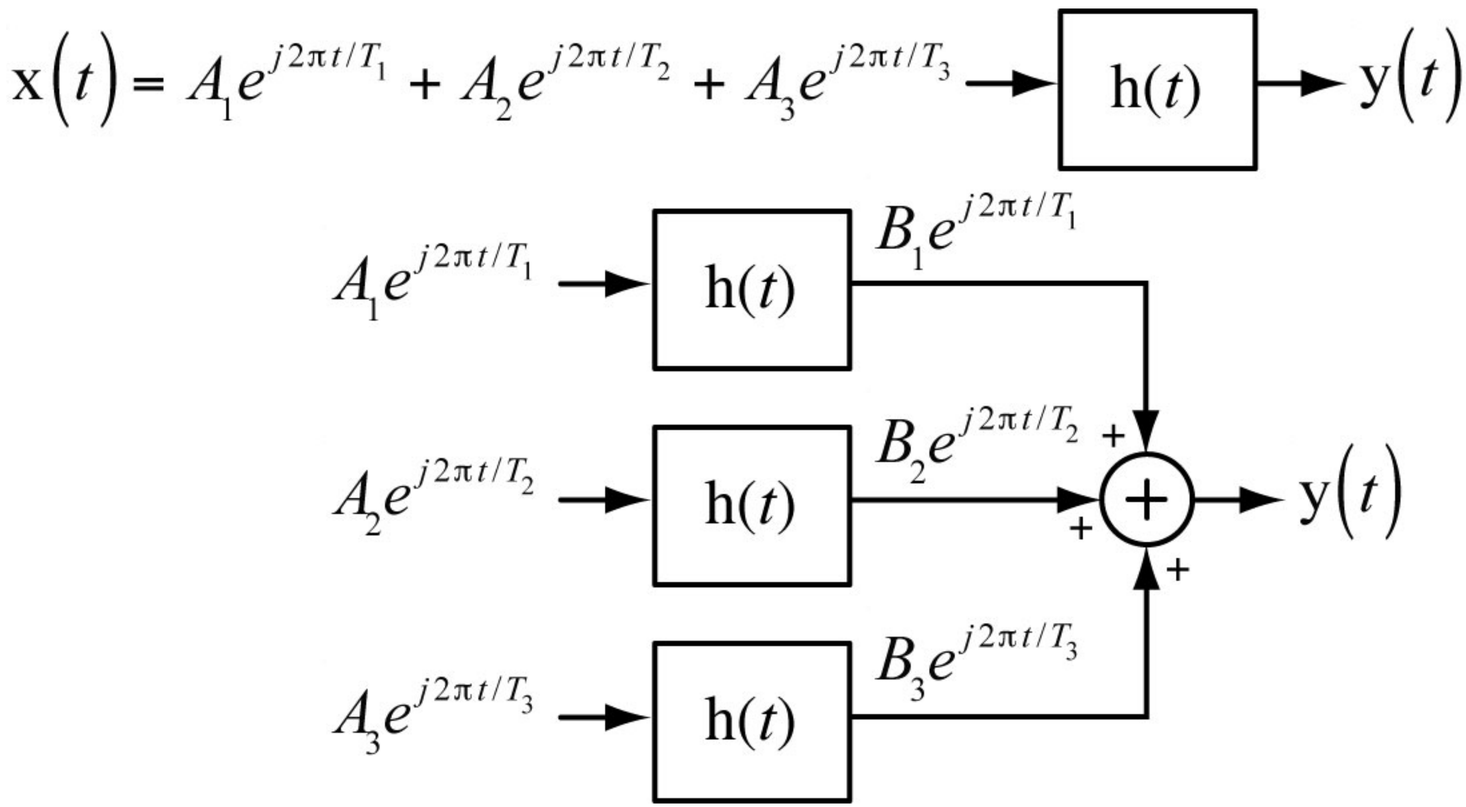
The Continuous-Time Fourier Series

Representing a Signal

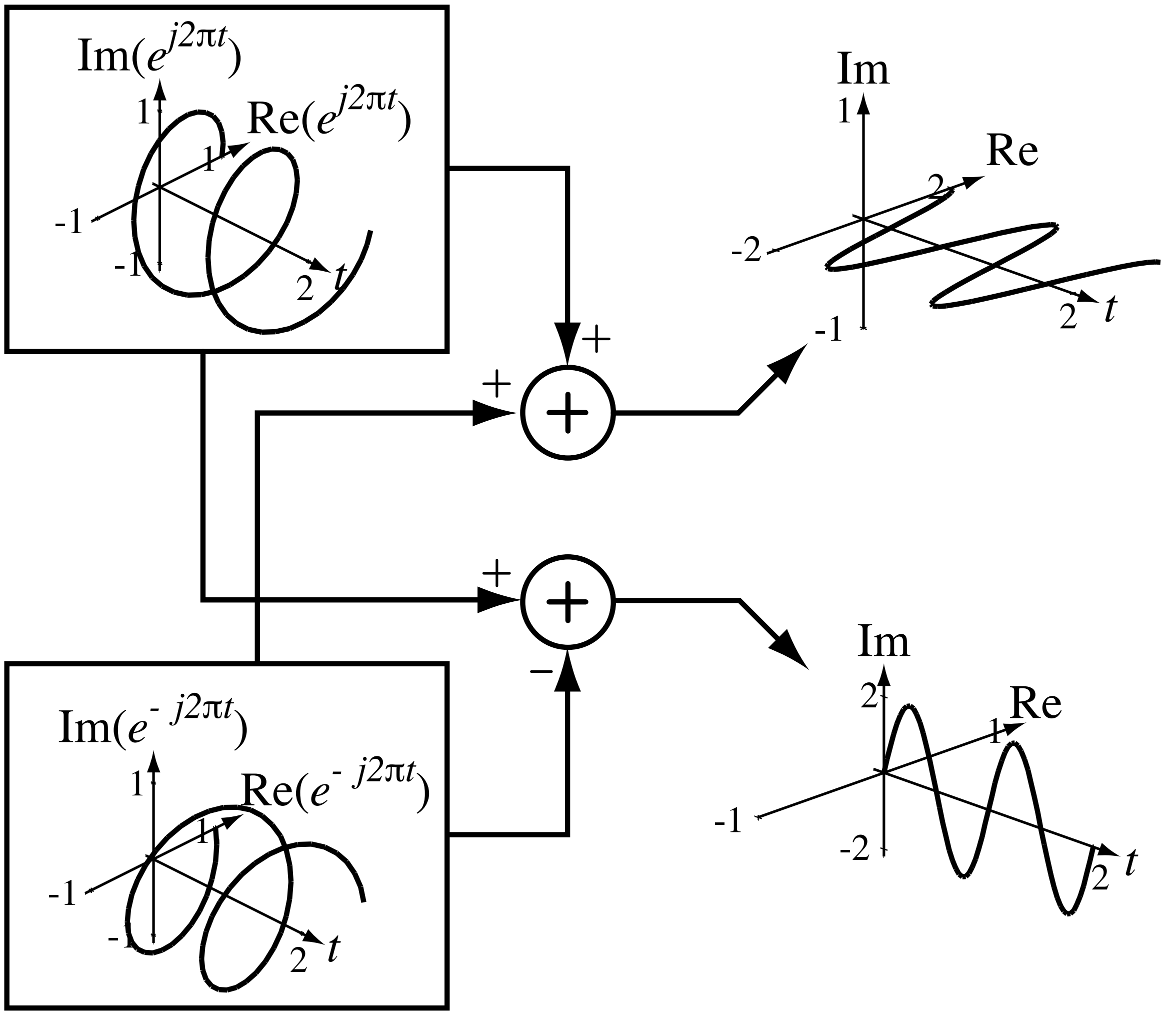
- The convolution method for finding the response of an LTI system to an excitation takes advantage of the linearity and time-invariance of the system and represents the excitation as a linear combination of <u>impulses</u> and the response as a linear combination of <u>impulse responses</u>.
- The Fourier series represents an excitation as a linear combination of <u>complex sinusoids</u> and the response as a linear combination of <u>complex sinusoid responses</u>.

Linearity and Superposition

If an excitation can be expressed as a linear combination of complex sinusoids, the response of an LTI system can be expressed as a linear combination of responses to complex sinusoids.



Real and Complex Sinusoids



$$\cos(x) = \frac{e^{jx} + e^{-jx}}{2}$$

$$\sin(x) = \frac{e^{jx} - e^{-jx}}{j2}$$

Real and Complex Sinusoids

Let $\mathrm{x}(t) = 4\cos(3t) - 7\sin(2t)$ and express $\mathrm{x}(t)$ as a linear combination of complex sinusoids.

$$\mathrm{x}(t) = 4\frac{e^{j3t} + e^{-j3t}}{2} - 7\frac{e^{j2t} - e^{-j2t}}{j2} = 2\left(e^{j3t} + e^{-3t}\right) + j3.5\left(e^{j2t} - e^{-j2t}\right)$$

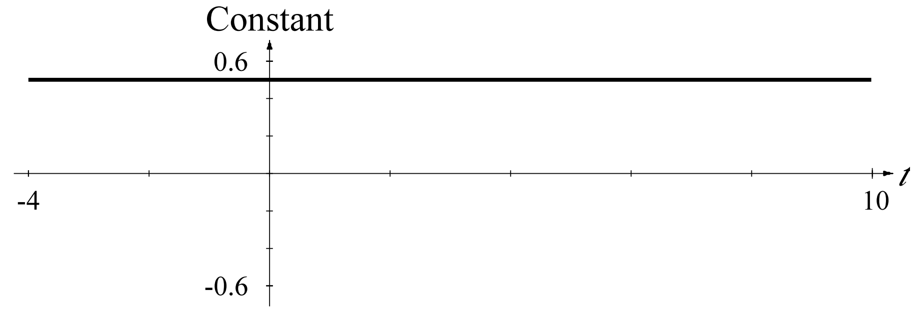
This is a simple example. Any periodic signal with engineering usefulness can also be represented as a linear combination of complex sinusoids.

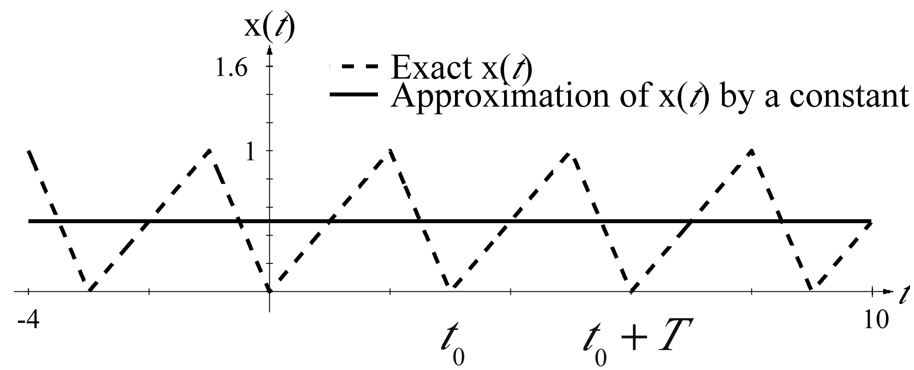
Jean Baptiste Joseph Fourier

3/21/1768 - 5/16/1830

Conceptual Overview

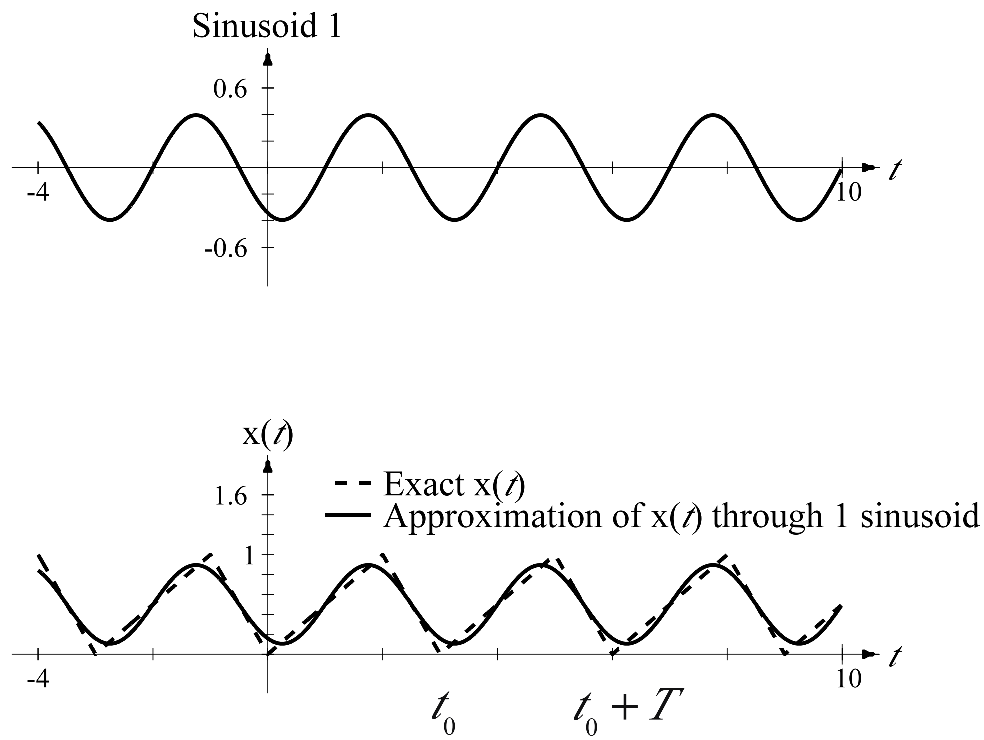
The Fourier series represents a signal as a sum of sinusoids. The best approximation to the dashed-line signal below using only a constant is the solid line. (A constant is a cosine of zero frequency.)





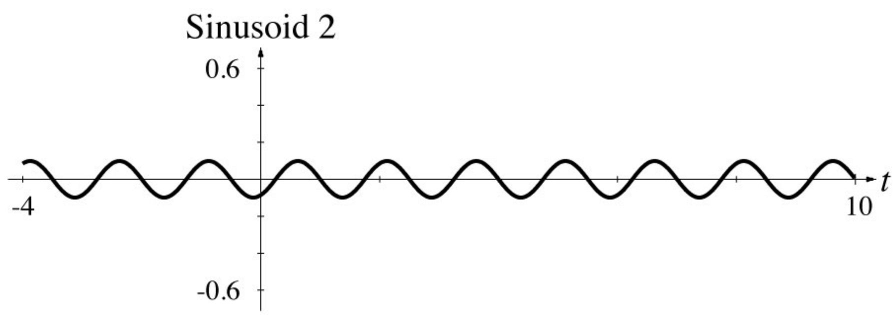
Conceptual Overview

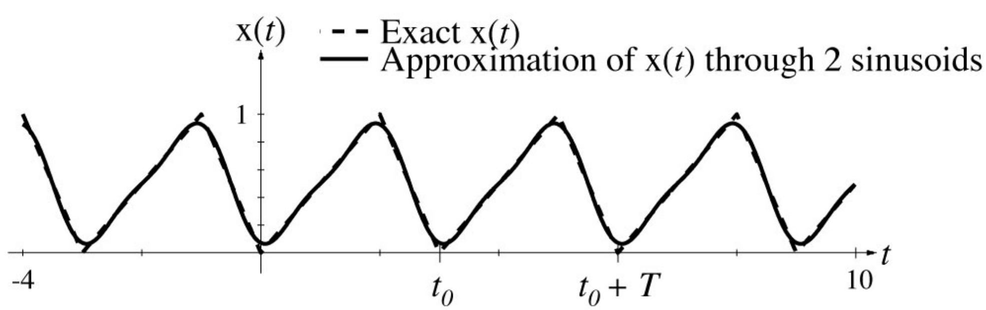
The best approximation to the dashed-line signal using a constant plus one real sinusoid of the same fundamental frequency as the dashed-line signal is the solid line.



Conceptual Overview

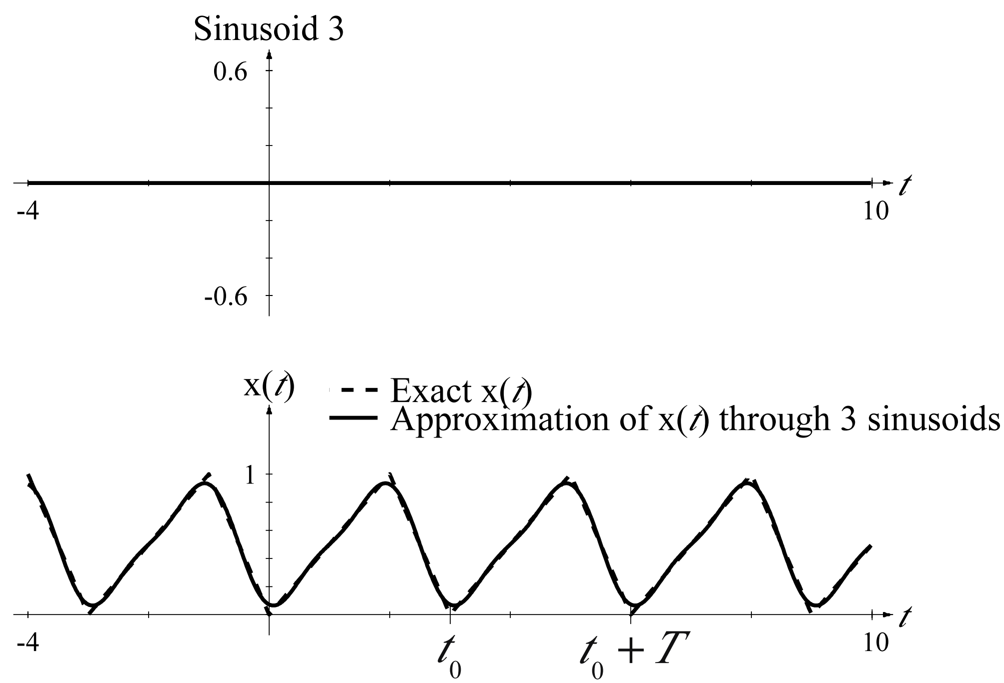
The best approximation to the dashed-line signal using a constant plus one sinusoid of the same fundamental frequency as the dashed-line signal plus another sinusoid of twice the fundamental frequency of the dashed-line signal is the solid line. The frequency of this second sinusoid is the **second harmonic** of the fundamental frequency.





