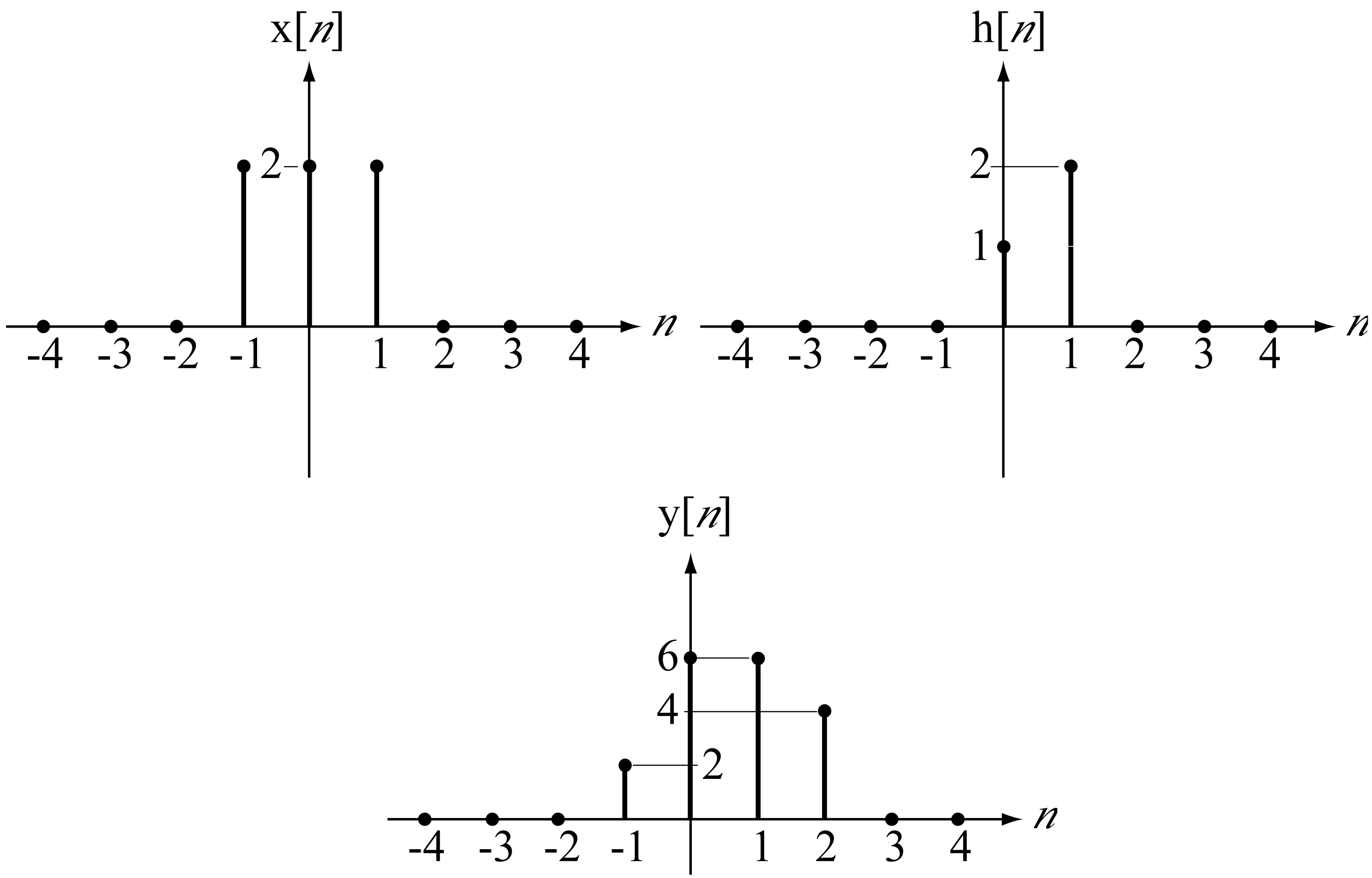


y[1] = 6



y[2] = 4

A Convolution Sum Example



Convolution Sum Properties

Convolution is defined mathematically by

$$y[n] = x[n] * h[n] = \sum_{m=-\infty}^{\infty} x[m]h[n-m]$$

The following properties can be proven from the definition.

$$x[n] * A\delta[n-n_0] = Ax[n-n_0]$$

Let $y[n] = x[n] * h[n]$ then

$$y[n-n_0] = x[n] * h[n-n_0] = x[n-n_0] * h[n]$$

$$y[n] - y[n-1] = x[n] * (h[n] - h[n-1]) = (x[n] - x[n-1]) * h[n]$$

and the sum of the impulse strengths in y is the product of the sum of the impulse strengths in x and the sum of the impulse strengths in h.

Convolution Sum Properties (continued)