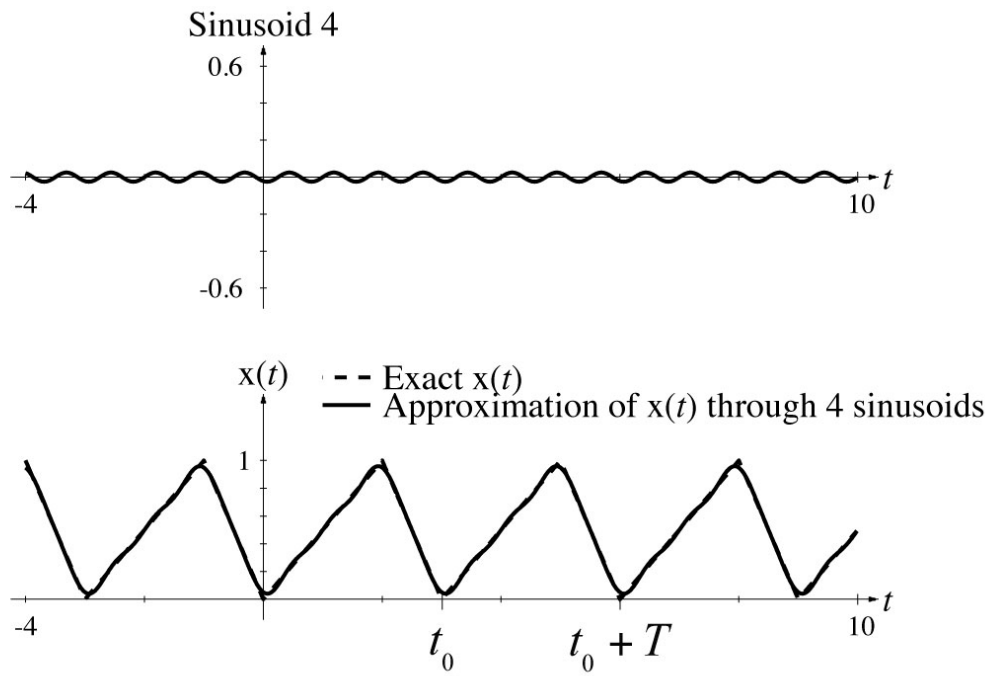
Conceptual Overview

The best approximation to the dashed-line signal using a constant plus three harmonics is the solid line. In this case (but not in general), the third harmonic has zero amplitude. This means that no sinusoid of three times the fundamental frequency improves the approximation.



Conceptual Overview

The best approximation to the dashed-line signal using a constant plus four harmonics is the solid line. This is a good approximation that gets better with the addition of more harmonics.



Continuous-Time Fourier Series Definition

The Fourier series representation of a signal $\mathrm{x}(t)$ over a time $t_0 < t < t_0 + T$ is

$$\mathrm{x}(t) = \sum_{k=-\infty}^{\infty} \mathrm{c_x}[k] e^{j2\pi kt/T}$$

where $\mathrm{c_x}[k]$ is the **harmonic function** and k is the **harmonic number**. The harmonic function can be found from the signal using the principle of **orthogonality**.

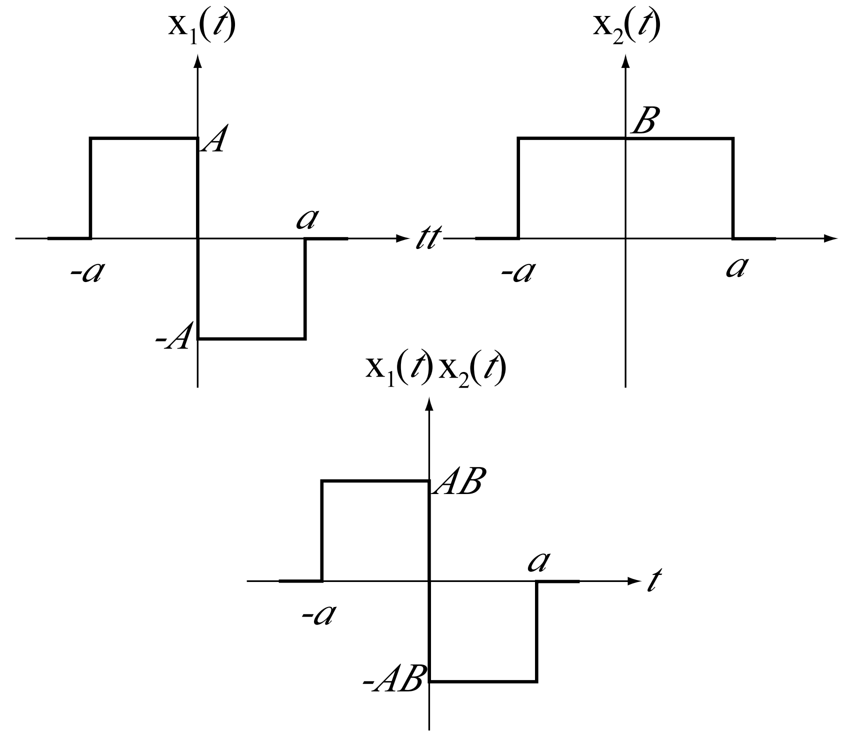
Orthogonality

The **inner product** of two functions over a range of times $t_0 < t < t_0 + T$ is defined by

$$\underbrace{\left(\mathrm{x}_1(t), \mathrm{x}_2(t)\right)}_{\text{inner product}} = \int_{t_0}^{t_0+T} \mathrm{x}_1(t)\,\mathrm{x}_2^*(t)\,dt$$

If this inner product is zero, the two functions are **orthogonal** over the interval $t_0 < t < t_0 + T$.

$\mathrm{x}_1(t)$ and $\mathrm{x}_2(t)$ are orthogonal on the interval $-a < t < a$



Orthogonality

The inner product of two complex sinusoids

$e^{j2\pi kt/T}$ and $e^{j2\pi qt/T}$ $\left(k \text{ and } q \text{ integers}\right)$on the interval $t_0 < t < t_0 + T$ is

$$\left(e^{j2\pi kt/T}, e^{j2\pi qt/T}\right) = \int_{t_0}^{t_0+T} e^{j2\pi kt/T} e^{-j2\pi qt/T}\, dt = \int_{t_0}^{t_0+T} e^{j2\pi(k-q)t/T}\, dt$$

Using Euler's identity

$$\left(e^{j2\pi kt/T}, e^{j2\pi qt/T}\right) = \int_{t_0}^{t_0+T} \left[\cos\left(2\pi \frac{k-q}{T} t\right) + j\sin\left(2\pi \frac{k-q}{T} t\right)\right] dt$$

Orthogonality

$$\left(e^{j2\pi kt/T}, e^{j2\pi qt/T}\right) = \int_{t_0}^{t_0+T} \left[\cos\left(2\pi \frac{k-q}{T} t\right) + j\sin\left(2\pi \frac{k-q}{T} t\right)\right] dt$$

If $k = q$,

$$\left(e^{j2\pi kt/T}, e^{j2\pi qt/T}\right) = \int_{t_0}^{t_0+T} \left[\cos(0) + j\sin(0)\right] dt = \int_{t_0}^{t_0+T} dt = T.$$

If $k \neq q$, the integral

$$\left(e^{j2\pi kt/T}, e^{j2\pi qt/T}\right) = \int_{t_0}^{t_0+T} \left[\cos\left(2\pi \frac{k-q}{T} t\right) + j\sin\left(2\pi \frac{k-q}{T} t\right)\right] dt$$

is over a non-zero, integer number of cycles of a cosine and a sine and is therefore zero.

Orthogonality

Therefore $e^{j2\pi kt/T}$ and $e^{j2\pi qt/T}$ are orthogonal if k and q are not equal.

Now multiply the Fourier series expression $\mathrm{x}(t) = \sum_{k=-\infty}^{\infty} \mathrm{c}_{\mathrm{x}}[k] e^{j2\pi kt/T}$

by $e^{-j2\pi qt/T}$ (q an integer)

$$\mathrm{x}(t) e^{-j2\pi qt/T} = \sum_{k=-\infty}^{\infty} \mathrm{c}_{\mathrm{x}}[k] e^{j2\pi(k-q)t/T}$$

and integrate both sides over the interval $t_0 \le t < t_0 + T$

$$\int_{t_0}^{t_0+T} \mathrm{x}(t) e^{-j2\pi qt/T} \, dt = \int_{t_0}^{t_0+T} \left[\sum_{k=-\infty}^{\infty} \mathrm{c}_{\mathrm{x}}[k] e^{j2\pi(k-q)t/T} \right] dt.$$

Orthogonality

Since k and t are independent variables

$$\int_{t_0}^{t_0+T} \mathrm{x}(t) e^{-j2\pi qt/T}\, dt = \sum_{k=-\infty}^{\infty} \mathrm{c_x}[k] \underbrace{\int_{t_0}^{t_0+T} e^{j2\pi(k-q)t/T}\, dt}_{=0 \text{ if } k\neq q,\ =T \text{ if } k=q}$$

Therefore $\displaystyle\int_{t_0}^{t_0+T} \mathrm{x}(t) e^{-j2\pi qt/T}\, dt = \mathrm{c_x}[q]T$ and $\displaystyle\mathrm{c_x}[q] = \frac{1}{T}\int_{t_0}^{t_0+T} \mathrm{x}(t) e^{-j2\pi qt/T}\, dt$

implying that $\displaystyle\mathrm{c_x}[k] = \frac{1}{T}\int_{t_0}^{t_0+T} \mathrm{x}(t) e^{-j2\pi kt/T}\, dt.$

Continuous-Time Fourier Series Definition

Summarizing

$$x(t)=\sum_{k=-\infty}^{\infty} c_x[k]e^{j2\pi kt/T} \quad \text{and} \quad c_x[k]=\frac{1}{T}\int_{t_0}^{t_0+T} x(t)e^{-j2\pi kt/T}\,dt.$$

The signal and its harmonic function form a **Fourier series pair** $x(t)\xleftrightarrow[T]{\mathcal{FS}}c_x[k]$ where T is the representation time and, therefore, the fundamental period of the continuous-time Fourier series (CTFS) representation of $x(t)$. If T is also a period of $x(t)$, the CTFS representation of $x(t)$ is valid for all time. This is, by far, the most common use of the CTFS in engineering applications. If T is not a period of $x(t)$, the CTFS representation is generally valid only in the interval $t_0 \le t < t_0 + T$.

CTFS of a Real Function

It can be shown that the continuous-time Fourier series (CTFS) harmonic function of any real-valued function $\mathrm{x}(t)$ has the property that $\mathrm{c}_{\mathrm{x}}[k] = \mathrm{c}_{x}^{*}[-k]$.

One implication of this fact is that, for real-valued functions, the <u>magnitudes</u> of their harmonic functions are <u>even functions</u> and their <u>phases</u> can be expressed as <u>odd functions</u> of harmonic number k.

The Trigonometric CTFS

The fact that, for a real-valued function $\mathrm{x}(t)$

$$\mathrm{c_x}[k] = \mathrm{c_x}^*[-k]$$

also leads to the definition of an alternate form of the CTFS, the so-called **trigonometric form**.

$$\mathrm{x}(t) = \mathrm{a_x}[0] + \sum_{k=1}^{\infty} \left\{ \mathrm{a_x}[k]\cos(2\pi kt/T) + \mathrm{b_x}[k]\sin(2\pi kt/T) \right\}$$

where

$$\mathrm{a_x}[k] = \frac{2}{T}\int_{t_0}^{t_0+T} \mathrm{x}(t)\cos(2\pi kt/T)\,dt$$

$$\mathrm{b_x}[k] = \frac{2}{T}\int_{t_0}^{t_0+T} \mathrm{x}(t)\sin(2\pi kt/T)\,dt$$

The Trigonometric CTFS

Since both the complex and trigonometric forms of the CTFS represent a signal, there must be relationships between these different types of harmonic functions.

Those relationships are

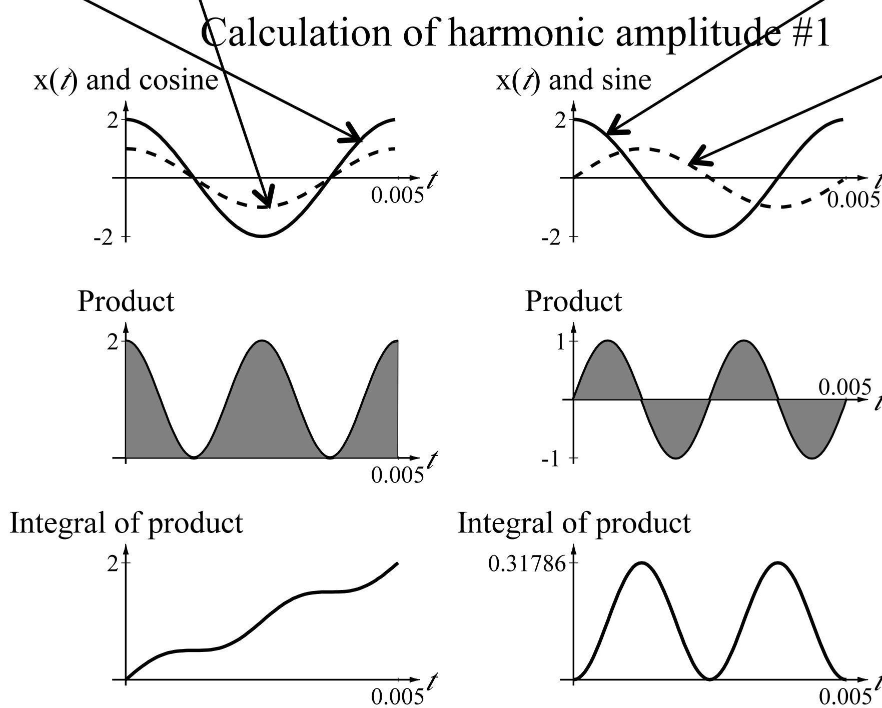
$$\left\{ \begin{array}{c} \mathrm{a_x}[0] = \mathrm{c_x}[0] \\ \mathrm{b_x}[0] = 0 \\ \mathrm{a_x}[k] = \mathrm{c_x}[k] + \mathrm{c_x}^*[k] \\ \mathrm{b_x}[k] = j\left(\mathrm{c_x}[k] - \mathrm{c_x}^*[k]\right) \end{array} \right\} \;, \quad k = 1,2,3,\cdots$$

$$\left\{ \begin{array}{c} \mathrm{c_x}[0] = \mathrm{a_x}[0] \\ \mathrm{c_x}[k] = \dfrac{\mathrm{a_x}[k] - j\mathrm{b_x}[k]}{2} \\ \mathrm{c_x}[-k] = \mathrm{c_x}^*[k] = \dfrac{\mathrm{a_x}[k] + j\mathrm{b_x}[k]}{2} \end{array} \right\} \;, \quad k = 1,2,3,\cdots$$

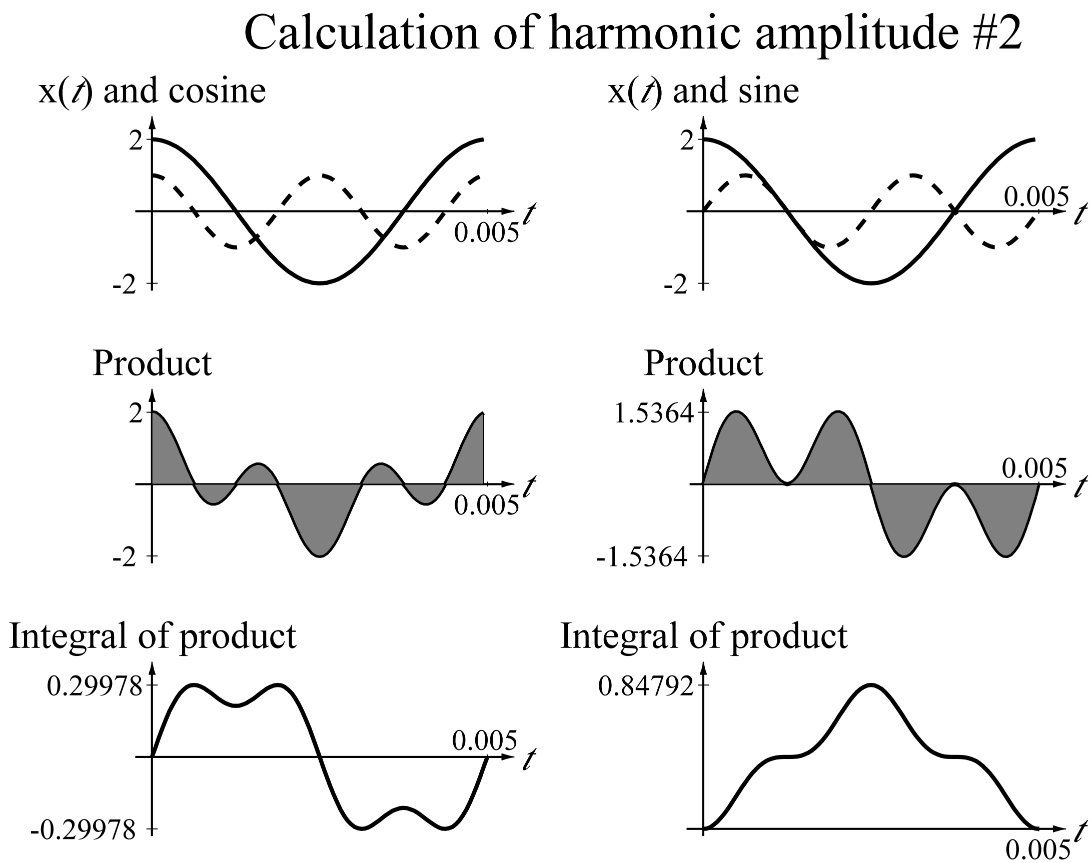
CTFS Example #1

Let a signal be defined by $x(t) = 2\cos(400\pi t)$ and let $T = 5$ ms which is the same as T_0.

$$a_x[k] = \frac{2}{T}\int_{t_0}^{t_0+T} \underbrace{x(t)}\underbrace{\cos(2\pi kt/T)}dt \quad b_x[k] = \frac{2}{T}\int_{t_0}^{t_0+T} \underbrace{x(t)}\underbrace{\sin(2\pi kt/T)}dt$$

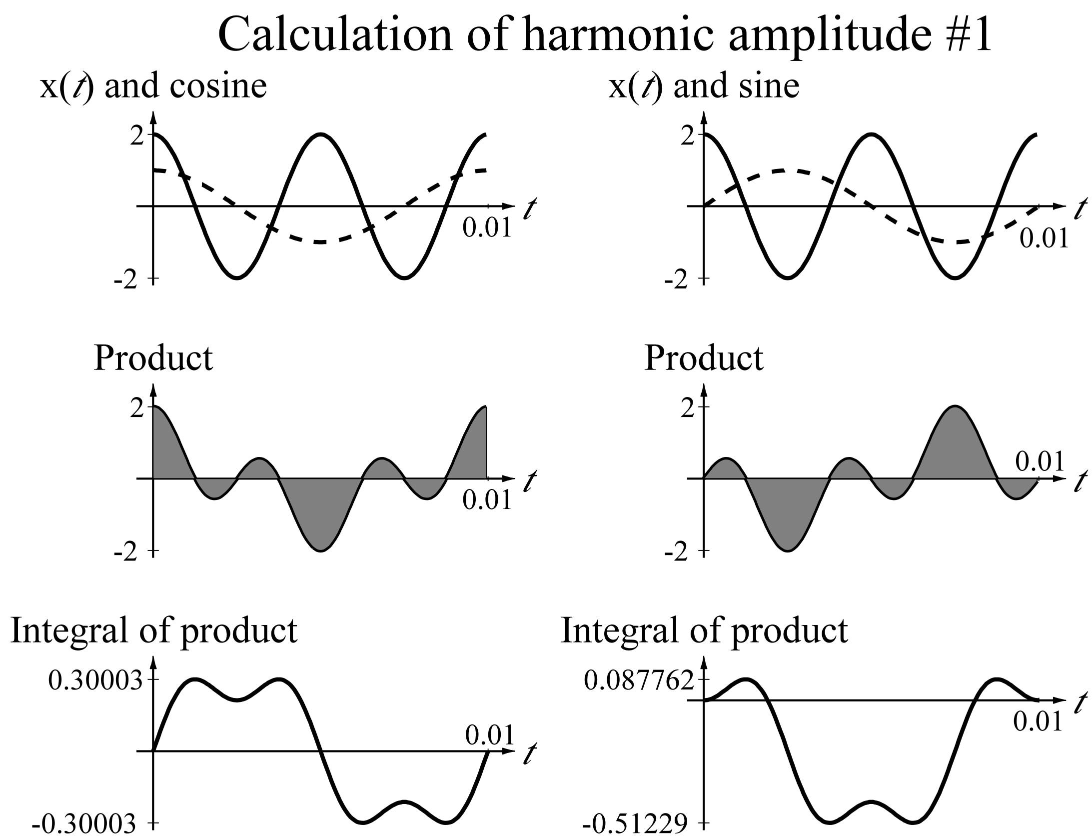


CTFS Example #1

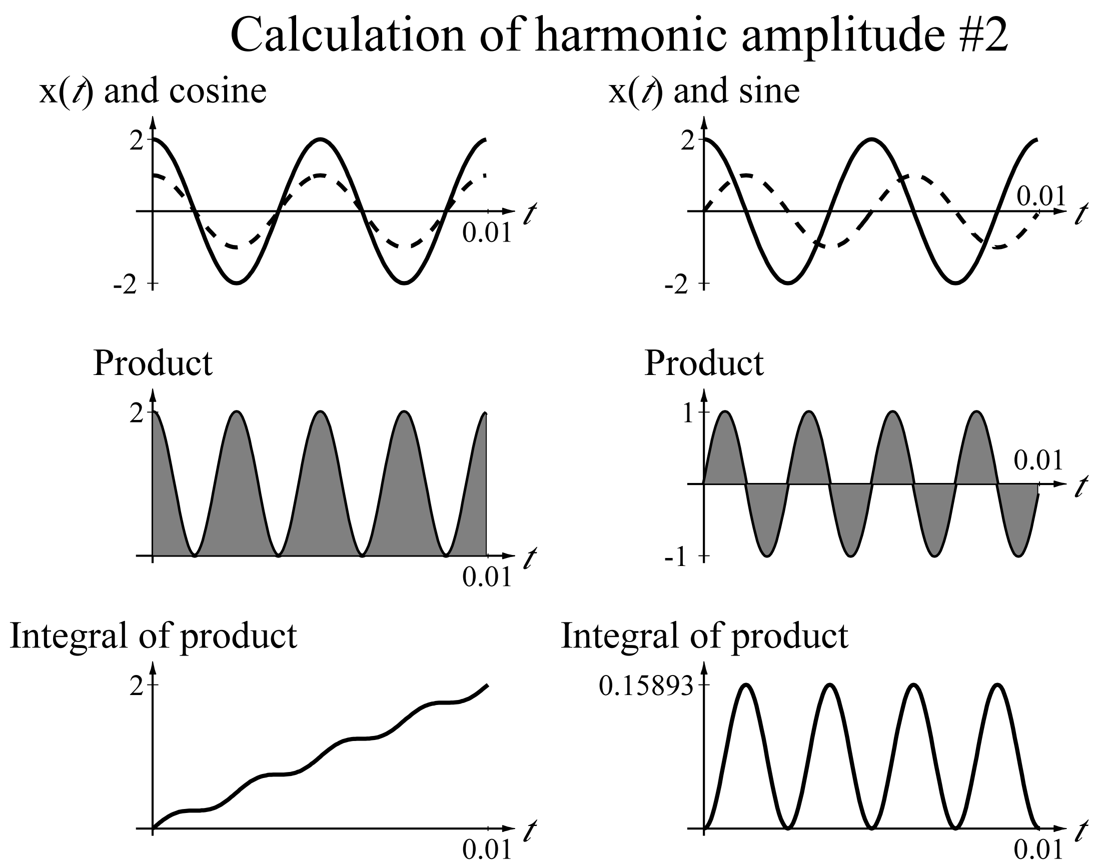


CTFS Example #2

Let a signal be defined by $\mathrm{x}(t) = 2\cos(400\pi t)$ and let $T = 10$ ms which is $2T_0$.

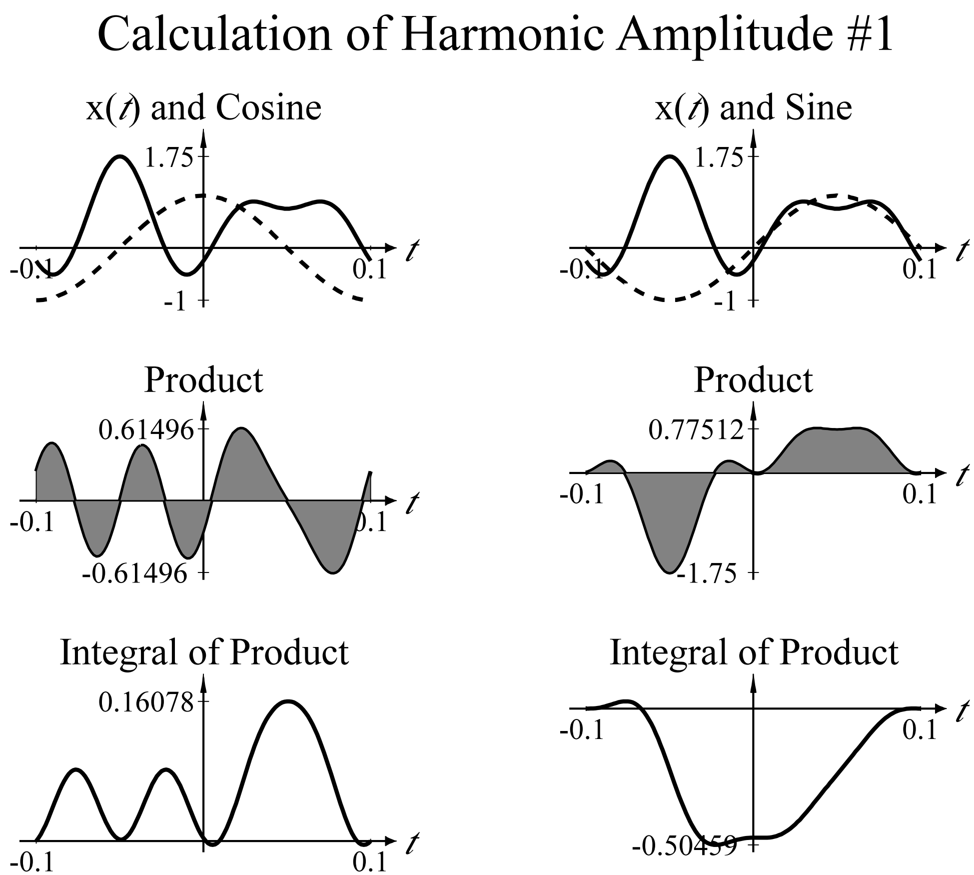


CTFS Example #2

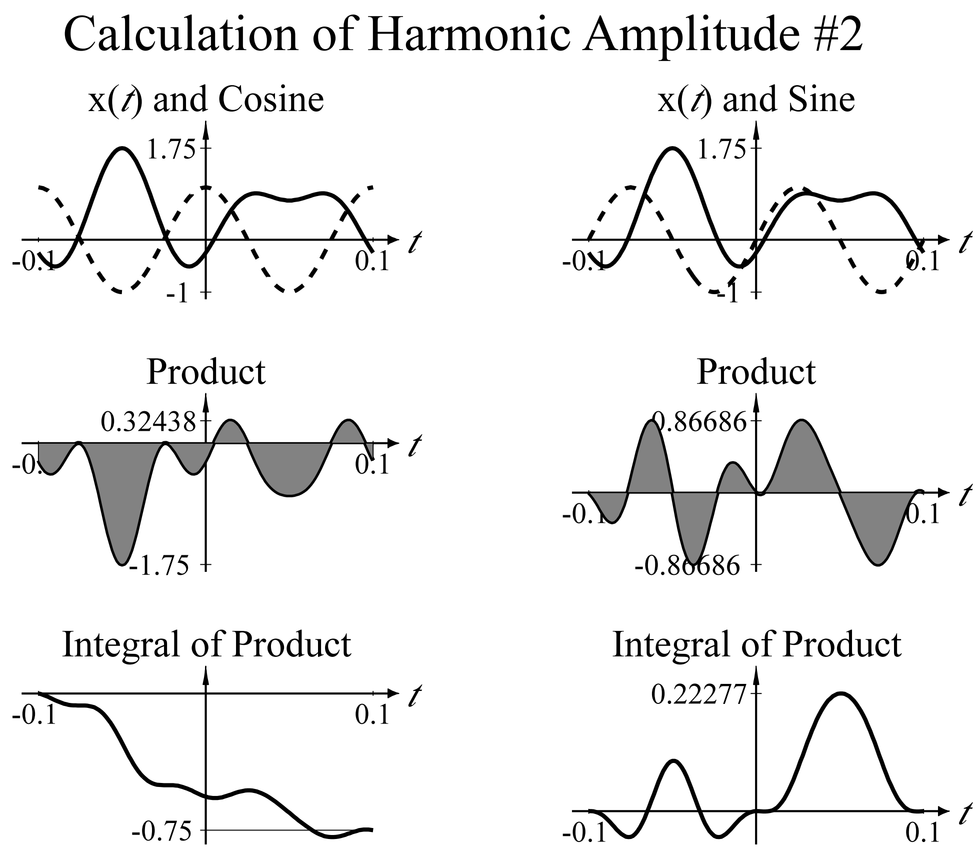


CTFS Example #3

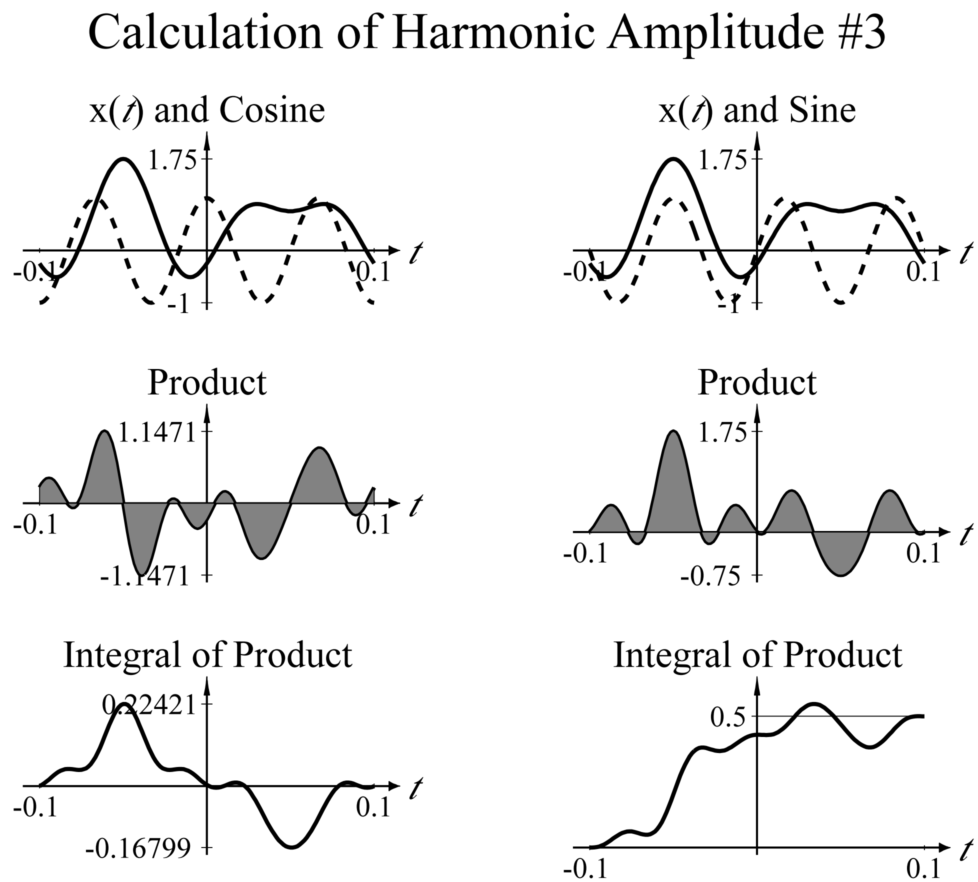
Let $x(t) = 1/2 - (3/4)\cos(20\pi t) + (1/2)\sin(30\pi t)$ and let $T = 200$ ms.