Commutativity

$$x[n] * y[n] = y[n] * x[n]$$

Associativity

$$(x[n] * y[n]) * z[n] = x[n] * (y[n] * z[n])$$

Distributivity

$$(x[n] + y[n]) * z[n] = x[n] * z[n] + y[n] * z[n]$$

Numerical Convolution

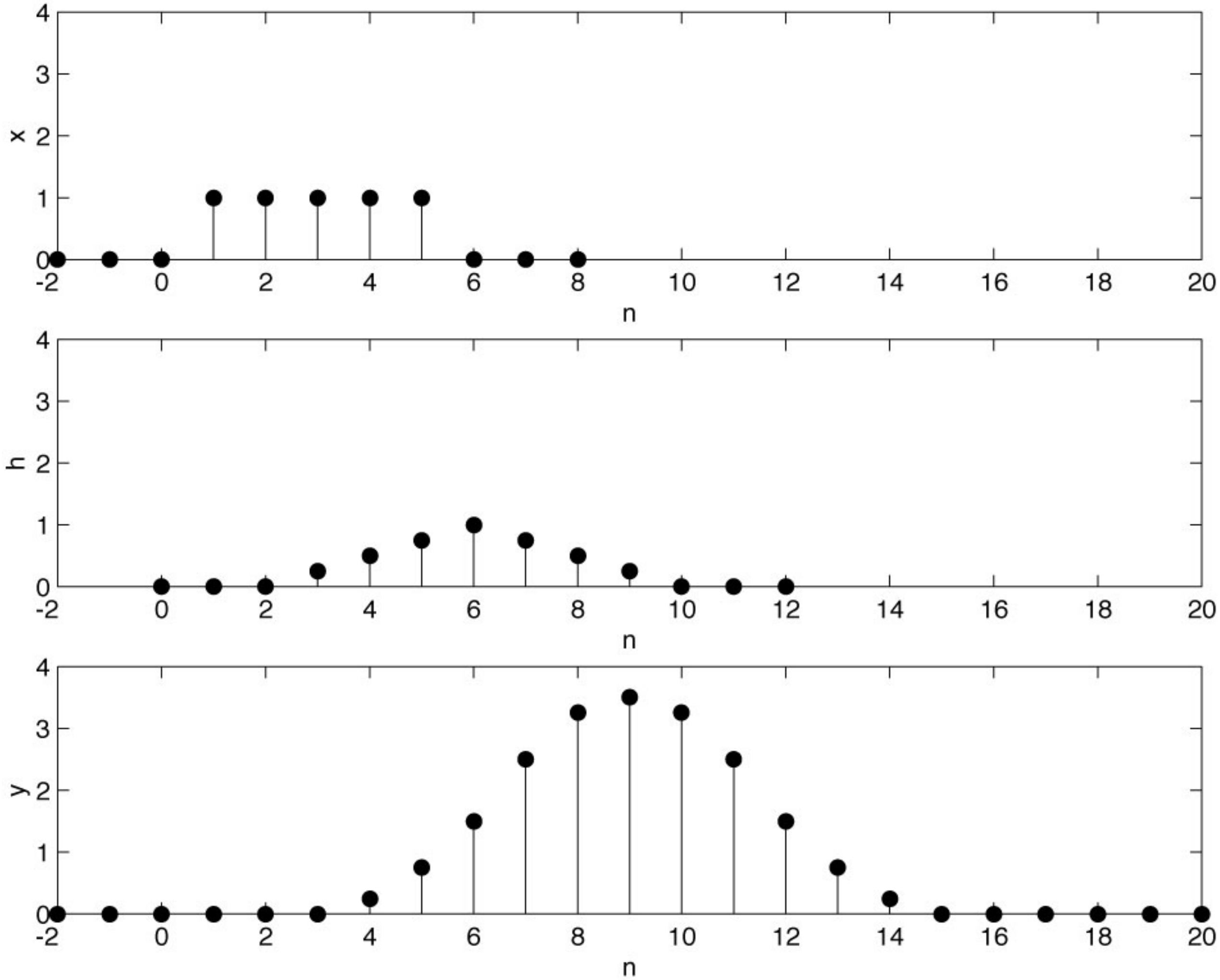
MATLAB has a command `conv` that computes a convolution sum. The syntax is `y = conv(x,h)`. MATLAB can only convolve time-limited signals and the vectors x and h should contain all the non-zero values of the signals they represent. If the time of the first element in `x` is n_{x0} and the time of the first element of `h` is n_{h0}, the time of the first element of `y` is $n_{x0} + n_{h0}$. If the time of the last element in `x` is n_{x1} and the time of the last element of `h` is n_{h1}, the length of `x` is $n_{x1} - n_{x0} + 1$ and the length of `h` is $n_{h1} - n_{h0} + 1$. So the extent of `y` is in the range $n_{x0} + n_{h0} \le n < n_{x1} + n_{h1}$ and its length is

$$n_{x1} + n_{h1} - \left(n_{x0} + n_{h0}\right) + 1 = \underbrace{n_{x1} - n_{x0} + 1}_{\text{length of x}} + \underbrace{n_{h1} - n_{h0} + 1}_{\text{length of h}} - 1$$

Numerical Convolution

```
nx = -2:8 ; nh = 0:12 ;    %  Set time vectors for x and h
x = usD(nx-1) - usD(nx-6) ; %  Compute values of x
h = tri((nh-6)/4) ;        %  Compute values of h
y = conv(x,h) ;            %  Compute the convolution of x with h
%
%  Generate a discrete-time vector for y
%
ny = (nx(1) + nh(1)) + (0:(length(nx) + length(nh) - 2)) ;
%
%  Graph the results
%
subplot(3,1,1) ; stem(nx,x,'k','filled') ;
xlabel('n') ; ylabel('x') ; axis([-2,20,0,4]) ;
subplot(3,1,2) ; stem(nh,h,'k','filled') ;
xlabel('n') ; ylabel('h') ; axis([-2,20,0,4]) ;
subplot(3,1,3) ; stem(ny,y,'k','filled') ;
xlabel('n') ; ylabel('y') ; axis([-2,20,0,4]) ;
```


Numerical Convolution



Numerical Convolution

Continuous-time convolution can be approximated using the `conv` function in MATLAB.

$$\mathrm{y}(t) = \mathrm{x}(t) * \mathrm{h}(t) = \int_{-\infty}^{\infty} \mathrm{x}(\tau)\mathrm{h}(t-\tau)\,d\tau$$

Approximate $\mathrm{x}(t)$ and $\mathrm{h}(t)$ each as a sequence of rectangles of width T_s.

$$\mathrm{x}(t) \cong \sum_{n=-\infty}^{\infty} \mathrm{x}(nT_s)\,\mathrm{rect}\left(\frac{t - nT_s - T_s/2}{T_s}\right)$$

$$\mathrm{h}(t) \cong \sum_{n=-\infty}^{\infty} \mathrm{h}(nT_s)\,\mathrm{rect}\left(\frac{t - nT_s - T_s/2}{T_s}\right)$$

Numerical Convolution

The integral can be approximated at discrete points in time as

$$\mathrm{y}(nT_s) \cong \sum_{m=-\infty}^{\infty} \mathrm{x}(mT_s)\mathrm{h}((n-m)T_s)T_s$$

This can be expressed in terms of a convolution sum as

$$\mathrm{y}(nT_s) \cong T_s \sum_{m=-\infty}^{\infty} \mathrm{x}[m]\mathrm{h}[n-m] = T_s\,\mathrm{x}[n] * \mathrm{h}[n]$$