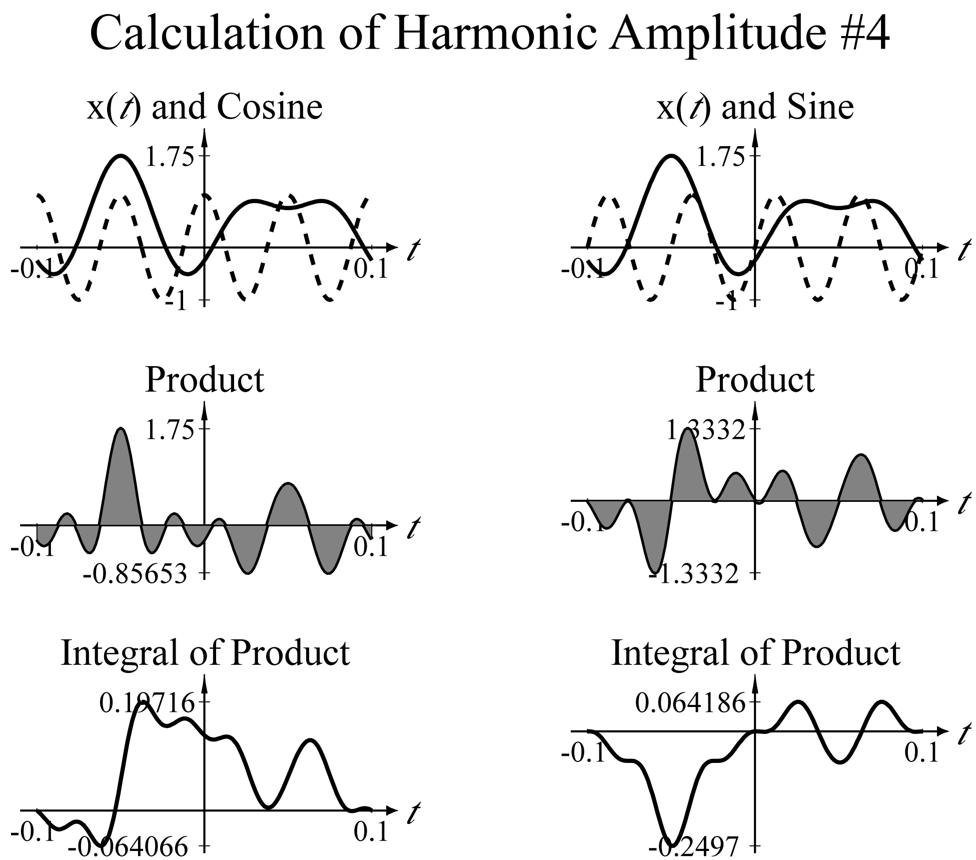
CTFS Example #3



CTFS Example #3



CTFS Example #3



Linearity of the CTFS

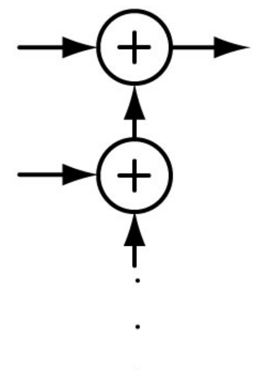
$$x(t) = x_1(t) + x_2(t) + \cdots$$

$$x(t) \xleftrightarrow[T]{\mathcal{FS}} c_x[k]$$

$$x_1(t) \xleftrightarrow[T]{\mathcal{FS}} c_{x1}[k]$$

$$x_2(t) \xleftrightarrow[T]{\mathcal{FS}} c_{x2}[k]$$

$$\vdots \qquad \vdots$$



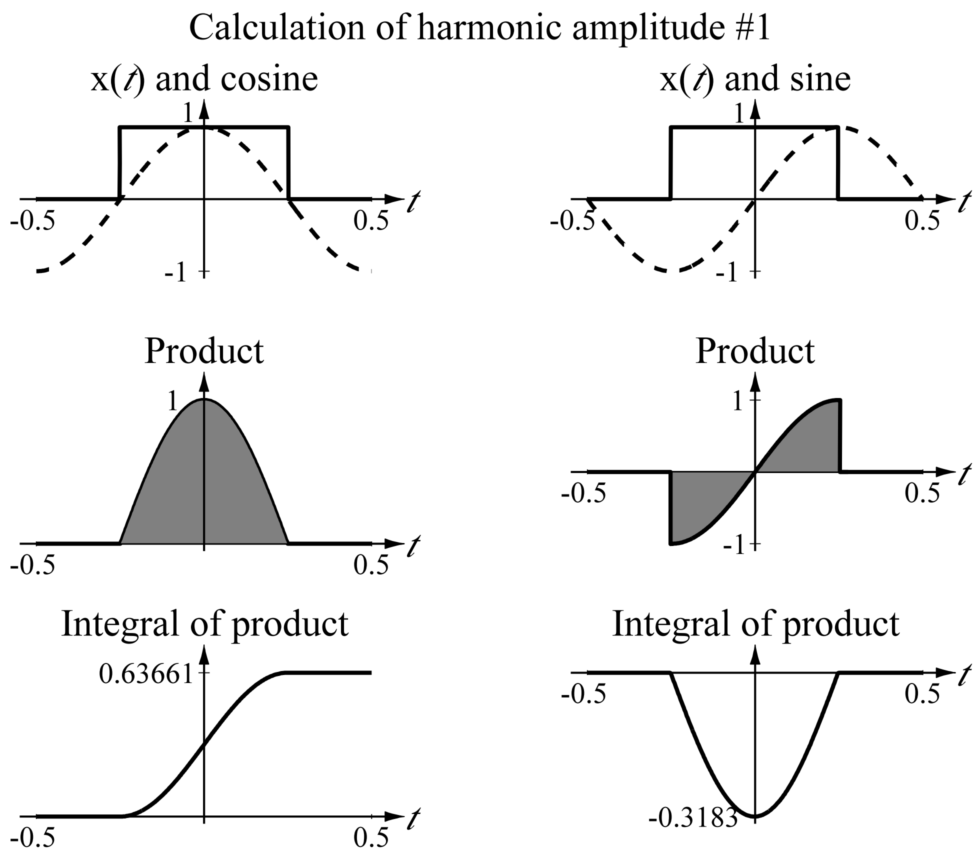
$c_x[k]$

These relations hold only if the harmonic functions of all the component functions are based on the same **representation time** T.

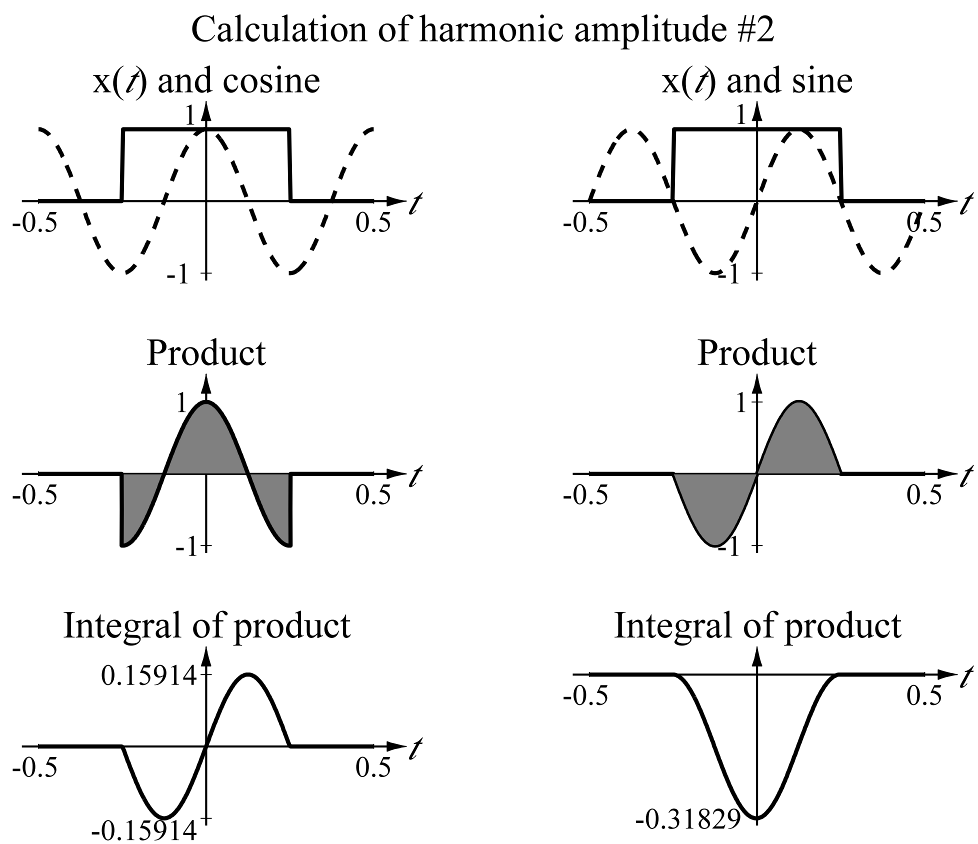
CTFS Example #4

Let the signal be a 50% duty-cycle square wave with an amplitude of one and a fundamental period $T_0 = 1$.

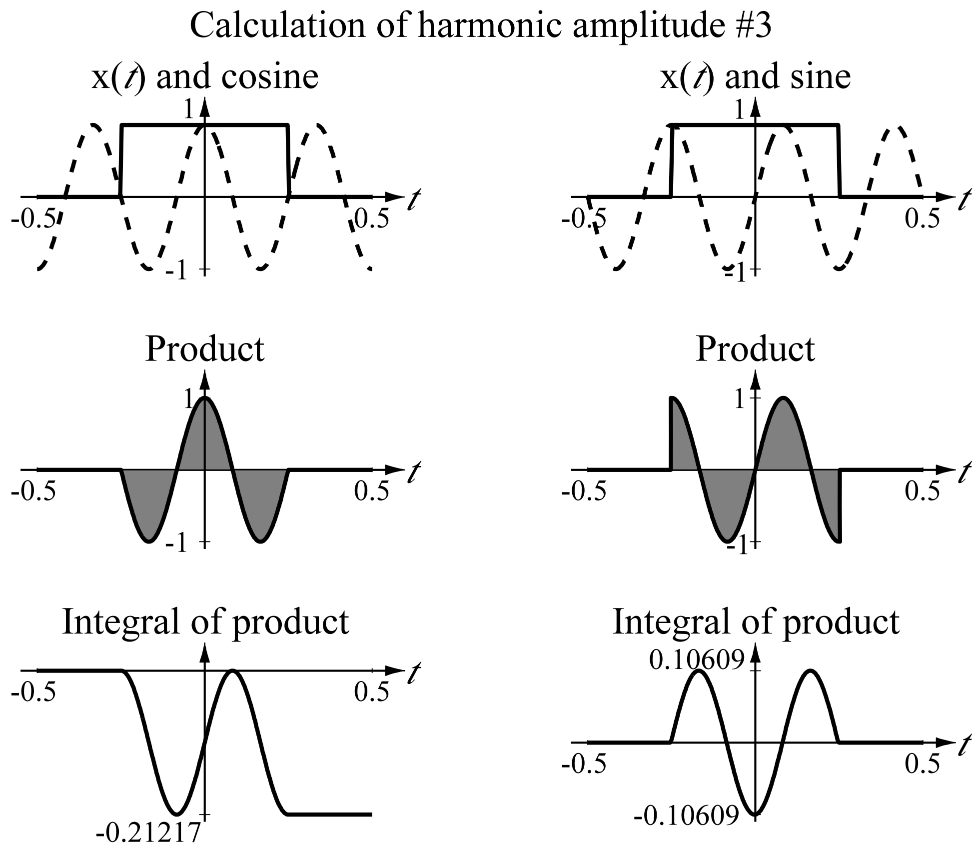
$$x(t) = \text{rect}(2t) * \delta_1(t)$$



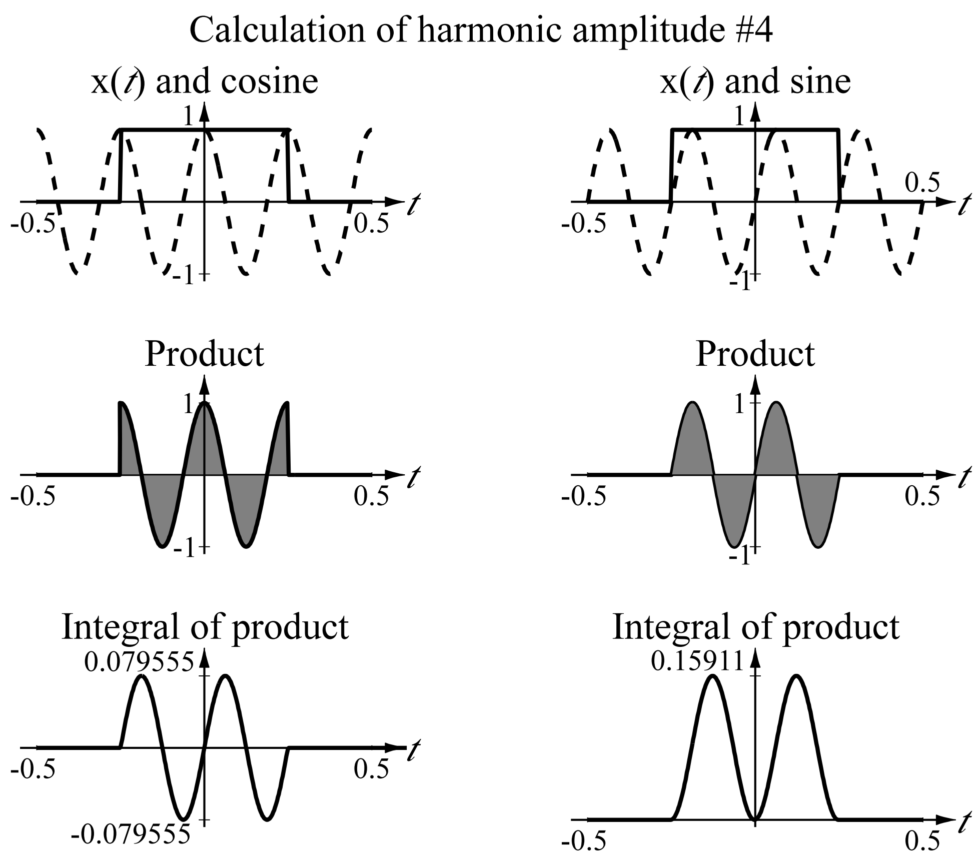
CTFS Example #4



CTFS Example #4

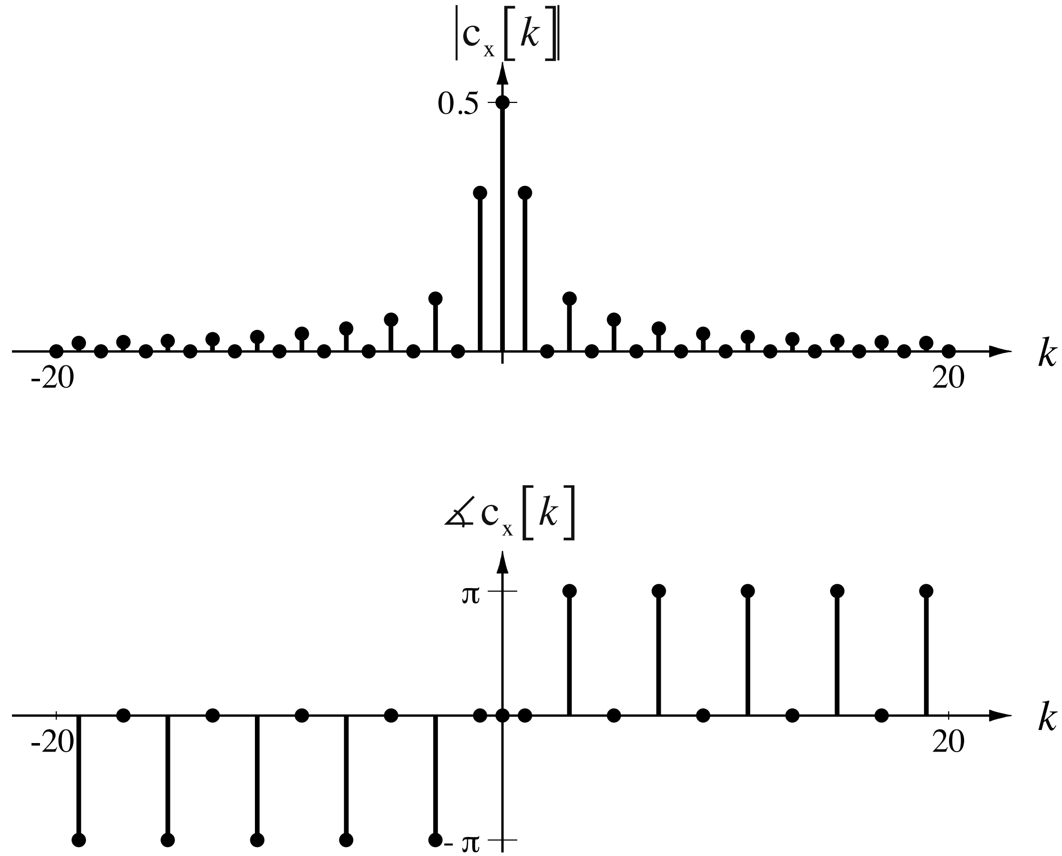


CTFS Example #4



CTFS Example #4

A graph of the magnitude and phase of the harmonic function as a function of harmonic number is a good way of illustrating it. Notice that the magnitude is an even function of k and the phase is an odd function of k.



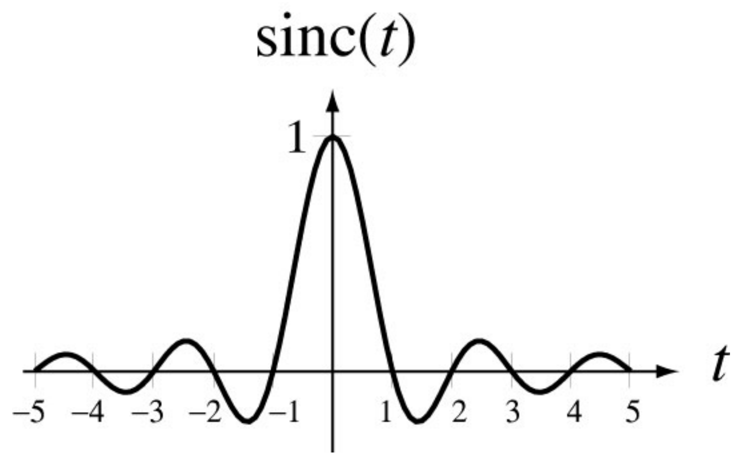
The Sinc Function

Let $\mathrm{x}(t) = A\,\mathrm{rect}(t/w) * \delta_{T_0}(t)$, $w < T_0$. Then

$$\mathrm{x}(t) = A\,\mathrm{rect}(t/w) * \delta_{T_0}(t) \xleftrightarrow[T_0]{\mathcal{FS}} \mathrm{c}_{\mathrm{x}}[k] = A\frac{\sin(\pi k w / T_0)}{\pi k}$$

The mathematical form $\dfrac{\sin(\pi x)}{\pi x}$ arises frequently enough

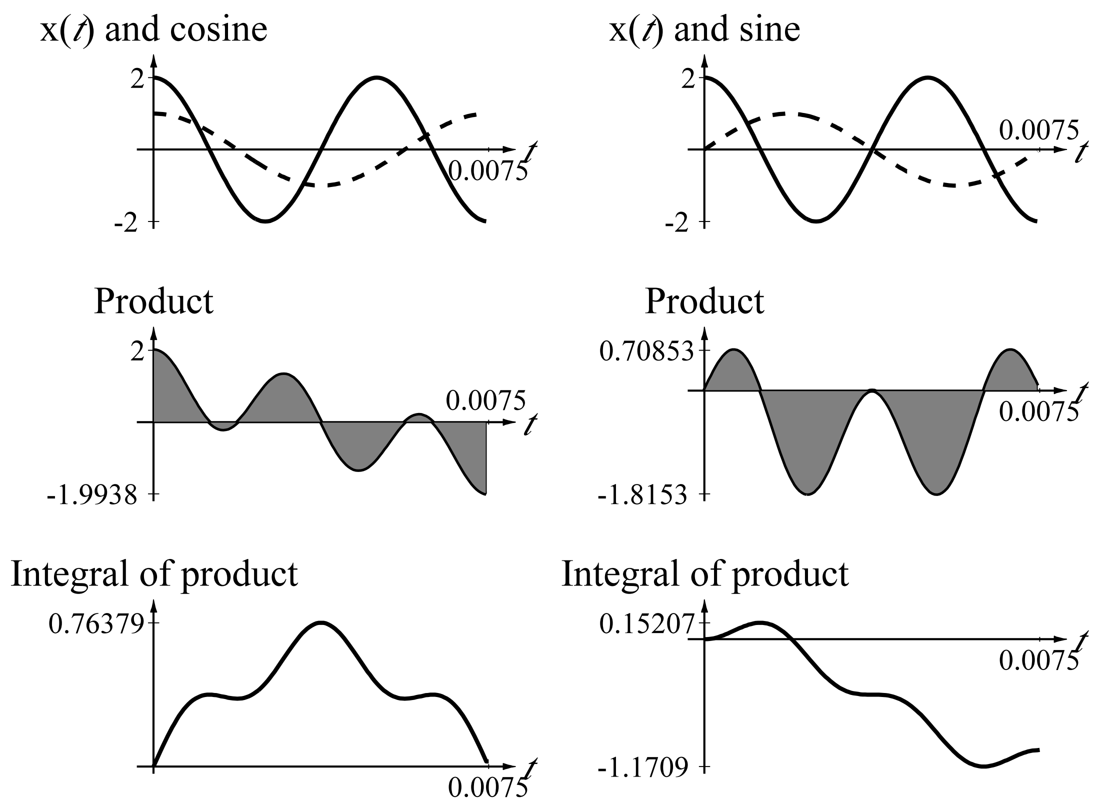
to be given its own name, "sinc". That is $\mathrm{sinc}(t) = \dfrac{\sin(\pi t)}{\pi t}$.



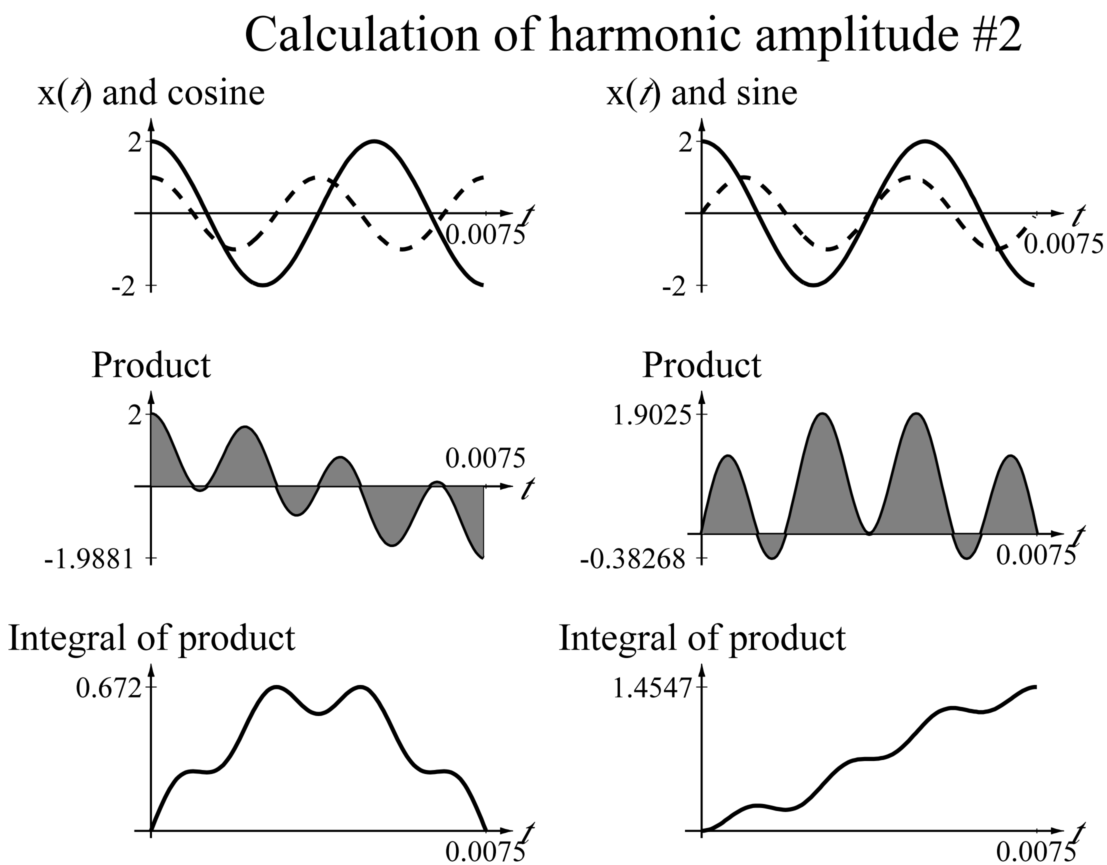
CTFS Example #5

Let $\mathrm{x}(t) = 2\cos(400\pi t)$ and let $T = 7.5$ ms which is 1.5 fundamental periods of this signal.

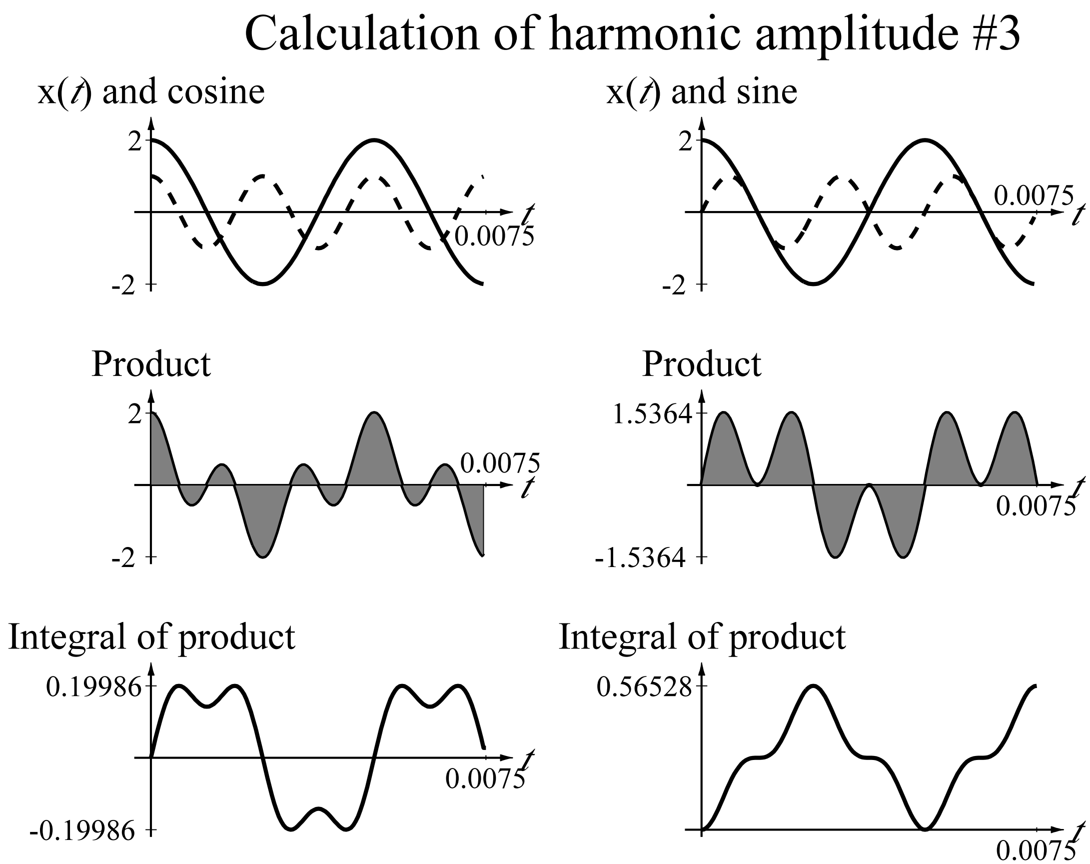
Calculation of harmonic amplitude #1



CTFS Example #5

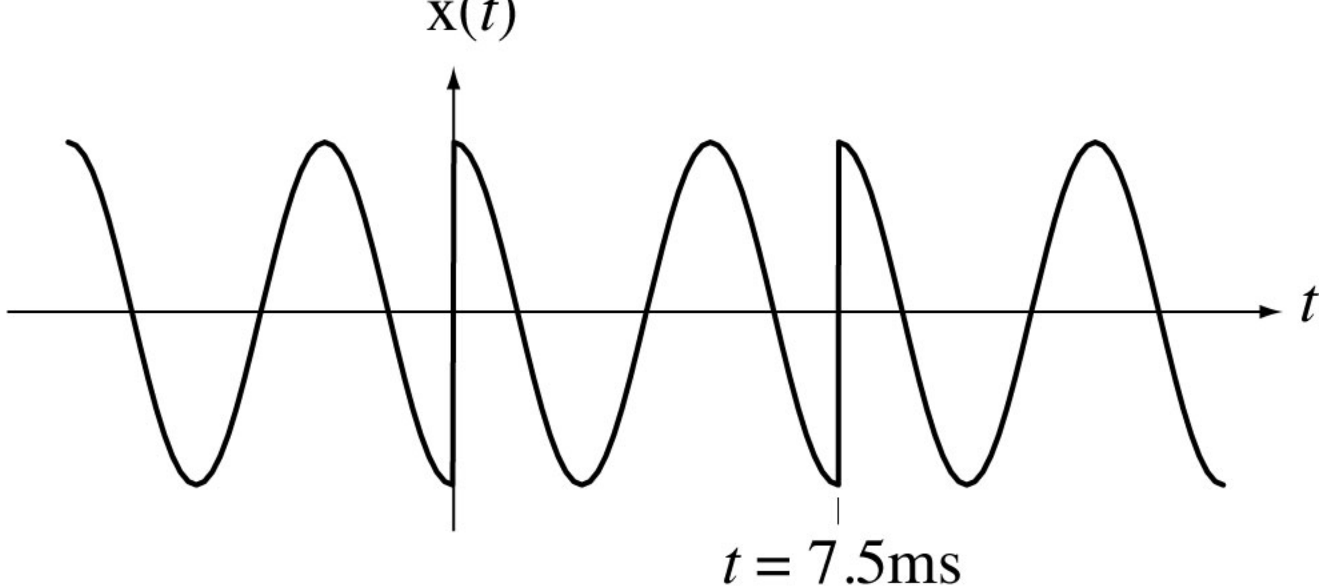


CTFS Example #5



CTFS Example #5

The CTFS representation of this cosine is the signal below, which is an odd function, and the discontinuities make the representation have significant higher harmonic content. Although correct in the time interval from zero to 7.5 ms, this is a very inelegant representation.



CTFS of Even and Odd Functions

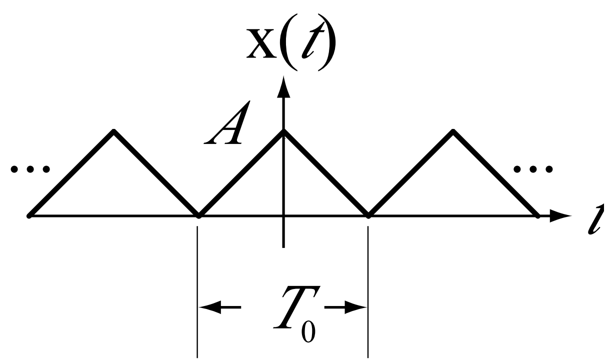
For an **even function**, the complex CTFS harmonic function $c_x[k]$ is **purely real** and the sine harmonic function $a_x[k]$ is zero.

For an **odd function**, the complex CTFS harmonic function $c_x[k]$ is **purely imaginary** and the cosine harmonic function $b_x[k]$ is zero.

Convergence of the CTFS

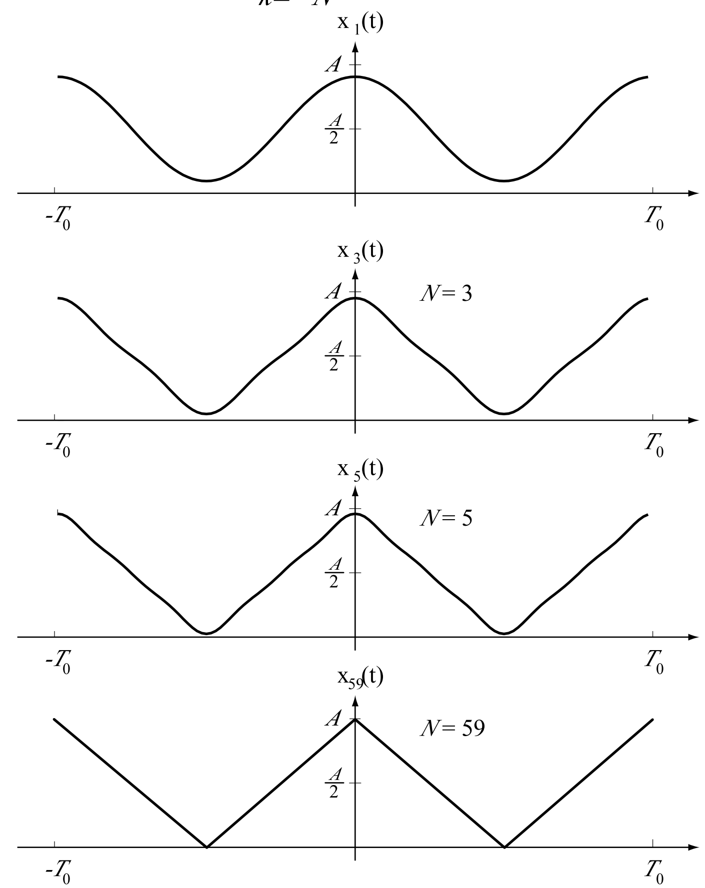
For continuous signals, **convergence** is exact at every point.