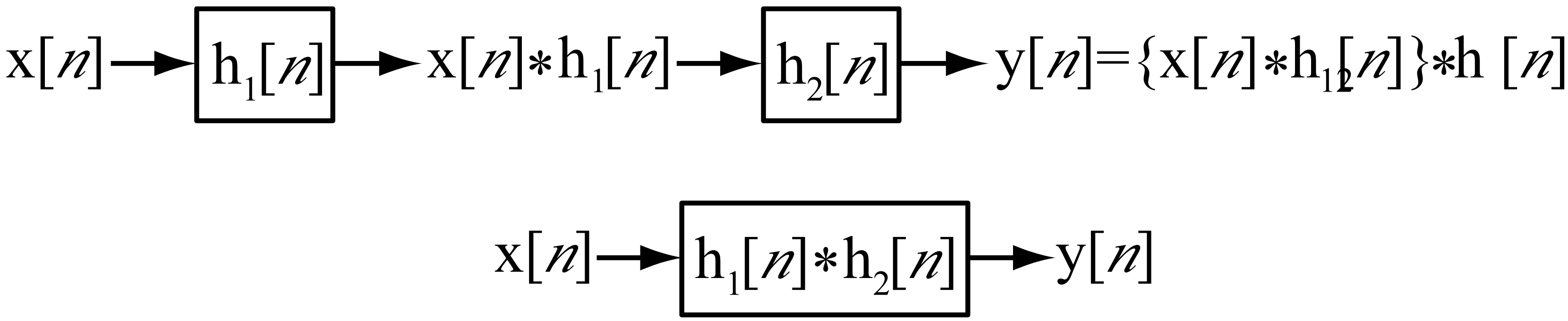
where $\mathrm{x}[n] = \mathrm{x}(nT_s)$ and $\mathrm{h}[n] = \mathrm{h}(nT_s)$.

Stability and Impulse Response

It can be shown that a discrete-time BIBO-stable system has an impulse response that is **absolutely summable.** That is,

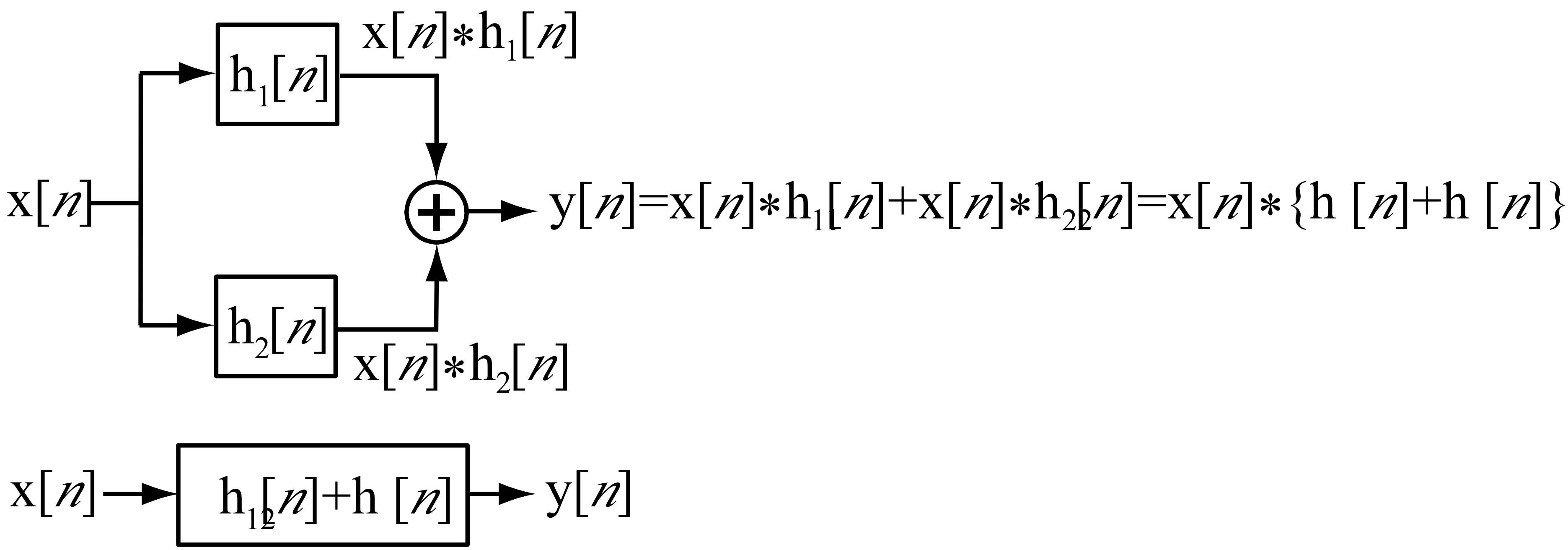
$$\sum_{n=-\infty}^{\infty} \left| \mathrm{h}[n] \right| \text{ is finite.}$$

System Interconnections



The **cascade connection** of two systems can be viewed as a single system whose impulse response is the convolution of the two individual system impulse responses. This is a direct consequence of the **associativity** property of convolution.

System Interconnections



The **parallel connection** of two systems can be viewed as a single system whose impulse response is the sum of the two individual system impulse responses. This is a direct consequence of the **distributivity** property of convolution.

Unit Impulse Response and Unit Sequence Response

In any LTI system let an excitation $\mathrm{x}[n]$ produce the response $\mathrm{y}[n]$. Then the excitation $\mathrm{x}[n]-\mathrm{x}[n-1]$ will produce the response $\mathrm{y}[n]-\mathrm{y}[n-1]$.

It follows then that the unit impulse response is the first backward difference of the unit sequence response and, conversely that the unit sequence response is the accumulation of the unit impulse response.

Systems Described by Difference Equations

The most common description of a discrete-time system is a difference equation of the general form

$$\sum_{k=0}^{N} a_k \, \mathrm{y}[n-k] = \sum_{k=0}^{M} b_k \, \mathrm{x}[n-k].$$

If $\mathrm{x}[n] = Xz^n$, $\mathrm{y}[n]$ has the form $\mathrm{y}[n] = Yz^n$ where X and Y are complex constants. Then $\mathrm{x}[n-k] = z^{-k} X z^n$ and

$\mathrm{y}[n-k] = z^{-k} Y z^n$ and $\sum_{k=0}^{N} a_k z^{-k} Y z^n = \sum_{k=0}^{M} b_k z^{-k} X z^n$. Rearranging

$$Yz^n \sum_{k=0}^{N} a_k z^{-k} = Xz^n \sum_{k=0}^{M} b_k z^{-k} \Rightarrow \frac{Y}{X} = \frac{\sum_{k=0}^{M} b_k z^{-k}}{\sum_{k=0}^{N} a_k z^{-k}}$$

Systems Described by Difference Equations

The transfer function is

$$\mathrm{H}(z)=\frac{\sum_{k=0}^{M} b_k z^{-k}}{\sum_{k=0}^{N} a_k z^{-k}}=\frac{b_0+b_1 z^{-1}+b_2 z^{-2}+\cdots+b_M z^{-M}}{a_0+a_1 z^{-1}+a_2 z^{-2}+\cdots+a_N z^{-N}}$$

or, alternately,

$$\mathrm{H}(z)=\frac{\sum_{k=0}^{M} b_k z^{-k}}{\sum_{k=0}^{N} a_k z^{-k}}=z^{N-M}\,\frac{b_0 z^{M}+b_1 z^{M-1}+\cdots+b_{M-1} z+b_M}{a_0 z^{N}+a_1 z^{N-1}+\cdots+a_{N-1} z+a_N}$$

The transfer function can be written directly from the difference equation and vice versa.

Frequency Response

Let $\mathrm{x}[n]=Xe^{j\Omega n}$. Then $\mathrm{y}[n]=Ye^{j\Omega n}$ and $\mathrm{x}[n-k]=e^{-j\Omega k}Xe^{j\Omega n}$ and $\mathrm{y}[n-k]=e^{-j\Omega k}Ye^{j\Omega n}$. Then the general difference equation description of a discrete-time system

$$\sum_{k=0}^{N} a_k \mathrm{y}[n-k]=\sum_{k=0}^{M} b_k \mathrm{x}[n-k]$$

becomes

$$Ye^{j\Omega n}\sum_{k=0}^{N} a_k e^{-j\Omega k} = Xe^{j\Omega n}\sum_{k=0}^{M} b_k e^{-j\Omega k}$$

Frequency Response

We can form the ratio $\mathrm{H}\left(e^{j\Omega}\right) = \dfrac{Y}{X} = \dfrac{\displaystyle\sum_{k=0}^{M} b_k e^{-j\Omega k}}{\displaystyle\sum_{k=0}^{N} a_k e^{-j\Omega k}}$

$\mathrm{H}\left(e^{j\Omega}\right)$ is the system's **frequency response**. It is the transfer function $\mathrm{H}(z)$ with z replaced by $e^{j\Omega}$.

$$\left|Y\right| e^{j\measuredangle Y} = \left|\mathrm{H}\left(e^{j\Omega}\right)\right| e^{j\measuredangle \mathrm{H}\left(e^{j\Omega}\right)} \left|X\right| e^{j\measuredangle X} = \left|\mathrm{H}\left(e^{j\Omega}\right)\right| \left|X\right| e^{j\left(\measuredangle \mathrm{H}\left(e^{j\Omega}\right) + \measuredangle X\right)}$$

$$\left|Y\right| = \left|\mathrm{H}\left(e^{j\Omega}\right)\right| \left|X\right| \quad \text{and} \quad \measuredangle Y = \measuredangle \mathrm{H}\left(e^{j\Omega}\right) + \measuredangle X$$

Frequency Response Example

Example

Let a system be described by the difference equation

$$8\,\mathrm{y}[n]+4\,\mathrm{y}[n-1]+\mathrm{y}[n-2]=\mathrm{x}[n]$$

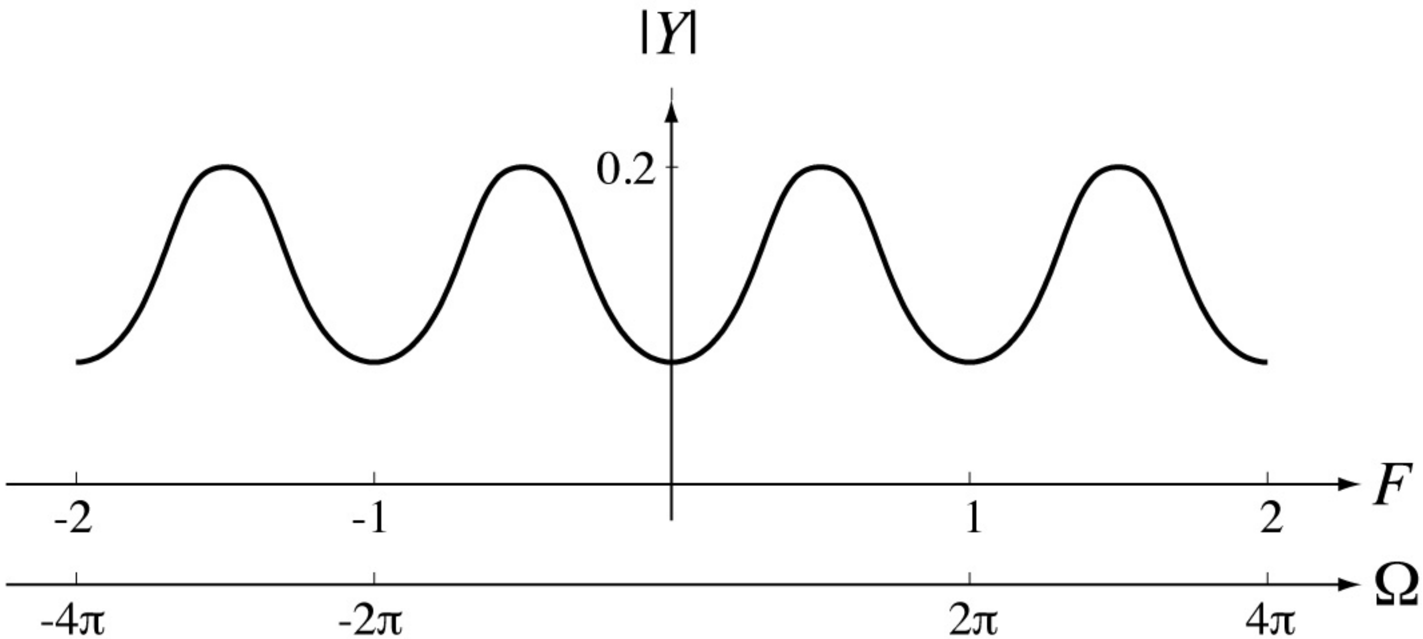
find the response to a unit-amplitude, complex-sinusoid at a radian frequency Ω and then graph the amplitude of the forced complex sinusoidal response versus cyclic frequency F and versus radian frequency Ω.