A Continuous Signal



Partial CTFS Sums

$$x_N(t) = \sum_{k=-N}^{N} c_x[k] e^{j2\pi kt/T_0}$$

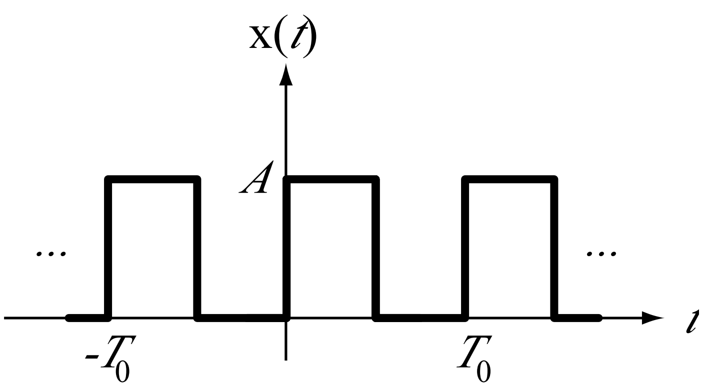


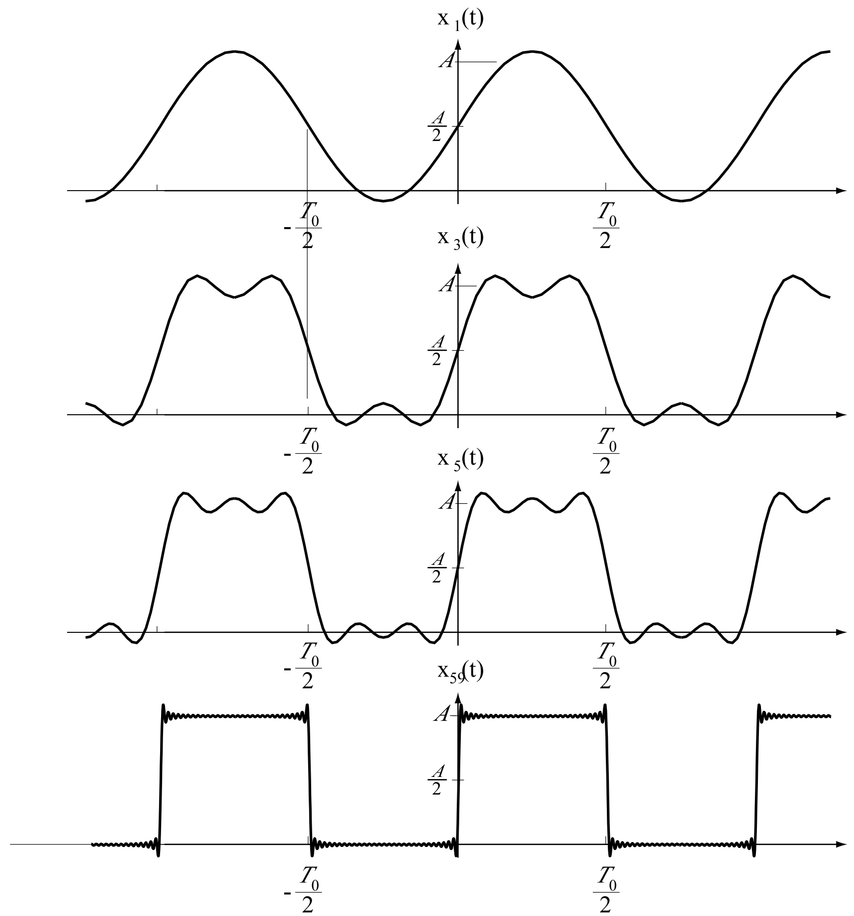
Convergence of the CTFS

Partial CTFS Sums

For discontinuous signals, convergence is exact at every point of continuity.

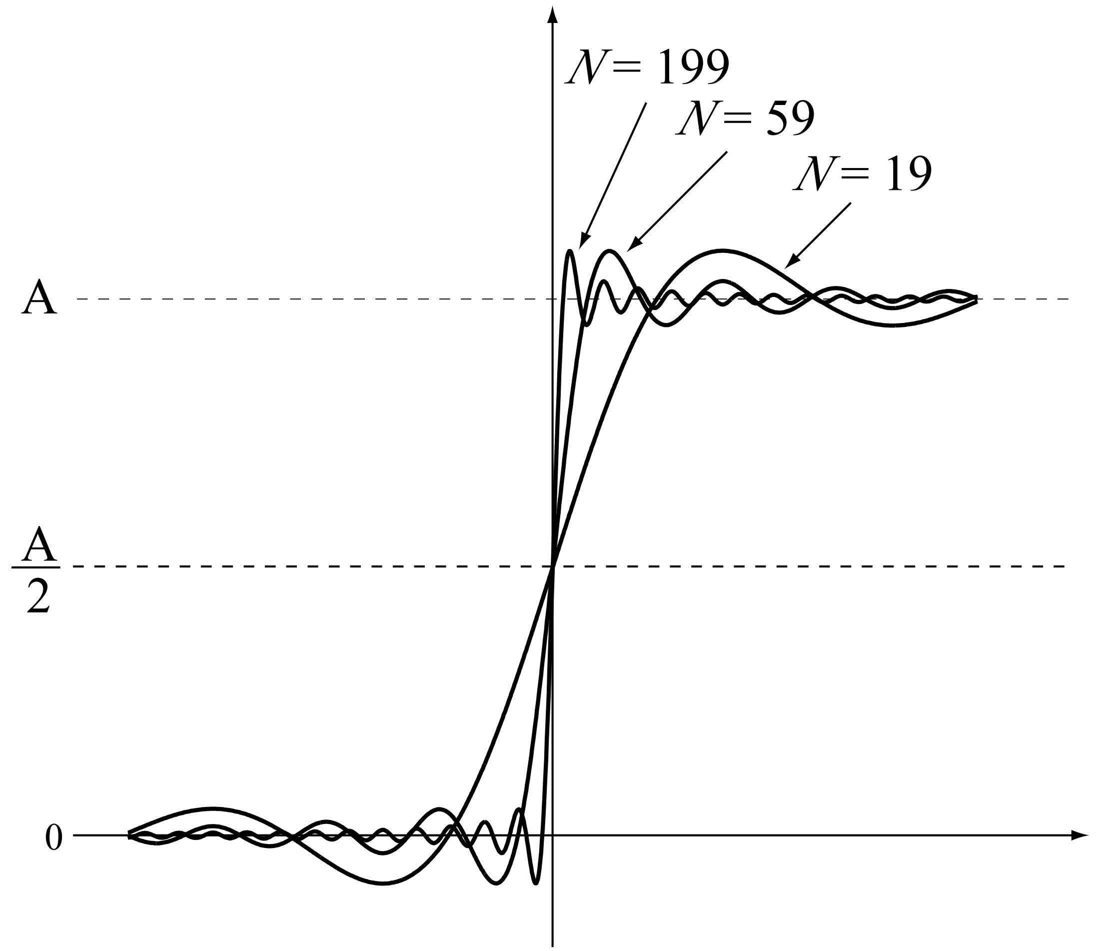
Discontinuous Signal





Convergence of the CTFS

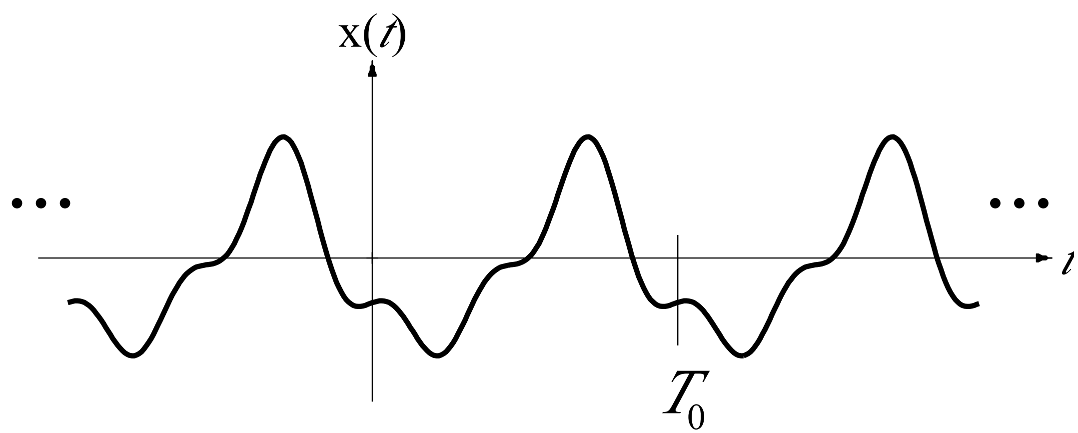
At points of discontinuity the Fourier series representation converges to the mid-point of the discontinuity.



Numerical Computation of the CTFS

How could we find the CTFS of a signal that has no known functional description?

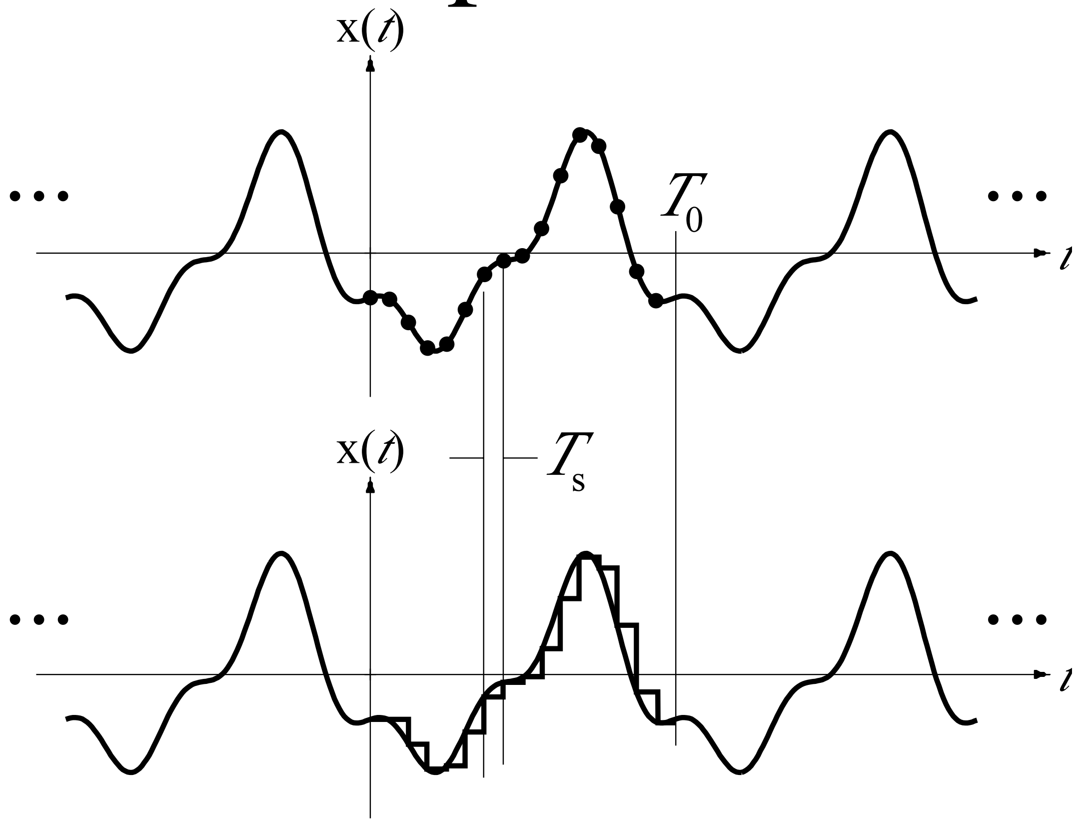
Numerically.



$$c_x[k] = \frac{1}{T}\int_T \underbrace{x(t)} e^{-j2\pi kt/T}\,dt$$

Unknown

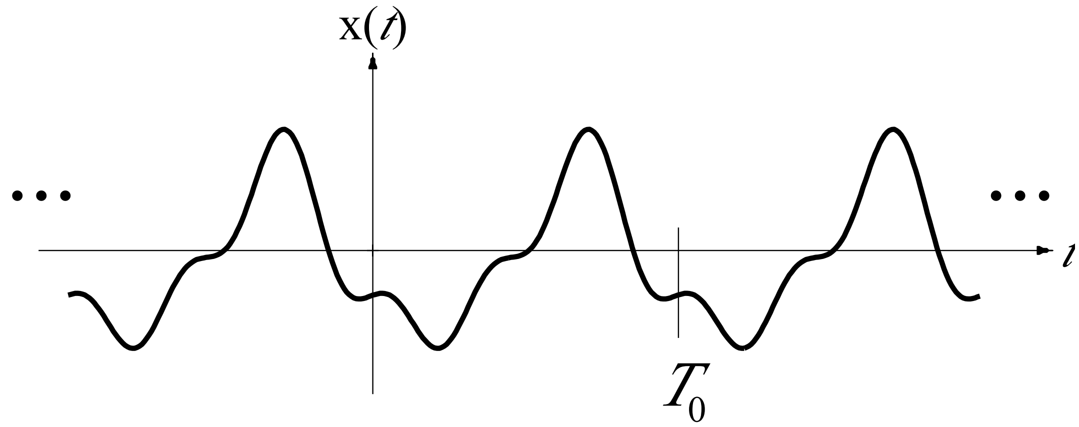
Numerical Computation of the CTFS



$$c_x[k] \cong \frac{1}{T}\sum_{n=0}^{N-1}\left[\int_{nT_s}^{(n+1)T_s} x(nT_s)e^{-j2\pi knT_s/T}\,dt\right]$$

Samples from x(t)

Numerical Computation of the CTFS



It can be shown that, for harmonic numbers $|k| << N$

$$c_x[k] \cong (1/N)\mathcal{DFT}\left(x(nT_s)\right) \quad , \quad |k| << N$$

where

$$\mathcal{DFT}\left(x(nT_s)\right) = \sum_{n=0}^{N-1} x(nT_s) e^{-j2\pi nk/N}$$

Numerical Computation of the CTFS

The **Discrete Fourier Transform**

$$\mathcal{DFT}\left(\mathrm{x}\left(nT_s\right)\right)=\sum_{n=0}^{N-1}\mathrm{x}\left(nT_s\right)e^{-j2\pi nk/N}$$

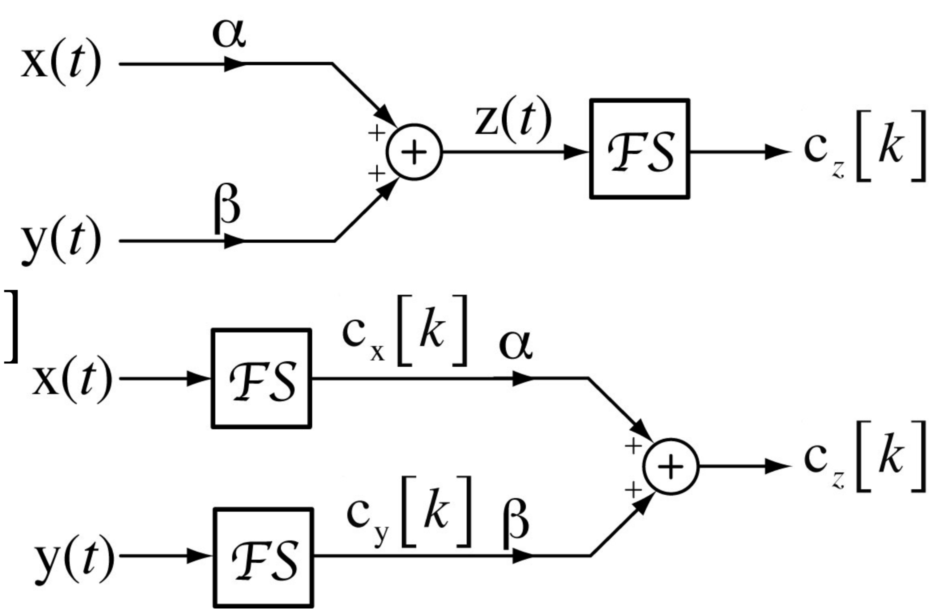
is an intrinsic function in most modern high-level computer languages.

CTFS Properties

Let a signal $x(t)$ have a fundamental period T_{0x} and let a signal $y(t)$ have a fundamental period T_{0y}. Let the CTFS harmonic functions, each using a common period T as the representation time, be $c_x[k]$ and $c_y[k]$. Then the following properties apply.

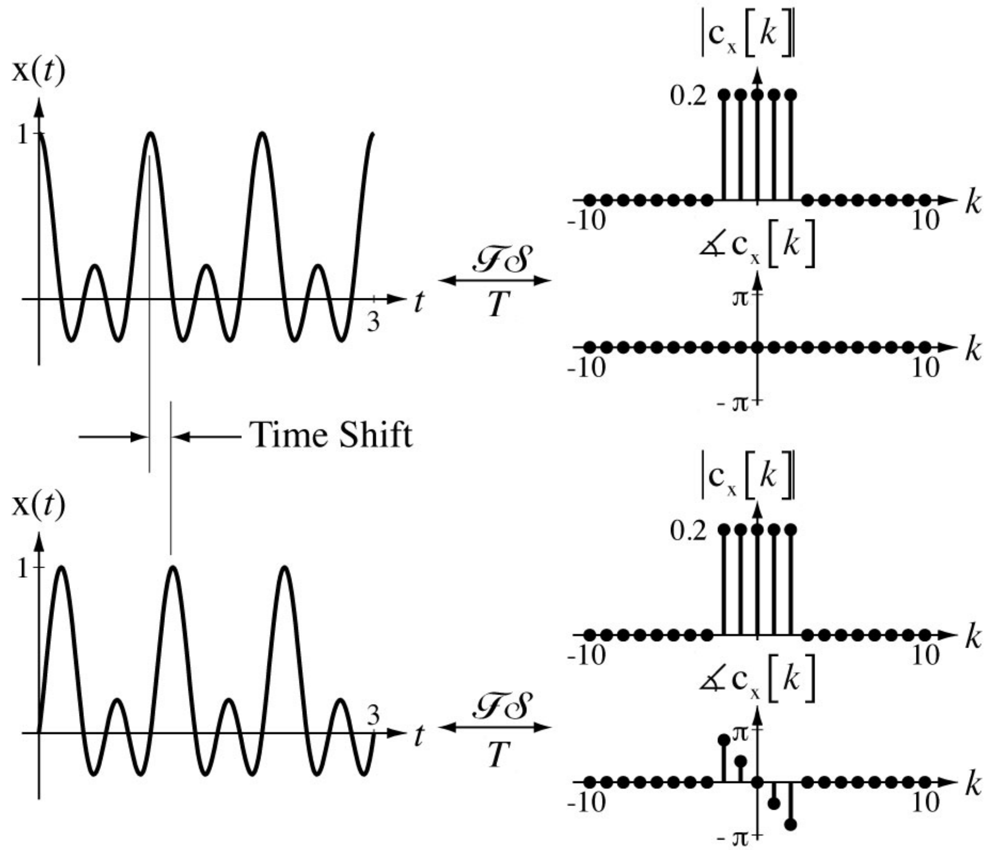
Linearity

$$\alpha x(t) + \beta y(t) \xleftrightarrow[T]{\mathcal{FS}} \alpha c_x[k] + \beta c_y[k]$$



CTFS Properties

Time Shifting $\mathrm{x}(t-t_0) \xleftrightarrow[T]{\mathcal{FS}} e^{-j2\pi k t_0/T}\,\mathrm{c_x}[k]$



CTFS Properties

Frequency Shifting (Harmonic Number Shifting)

$$e^{j2\pi k_0 t/T}\,\mathrm{x}(t) \xleftrightarrow[T]{\mathcal{FS}} \mathrm{c_x}[k-k_0]$$

A shift in frequency (harmonic number) corresponds to multiplication of the time function by a complex exponential.