$$8\,\mathrm{y}[n]+4\,\mathrm{y}[n-1]+\mathrm{y}[n-2]=e^{j\Omega n}$$

$$\mathrm{H}(z)=\frac{1}{8+4z^{-1}+z^{-2}}\Rightarrow \mathrm{H}\left(e^{j\Omega}\right)=\frac{1}{8+4e^{-j\Omega}+e^{-j\Omega}}$$

$$|Y|=\left|\mathrm{H}\left(e^{j\Omega}\right)\right||X| \quad \text{and} \quad \measuredangle Y=\measuredangle \mathrm{H}\left(e^{j\Omega}\right)+\measuredangle X$$

Frequency Response Example



If $\mathrm{y}'(t) - 3\,\mathrm{y}(t) = 4\,\mathrm{x}'(t) + 7\,\mathrm{x}(t)$ find the impulse response $\mathrm{h}(t)$.

$\mathrm{h}'(t) - 3\,\mathrm{h}(t) = 4\delta'(t) + 7\delta(t) \Rightarrow$ Eigenvalue is 3.

$\mathrm{h}(t) = Ke^{3t}\,\mathrm{u}(t) + K_\delta \delta(t)$

$$\int_{0^-}^{0^+} \mathrm{h}'(t)\,dt - 3\int_{0^-}^{0^+} \mathrm{h}(t)\,dt = 4\int_{0^-}^{0^+} \delta'(t)\,dt + 7\int_{0^-}^{0^+} \delta(t)\,dt$$

$$\underbrace{\mathrm{h}(0^+)}_{=K} - \underbrace{\mathrm{h}(0^-)}_{=0} - 3\left\{ \underbrace{\left[Ke^{3t}/3 \right]_0^{0^+}}_{=0} + K_\delta \left[\underbrace{\mathrm{u}(0^+)}_{=1} - \underbrace{\mathrm{u}(0^-)}_{=0} \right] \right\} = 4\left[\underbrace{\delta(0^+)}_{=0} - \underbrace{\delta(0^-)}_{=0} \right] + 7\left[\underbrace{\mathrm{u}(0^+)}_{=1} - \underbrace{\mathrm{u}(0^-)}_{=0} \right]$$

$K - 3K_\delta = 7$

$$\int_{0^-}^{0^+} \underbrace{\int_{-\infty}^{t} \mathrm{h}'(\lambda)\,d\lambda}_{=\mathrm{h}(t)}\,dt - 3\int_{0^-}^{0^+} \underbrace{\int_{-\infty}^{t} \mathrm{h}(\lambda)\,d\lambda}_{=Ke^{3t}\,\mathrm{u}(t)/3 + K_\delta\,\mathrm{u}(t)}\,dt = 4\int_{0^-}^{0^+} \underbrace{\int_{-\infty}^{t} \delta'(\lambda)\,d\lambda}_{=\delta(t)}\,dt + 7\int_{0^-}^{0^+} \underbrace{\int_{-\infty}^{t} \delta(\lambda)\,d\lambda}_{=\mathrm{u}(t)}\,dt$$

$K_\delta - 3(0) = 4 + 7(0) \Rightarrow K_\delta = 4 \Rightarrow K = 19$

$\mathrm{h}(t) = 19e^{3t}\,\mathrm{u}(t) + 4\delta(t) \Rightarrow \mathrm{h}'(t) = 19e^{3t}\delta(t) + 57e^{3t}\,\mathrm{u}(t) + 4\delta'(t)$

$$\mathrm{h}'(t) - 3\,\mathrm{h}(t) = 4\delta'(t) + 7\delta(t) \Rightarrow 19\underbrace{e^{3t}\delta(t)}_{=\delta(t)} + 57e^{3t}\,\mathrm{u}(t) + 4\delta'(t) - 57e^{3t}\,\mathrm{u}(t) - 12\delta(t) = 4\delta'(t) + 7\delta(t)$$

$4\delta'(t) + 7\delta(t) = 4\delta'(t) + 7\delta(t)$ Check.

If $2\,\mathrm{y}''(t)+5\,\mathrm{y}'(t)=4\,\mathrm{x}(t)$ find the impulse response $\mathrm{h}(t)$.

$2\,\mathrm{h}''(t)+5\,\mathrm{h}'(t)=4\delta(t) \Rightarrow$ Eigenvalues are $-5/2$ and 0.

$$\mathrm{h}(t)=\left(K_1 e^{-5t/2}+K_2\right)\mathrm{u}(t)$$

$$2\int_{0^-}^{0^+}\mathrm{h}''(t)\,dt+5\int_{0^-}^{0^+}\mathrm{h}'(t)\,dt=4\int_{0^-}^{0^+}\delta(t)\,dt$$

$$2\left[\underbrace{\mathrm{h}'\left(0^+\right)}_{=-5K_1/2}-\underbrace{\mathrm{h}'\left(0^-\right)}_{=0}\right]+5\left[\underbrace{\mathrm{h}\left(0^+\right)}_{=K_1+K_2}-\underbrace{\mathrm{h}\left(0^-\right)}_{0}\right]=4\left[\underbrace{\mathrm{u}\left(0^+\right)}_{1}-\underbrace{\mathrm{u}\left(0^-\right)}_{0}\right]$$

$$-5K_1+5K_1+5K_2=4 \Rightarrow K_2=4/5$$

$$2\int_{0^-}^{0^+}\underbrace{\int_{-\infty}^{t}\mathrm{h}''(\lambda)\,d\lambda}_{=\mathrm{h}'(t)}\,dt+5\int_{0^-}^{0^+}\underbrace{\int_{-\infty}^{t}\mathrm{h}'(\lambda)\,d\lambda}_{=\mathrm{h}(t)}\,dt=4\int_{0^-}^{0^+}\underbrace{\int_{-\infty}^{t}\delta(\lambda)\,d\lambda}_{\mathrm{u}(t)}\,dt$$