Time Reversal

$$\mathrm{x}(-t) \xleftrightarrow[T]{\mathcal{FS}} \mathrm{c_x}[-k]$$

CTFS Properties

Time Scaling

Let $z(t) = x(at)$, $a > 0$

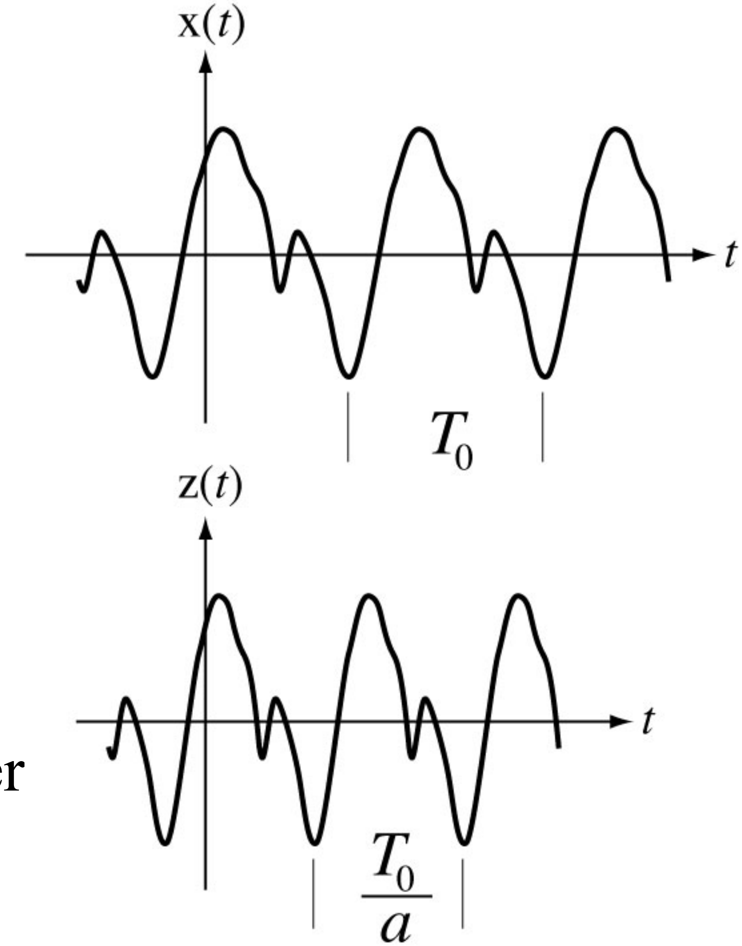
Case 1. $T = T_{0x} / a = T_{0z}$ for $z(t)$

$$c_z[k] = c_x[k]$$

Case 2. $T = T_{0x}$ for $z(t)$

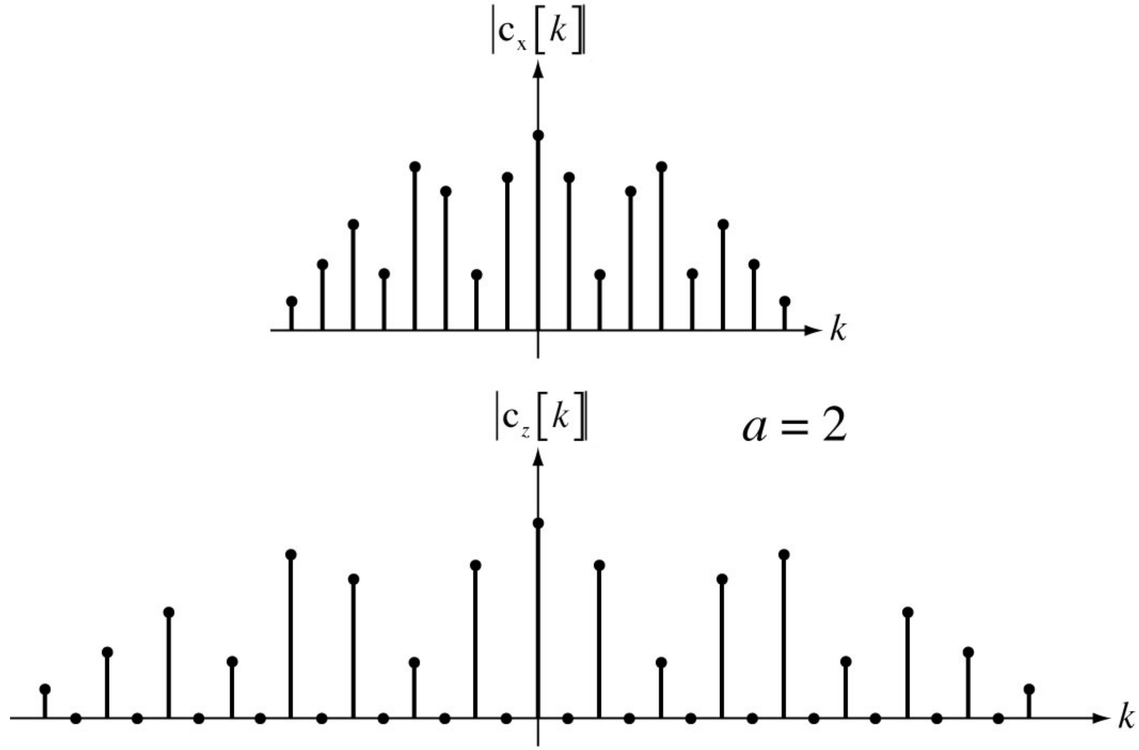
If a is an integer,

$$c_z[k] = \begin{cases} c_x[k/a] & , \ k/a \text{ an integer} \\ 0 & , \ \text{otherwise} \end{cases}$$



CTFS Properties

Time Scaling (continued)



CTFS Properties

Change of Representation Time

With $T = T_{0x}$, $\mathrm{x}(t) \xleftrightarrow[T]{\mathcal{FS}} \mathrm{c_x}[k]$

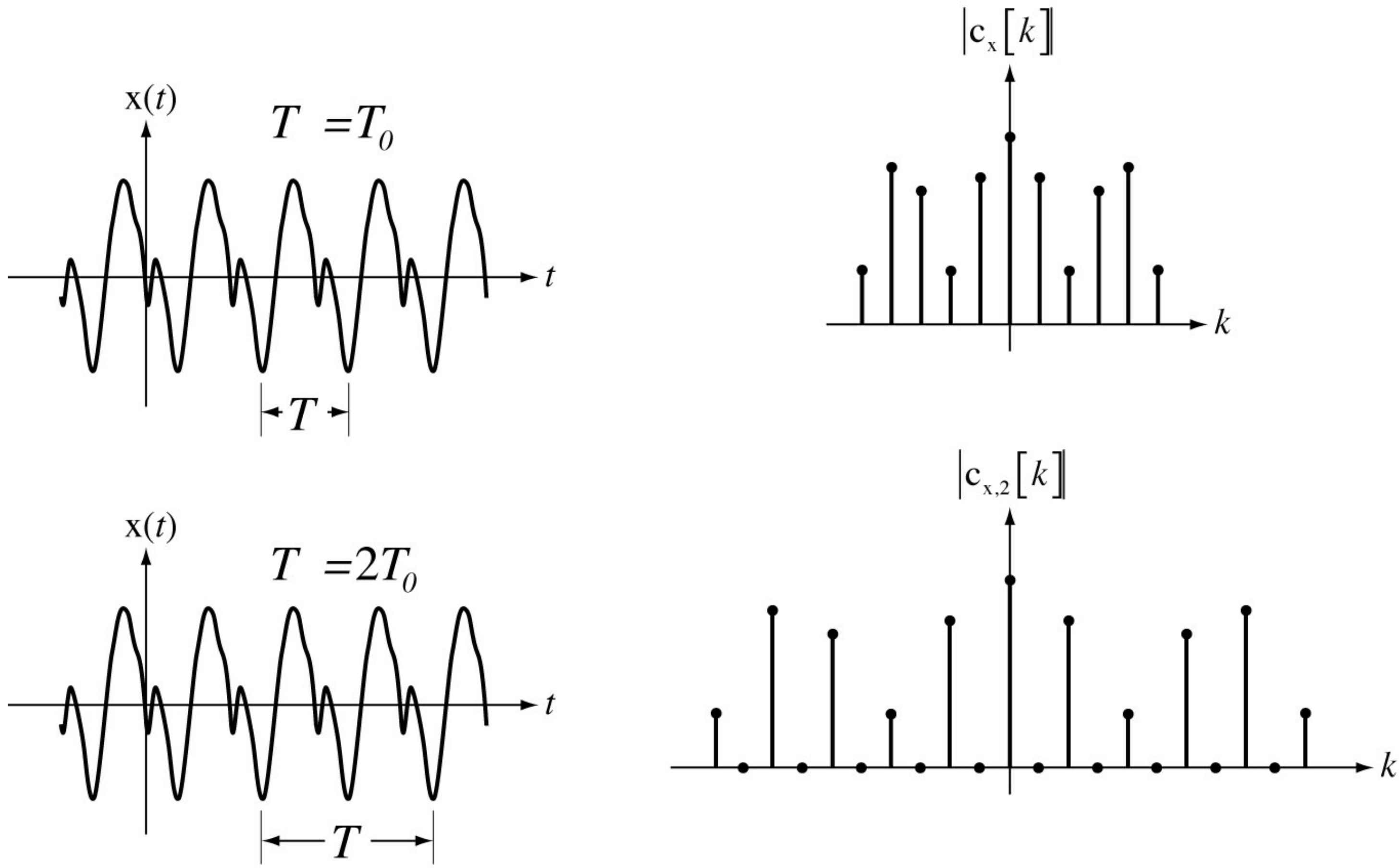
With $T = mT_{0x}$, $\mathrm{x}(t) \xleftrightarrow[T]{\mathcal{FS}} \mathrm{c}_{\mathrm{x},m}[k]$

$$\mathrm{c}_{\mathrm{x},m}[k] = \begin{cases} \mathrm{c_x}[k/m] & , \ k/m \text{ an integer} \\ 0 & , \ \text{otherwise} \end{cases}$$

(m is any positive integer)

CTFS Properties

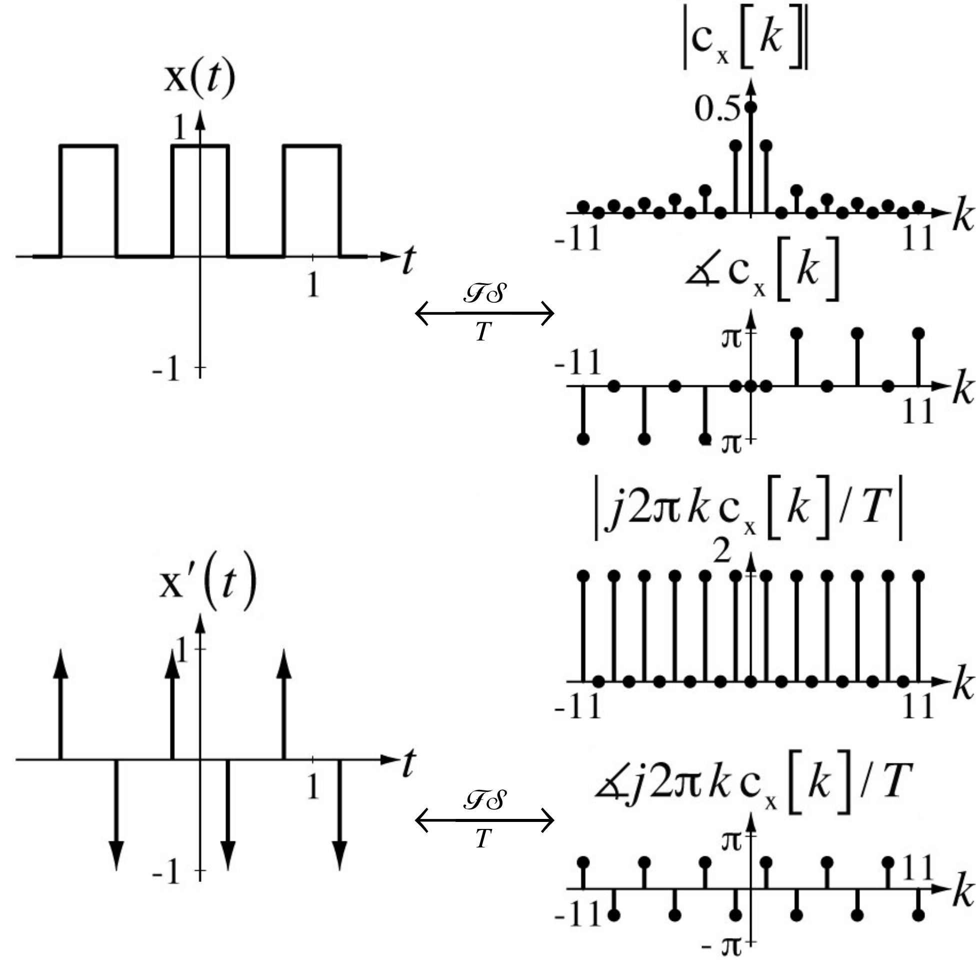
Change of Representation Time



CTFS Properties

Time Differentiation

$$\frac{d}{dt}\left(\mathrm{x}(t)\right) \xleftrightarrow[T]{\mathcal{FS}} j2\pi k\,\mathrm{c_x}[k]/T$$



CTFS Properties

Time Integration

Case 1. $c_x[0] = 0$

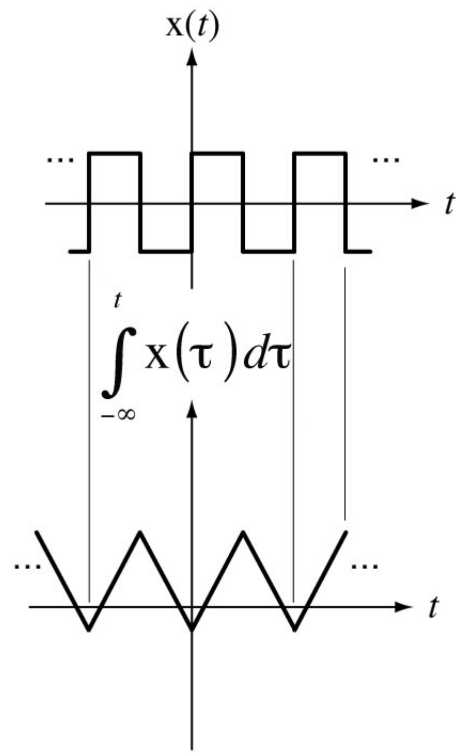
$$\int_{-\infty}^{t} x(\lambda)\, d\lambda \xleftrightarrow[T]{\mathcal{FS}} \frac{c_x[k]}{j2\pi k / T} \quad , k \neq 0$$

Case 2. $c_x[0] \neq 0$

$$\int_{-\infty}^{t} x(\lambda)\, d\lambda \text{ is not periodic}$$

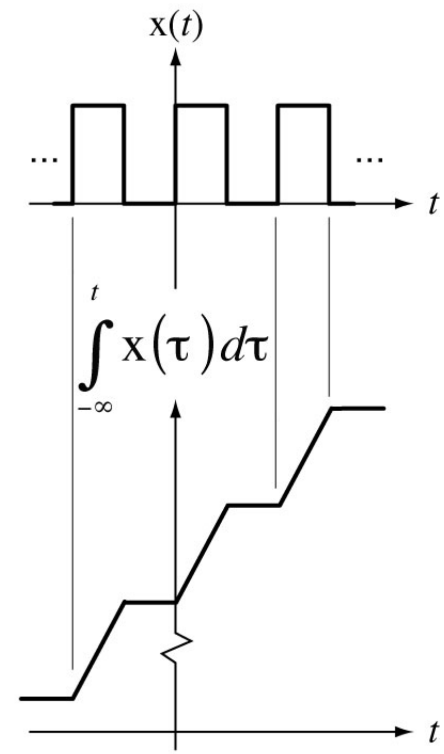
Case 1

$c_x[0] = 0$



Case 2

$c_x[0] \neq 0$



CTFS Properties

Multiplication - Convolution Duality

$$\mathrm{x}(t)\mathrm{y}(t) \xleftrightarrow[T]{\mathcal{FS}} \mathrm{c_x}[k] * \mathrm{c_y}[k]$$