$$2\left(K_1+K_2\right)+5\underbrace{\left[-2K_1e^{-5t/2}/5+K_2t\right]_0^{0^+}}_{=0}=0 \Rightarrow K_1+K_2=0 \Rightarrow K_1=-K_2=-4/5$$

$$\mathrm{h}(t)=(4/5)\left(1-e^{-5t/2}\right)\mathrm{u}(t) \Rightarrow \mathrm{h}'(t)=(4/5)\left[\underbrace{\left(1-e^{-5t/2}\right)\delta(t)}_{=0}+\left(5e^{-5t/2}/2\right)\mathrm{u}(t)\right]=2e^{-5t/2}\,\mathrm{u}(t)$$

$$\mathrm{h}''(t)=2\left[\underbrace{e^{-5t/2}\delta(t)}_{=\delta(t)}-\left(5e^{-5t/2}/2\right)\mathrm{u}(t)\right]=2\left[\delta(t)-\left(5e^{-5t/2}/2\right)\mathrm{u}(t)\right]$$

$2\,\mathrm{h}''(t)+5\,\mathrm{h}'(t)=4\delta(t) \Rightarrow 4\left[\delta(t)-\left(5e^{-5t/2}/2\right)\mathrm{u}(t)\right]+10e^{-5t/2}\,\mathrm{u}(t)=4\delta(t) \Rightarrow 4\delta(t)=4\delta(t)$ Check.

If $2\,\mathrm{y}[n]-\mathrm{y}[n-1]=3\,\mathrm{x}[n-1]+\mathrm{x}[n-2]$ find the impulse response $\mathrm{h}[n]$.

$2\,\mathrm{h}[n]-\mathrm{h}[n-1]=3\delta[n-1]+\delta[n-2] \Rightarrow$ Eigenvalue is $1/2$.

Let $2\mathrm{h}_0[n]-\mathrm{h}_0[n-1]=\delta[n]$. Then $\mathrm{h}_0[n]=K(1/2)^n\,\mathrm{u}[n]$.

$$2\underbrace{\mathrm{h}_0[0]}_{=K}-\underbrace{\mathrm{h}_0[-1]}_{=0}=\underbrace{\delta[0]}_{=1} \Rightarrow K=1/2 \text{ and } \mathrm{h}_0[n]=(1/2)^{n+1}\,\mathrm{u}[n].$$

Using superposition and time invariance, if $\mathrm{h}_0[n]=(1/2)^{n+1}\,\mathrm{u}[n]$

then $\mathrm{h}[n]=3\mathrm{h}_0[n-1]+\mathrm{h}_0[n-2]=3(1/2)^n\,\mathrm{u}[n-1]+(1/2)^{n-1}\,\mathrm{u}[n-2]$

$\mathrm{h}[n]=(1/2)^n\left(3\,\mathrm{u}[n-1]+2\,\mathrm{u}[n-2]\right)$.

The first few values of $\mathrm{h}[n]$ are

n	0	1	2	3	4
$\mathrm{h}[n]$	0	3/2	5/4	5/8	5/16

We can find these values also by direct iteration on

$\mathrm{h}[n]=(1/2)\left(3\delta[n-1]+\delta[n-2]+\mathrm{h}[n-1]\right)$ and we get

n	0	1	2	3	4
$\mathrm{h}[n]$	0	3/2	5/4	5/8	5/16

confirming the validity of the solution.

If $\mathrm{x}(t)=\delta(t-1)-3\delta(t+2)$ and $\mathrm{h}(t)=4\,\mathrm{rect}(t/5)$ and $\mathrm{y}(t)=\mathrm{x}(t)*\mathrm{h}(t)$ find the signal energy of $\mathrm{y}(t)$ E_{y}.

$$\mathrm{y}(t)=\left[\delta(t-1)-3\delta(t+2)\right]*4\,\mathrm{rect}(t/5)=4\left[\delta(t-1)*\mathrm{rect}(t/5)-3\delta(t+2)*\mathrm{rect}(t/5)\right]$$

$$\mathrm{y}(t)=4\left[\mathrm{rect}((t-1)/5)-3\,\mathrm{rect}((t+2)/5)\right]$$

$$E_{\mathrm{y}}=\int_{-\infty}^{\infty}\left|\mathrm{y}(t)\right|^2 dt=16\int_{-\infty}^{\infty}\left[\mathrm{rect}((t-1)/5)-3\,\mathrm{rect}((t+2)/5)\right]^2 dt$$

$$E_{\mathrm{y}}=16\int_{-\infty}^{\infty}\left[\mathrm{rect}^2((t-1)/5)+9\,\mathrm{rect}^2((t+2)/5)-6\,\mathrm{rect}((t-1)/5)\mathrm{rect}((t+2)/5)\right]dt$$

$$E_{\mathrm{y}}=16\left[\int_{-\infty}^{\infty}\mathrm{rect}^2((t-1)/5)dt+9\int_{-\infty}^{\infty}\mathrm{rect}^2((t+2)/5)dt-6\int_{-\infty}^{\infty}\mathrm{rect}((t-1)/5)\mathrm{rect}((t+2)/5)dt\right]$$

$$E_{\mathrm{y}}=16\left[\int_{-3/2}^{7/2}dt+9\int_{-9/2}^{1/2}dt-6\int_{-3/2}^{1/2}dt\right]=16(5+45-12)=608$$

$$\mathrm{y}(t)=\begin{cases}-12 & , \quad -9/2<t<-3/2\\ -8 & , \quad -3/2<t<1/2\\ 4 & , \quad 1/2<t<7/2\end{cases}\Rightarrow \mathrm{y}^2(t)=\begin{cases}144 & , \quad -9/2<t<-3/2\\ 64 & , \quad -3/2<t<1/2\\ 16 & , \quad 1/2<t<7/2\end{cases}$$

$E_{\mathrm{y}}=144\times 3+64\times 2+16\times 3=608$ Check.

If $\mathrm{x}(t)=\cos(200\pi t)\mathrm{u}(t)$ and $\mathrm{h}(t)=e^{-100t}\mathrm{u}(t)$ and $\mathrm{y}(t)=\mathrm{x}(t)*\mathrm{h}(t)$ find $\mathrm{y}(t)$.

$$\mathrm{y}(t)=\int_{-\infty}^{\infty}\mathrm{x}(\tau)\mathrm{h}(t-\tau)d\tau=\int_{-\infty}^{\infty}\cos(200\pi\tau)\mathrm{u}(\tau)e^{-100(t-\tau)}\mathrm{u}(t-\tau)d\tau$$

$$\mathrm{y}(t)=e^{-100t}\int_{0}^{t}e^{100\tau}\cos(200\pi\tau)d\tau=\frac{e^{-100t}}{2}\int_{0}^{t}e^{100\tau}\left(e^{j200\pi\tau}+e^{-j200\pi\tau}\right)d\tau\ \ ,\ \ t>0\ ;\ \mathrm{y}(t)=0\ ,\ t<0$$

$$\mathrm{y}(t)=\frac{e^{-100t}}{2}\mathrm{u}(t)\int_{0}^{t}\left(e^{(100+j200\pi)\tau}+e^{(100-j200\pi)\tau}\right)d\tau=\frac{e^{-100t}}{2}\mathrm{u}(t)\left[\frac{e^{(100+j200\pi)\tau}}{100+j200\pi}+\frac{e^{(100-j200\pi)\tau}}{100-j200\pi}\right]_{0}^{t}$$

$$\mathrm{y}(t)=\frac{e^{-100t}}{2}\left(\frac{e^{(100+j200\pi)t}-1}{100+j200\pi}+\frac{e^{(100-j200\pi)t}-1}{100-j200\pi}\right)\mathrm{u}(t)$$

$$=\frac{e^{-100t}}{2}\frac{100\left(e^{(100+j200\pi)t}+e^{(100-j200\pi)t}-2\right)+j200\left(e^{(100-j200\pi)t}-e^{(100+j200\pi)t}\right)}{100^{2}+(200\pi)^{2}}\mathrm{u}(t)$$

$$=\frac{e^{-100t}}{2}\frac{200\left(e^{100t}\cos(200\pi t)-1\right)+j200\left(-j2e^{100t}\sin(200\pi t)\right)}{404784.2}\mathrm{u}(t)$$

$$=e^{-100t}\frac{100\left(e^{100t}\cos(200\pi t)-1\right)+200e^{100t}\sin(200\pi t)}{404784.2}\mathrm{u}(t)$$

$$=\frac{\cos(200\pi t)+2\sin(200\pi t)-e^{-100t}}{4047.842}\mathrm{u}(t)$$

If $\mathrm{x}(t)=e^{-20t}\cos(200\pi t)\mathrm{u}(t)$ and $\mathrm{h}(t)=e^{-100t}\,\mathrm{u}(t)$ and $\mathrm{y}(t)=\mathrm{x}(t)*\mathrm{h}(t)$ find $\mathrm{y}(t)$.

$$\mathrm{y}(t)=\int_{-\infty}^{\infty}\mathrm{x}(\tau)\mathrm{h}(t-\tau)d\tau=\int_{-\infty}^{\infty}e^{-20\tau}\cos(200\pi\tau)\mathrm{u}(\tau)e^{-100(t-\tau)}\,\mathrm{u}(t-\tau)d\tau$$

$$\mathrm{y}(t)=e^{-100t}\int_{0}^{t}e^{80\tau}\cos(200\pi\tau)d\tau=\frac{e^{-100t}}{2}\int_{0}^{t}e^{80\tau}\left(e^{j200\pi\tau}+e^{-j200\pi\tau}\right)d\tau \;\;,\;\; t>0\;;\;\mathrm{y}(t)=0\;,\;t<0$$

$$\mathrm{y}(t)=\frac{e^{-100t}}{2}\mathrm{u}(t)\int_{0}^{t}\left(e^{(80+j200\pi)\tau}+e^{(80-j200\pi)\tau}\right)d\tau=\frac{e^{-100t}}{2}\mathrm{u}(t)\left[\frac{e^{(80+j200\pi)\tau}}{80+j200\pi}+\frac{e^{(80-j200\pi)\tau}}{80-j200\pi}\right]_{0}^{t}$$

$$\mathrm{y}(t)=\frac{e^{-100t}}{2}\left(\frac{e^{(80+j200\pi)t}-1}{80+j200\pi}+\frac{e^{(80-j200\pi)t}-1}{80-j200\pi}\right)\mathrm{u}(t)$$

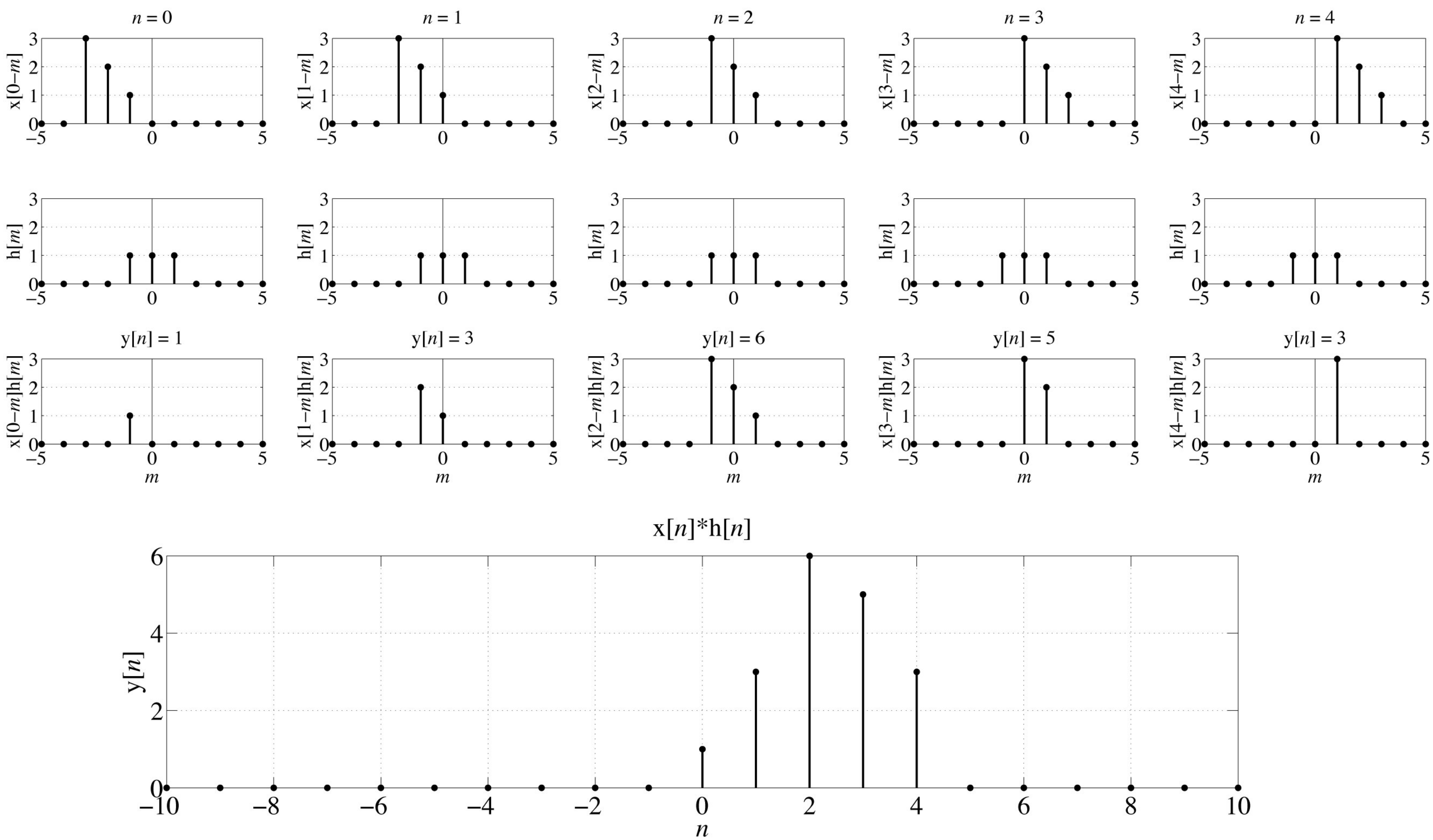
$$=\frac{e^{-100t}}{2}\frac{80\left(e^{(80+j200\pi)t}+e^{(80-j200\pi)t}-2\right)+j200\left(e^{(80-j200\pi)t}-e^{(80+j200\pi)t}\right)}{80^2+(200\pi)^2}\mathrm{u}(t)$$

$$=\frac{e^{-100t}}{2}\frac{160\left(e^{80t}\cos(200\pi t)-1\right)+j200\left(-j2e^{80t}\sin(200\pi t)\right)}{401184.2}\mathrm{u}(t)$$

$$=e^{-100t}\frac{80\left(e^{80t}\cos(200\pi t)-1\right)+200e^{80t}\sin(200\pi t)}{401184.2}\mathrm{u}(t)$$

$$=\frac{e^{-20t}\left[0.8\cos(200\pi t)+2\sin(200\pi t)\right]-0.8e^{-100t}}{4011.842}\mathrm{u}(t)$$

If $x[n] = \text{ramp}[n]u[3-n]$ and $h[n] = u[n+1] - u[n-2]$ and $y[n] = x[n] * h[n]$ find the signal energy of $y[n]$.



$$E_y = 1^2 + 3^2 + 6^2 + 5^2 + 3^2 = 80$$

If $\mathrm{x}[n] = \mathrm{u}[n+4]$ and $\mathrm{h}[n] = -\mathrm{u}[n-1]$ and $\mathrm{y}[n] = \mathrm{x}[n] * \mathrm{h}[n]$, find $\mathrm{y}[n]$.

$$\mathrm{y}[n] = \sum_{m=-\infty}^{\infty} \mathrm{x}[m]\mathrm{h}[n-m] = \sum_{m=-\infty}^{\infty} \mathrm{u}[m+4](-\mathrm{u}[n-m-1])$$

$$= \begin{Bmatrix} -\sum_{m=-4}^{n-1} 1 & , & n \geq -3 \\ 0 & , & n < -3 \end{Bmatrix} = \left(-\sum_{m=-4}^{n-1} 1 \right) \mathrm{u}[n+3]$$

n	-4	-3	-2	-1	0
$\mathrm{y}[n]$	0	-1	-2	-3	-4

$$\cdots \Rightarrow \mathrm{y}[n] = -\mathrm{ramp}[n+4]$$

If $\mathrm{x}[n]=\mathrm{u}[n-2]-\mathrm{u}[n-6]$ and $\mathrm{h}[n]=\mathrm{u}[n+3]-\mathrm{u}[n-3]$ and $\mathrm{y}[n]=\mathrm{x}[n]*\mathrm{h}[n]$, find $\mathrm{y}[n]$.

$$\mathrm{y}[n]=\sum_{m=-\infty}^{\infty}\mathrm{x}[m]\mathrm{h}[n-m]$$

$$=\sum_{m=-\infty}^{\infty}\left(\mathrm{u}[m-2]-\mathrm{u}[m-6]\right)\left(\mathrm{u}[n-m+3]-\mathrm{u}[n-m-3]\right)$$

$$=\sum_{m=2}^{5}\left(\mathrm{u}[n-m+3]-\mathrm{u}[n-m-3]\right)$$

In words, for any value of n, add the impulses in $\left(\mathrm{u}[n-m+3]-\mathrm{u}[n-m-3]\right)$ for m ranging from 2 to 5. For example, let $n=0$. Then

$$\mathrm{y}[0]=\sum_{m=2}^{5}\left(\mathrm{u}[-m+3]-\mathrm{u}[-m-3]\right)$$

$$=\left(\mathrm{u}[1]-\mathrm{u}[-5]\right)+\left(\mathrm{u}[0]-\mathrm{u}[-6]\right)+\left(\mathrm{u}[-1]-\mathrm{u}[-7]\right)+\left(\mathrm{u}[-2]-\mathrm{u}[-8]\right)=2$$