(The harmonic functions $\mathrm{c_x}[k]$ and $\mathrm{c_y}[k]$ must be based on the same representation time T.)

$$\mathrm{x}(t) \circledast \mathrm{y}(t) \xleftrightarrow[T]{\mathcal{FS}} T\,\mathrm{c_x}[k]\mathrm{c_y}[k]$$

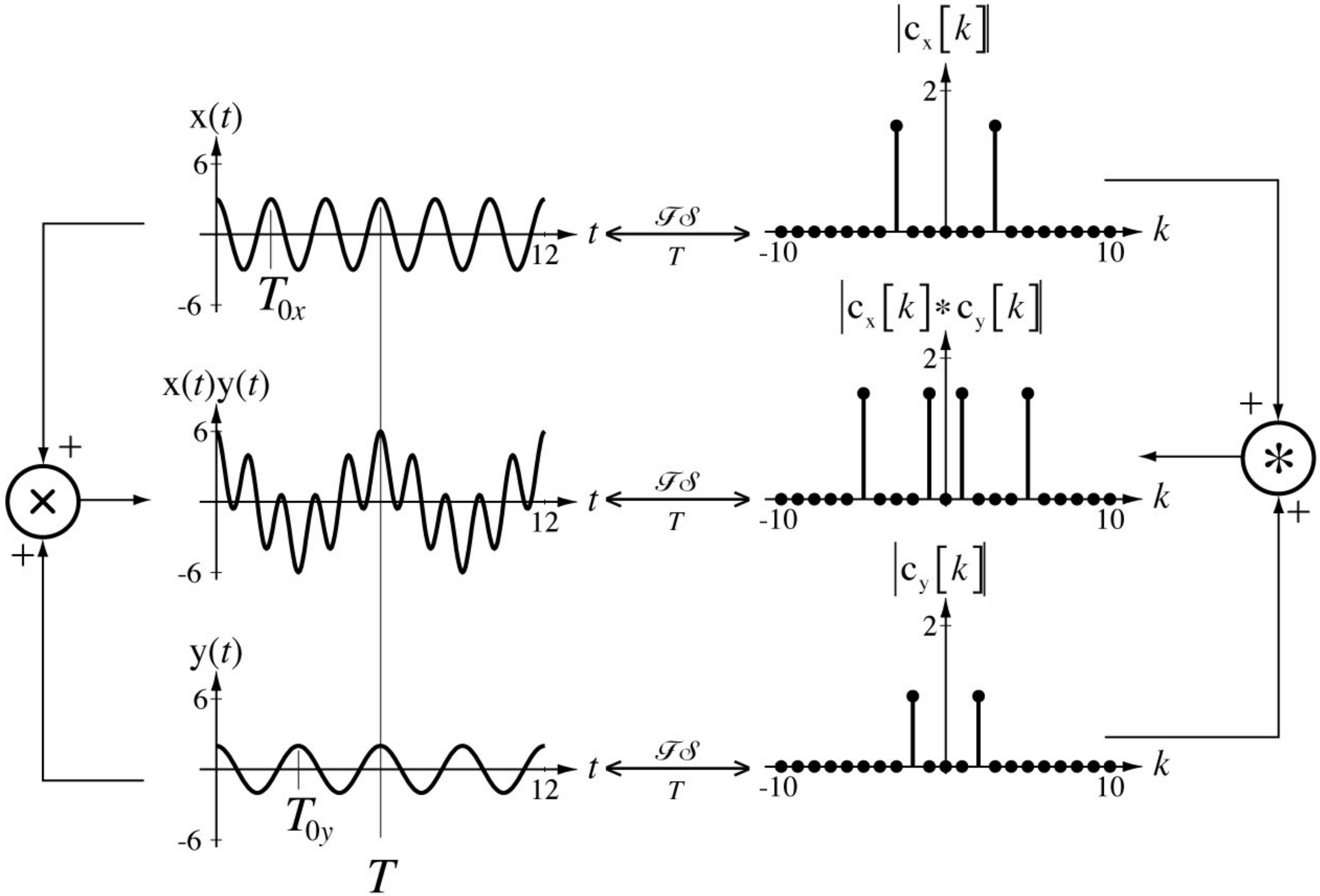
The symbol $\circledast$ indicates **periodic convolution**.

Periodic convolution is defined mathematically by

$$\mathrm{x}(t) \circledast \mathrm{y}(t) = \int_T \mathrm{x}(\tau)\mathrm{y}(t-\tau)\,d\tau$$

$\mathrm{x}(t) \circledast \mathrm{y}(t) = \mathrm{x}_{ap}(t) * \mathrm{y}(t)$ where $\mathrm{x}_{ap}(t)$ is any single period of $\mathrm{x}(t)$

CTFS Properties



CTFS Properties

Conjugation

$$\mathrm{x}^*(t) \xleftrightarrow[T]{\mathcal{FS}} \mathrm{c}_\mathrm{x}^*[-k]$$

Parseval's Theorem

$$\frac{1}{T}\int_T |\mathrm{x}(t)|^2\,dt = \sum_{k=-\infty}^{\infty} |\mathrm{c}_\mathrm{x}[k]|^2$$

The **average power** of a periodic signal is the sum of the average powers in its harmonic components.

Some Common CTFS Pairs

$$1 \xleftrightarrow[T]{\mathcal{FS}} \delta[k] \text{ , } T \text{ arbitrary}$$

$$\delta_{T_0}(t) \xleftrightarrow[mT_0]{\mathcal{FS}} \begin{cases} (1/T_0) & \text{, } k/m \text{ an integer} \\ 0 & \text{, otherwise} \end{cases}$$

$$e^{j2\pi qt/T_0} \xleftrightarrow[mT_0]{\mathcal{FS}} \delta[k-mq]$$

$$\sin(2\pi qt/T_0) \xleftrightarrow[mT_0]{\mathcal{FS}} (j/2)(\delta[k+mq]-\delta[k-mq])$$

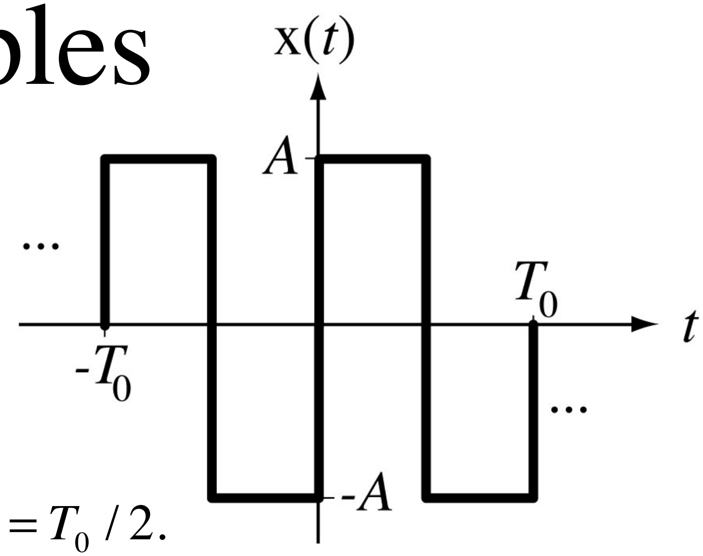
$$\cos(2\pi qt/T_0) \xleftrightarrow[mT_0]{\mathcal{FS}} (1/2)(\delta[k-mq]+\delta[k+mq])$$

$$\text{rect}(t/w) * \delta_{T_0}(t) \xleftrightarrow[mT_0]{\mathcal{FS}} (w/T_0)\text{sinc}(wk/mT_0)\delta_m[k]$$

$$\text{tri}(t/w) * \delta_{T_0}(t) \xleftrightarrow[mT_0]{\mathcal{FS}} (w/T_0)\text{sinc}^2(wk/mT_0)\delta_m[k]$$

$$(m \text{ an integer})$$

CTFS Examples



$$\mathrm{x}(t) = A\left\{2\,\mathrm{rect}\left(2\left(t - T_0/4\right)/T_0\right) * \delta_{T_0}(t) - 1\right\}$$

Find the CTFS harmonic function using $T = T_0$.

$$\mathrm{rect}(t/w) * \delta_{T_0}(t) \xleftrightarrow[mT_0]{\mathcal{FS}} (w/T_0)\,\mathrm{sinc}(wk/mT_0)\,\delta_m[k]$$

Since $T = T_0 \Rightarrow m = 1$. If $\mathrm{rect}(t/w) = \mathrm{rect}(2t/T_0)$ then $w = T_0/2$.

$$\mathrm{rect}(2t/T_0) * \delta_{T_0}(t) \xleftrightarrow[T_0]{\mathcal{FS}} (1/2)\,\mathrm{sinc}(k/2)\,\underbrace{\delta_1[k]}_{\substack{=1\text{ for}\\ \text{any integer }k}}$$

$$2\,\mathrm{rect}(2t/T_0) * \delta_{T_0}(t) \xleftrightarrow[T_0]{\mathcal{FS}} \mathrm{sinc}(k/2)$$

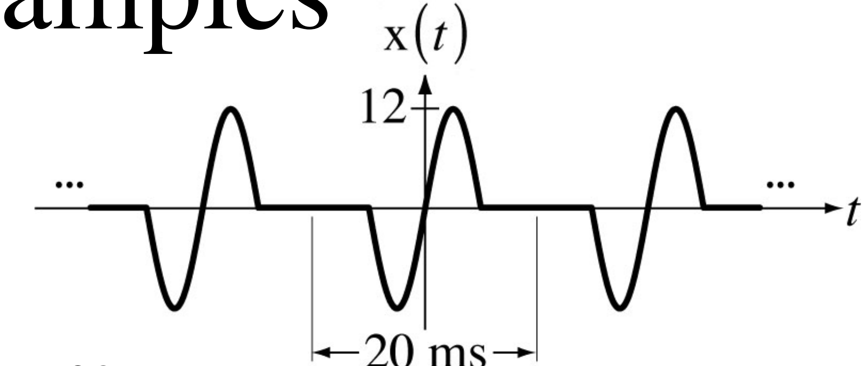
Using the time shifting property, $\mathrm{x}(t - t_0) \xleftrightarrow[T]{\mathcal{FS}} e^{-j2\pi k t_0/T}\,\mathrm{c_x}[k]$

$$2\,\mathrm{rect}\left(2\left(t - T_0/4\right)/T_0\right) * \delta_{T_0}(t) \xleftrightarrow[T_0]{\mathcal{FS}} \mathrm{sinc}(k/2)\,e^{-j\pi k/2}$$

Then, using $1 \xleftrightarrow[T]{\mathcal{FS}} \delta[k]$, T arbitrary , $1 \xleftrightarrow[T_0]{\mathcal{FS}} \delta[k]$ and

$$A\left\{2\,\mathrm{rect}\left(2\left(t - T_0/4\right)/T_0\right) * \delta_{T_0}(t) - 1\right\} \xleftrightarrow[T_0]{\mathcal{FS}} A\left\{\mathrm{sinc}(k/2)\,e^{-j\pi k/2} - \delta[k]\right\} = \mathrm{c_x}[k]$$

CTFS Examples



$$\mathrm{x}(t) = 12\sin(2\pi t/0.01)\left[\mathrm{rect}(t/0.01) * \delta_{0.02}(t)\right]$$

$$\mathrm{x}(t) = 12\sin(200\pi t)\left[\mathrm{rect}(100t) * \delta_{0.02}(t)\right]$$

Find the CTFS harmonic function of $\mathrm{x}(t)$ with $T = 20$ ms.

$$\sin(2\pi qt/T_0) \xleftrightarrow[mT_0]{\mathcal{FS}} (j/2)(\delta[k+mq] - \delta[k-mq])$$

$$\sin(200\pi t) \xleftrightarrow[2\times 0.01]{\mathcal{FS}} (j/2)(\delta[k+2\times 1] - \delta[k-2\times 1])$$

$$\mathrm{rect}(t/w) * \delta_{T_0}(t) \xleftrightarrow[mT_0]{\mathcal{FS}} (w/T_0)\mathrm{sinc}(wk/mT_0)\delta_m[k]$$

$$\mathrm{rect}(100t) * \delta_{0.02}(t) \xleftrightarrow[1\times 0.02]{\mathcal{FS}} (1/2)\mathrm{sinc}(k/2)$$

Using $\mathrm{x}(t)\mathrm{y}(t) \xleftrightarrow[T]{\mathcal{FS}} \mathrm{c_x}[k] * \mathrm{c_y}[k]$,

$$12\sin(200\pi t)\left[\mathrm{rect}(100t) * \delta_{0.02}(t)\right] \xleftrightarrow[0.02]{\mathcal{FS}} 12(j/2)(\delta[k+2] - \delta[k-2]) * (1/2)\mathrm{sinc}(k/2)$$

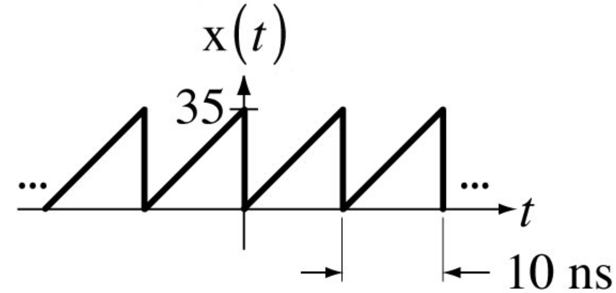
$$12\sin(200\pi t)\left[\mathrm{rect}(100t) * \delta_{0.02}(t)\right] \xleftrightarrow[0.02]{\mathcal{FS}} j3\left(\mathrm{sinc}((k+2)/2) - \mathrm{sinc}((k-2)/2)\right)$$

CTFS Examples

Find the CTFS harmonic function of $\mathrm{x}(t)$ with $T=10^{-8}$.

$$\mathrm{c_x}[k]=(1/T)\int_T \mathrm{x}(t)e^{-j2\pi kt/T}\,dt \Rightarrow \mathrm{c_x}[0]=10^8\int_0^{10^{-8}}\left(35\times10^8 t\right)dt=35/2$$

$$\mathrm{c_x}[k]=10^8\int_0^{10^{-8}}\left(35\times10^8 t\right)e^{-j2\pi\times10^8 kt}\,dt=35\times10^{16}\int_0^{10^{-8}} te^{-j2\pi\times10^8 kt}\,dt$$



$$\mathrm{c_x}[k]=35\times10^{16}\left\{\left[t\frac{e^{-j2\pi\times10^8 kt}}{-j2\pi\times10^8 k}\right]_0^{10^{-8}}-\int_0^{10^{-8}}\left(\frac{e^{-j2\pi\times10^8 kt}}{-j2\pi\times10^8 k}\right)dt\right\}$$

$$\mathrm{c_x}[k]=35\times10^{16}\left\{-\frac{e^{-j2\pi k}\times10^{-8}}{j2\pi\times10^8 k}-\left[\frac{e^{-j2\pi\times10^8 kt}}{\left(j2\pi\times10^8 k\right)^2}\right]_0^{10^{-8}}\right\}$$

$$\mathrm{c_x}[k]=35\times10^{16}\left[-\frac{10^{-16}e^{-j2\pi k}}{j2\pi k}+\frac{1-e^{-j2\pi k}}{\left(j2\pi k\right)^2\times10^{16}}\right]=35\,\frac{1-e^{-j2\pi k}-j2\pi ke^{-j2\pi k}}{\left(j2\pi k\right)^2}$$

$$\mathrm{c_x}[k]=35\begin{cases}1/2 & ,\ k=0\\ \dfrac{j}{2\pi k} & ,\ k\neq0\end{cases}$$

LTI Systems with Periodic Excitation

The differential equation describing an RC lowpass filter is

$$RC\,\mathrm{v}'_{out}(t) + \mathrm{v}_{out}(t) = \mathrm{v}_{in}(t)$$

If the excitation $\mathrm{v}_{in}(t)$ is periodic it can be expressed as a CTFS,

$$\mathrm{v}_{in}(t) = \sum_{k=-\infty}^{\infty} \mathrm{c}_{in}[k] e^{j2\pi kt/T}$$

The equation for the kth harmonic alone is

$$RC\,\mathrm{v}'_{out,k}(t) + \mathrm{v}_{out,k}(t) = \mathrm{v}_{in,k}(t) = \mathrm{c}_{in}[k] e^{j2\pi kt/T}$$

LTI Systems with Periodic Excitation

If the excitation is periodic, the response is also, with the same fundamental period. Therefore the response can be expressed as a CTFS also.

$$\mathrm{v}_{out,k}(t) = \mathrm{c}_{out}[k]e^{j2\pi kt/T}$$

Then the equation for the kth harmonic becomes

$$(j2k\pi RC/T)\mathrm{c}_{out}[k]e^{j2\pi kt/T} + \mathrm{c}_{out}[k]e^{j2\pi kt/T} = \mathrm{c}_{in}[k]e^{j2\pi kt/T}$$

Notice that what was once a <u>differential</u> equation is now an <u>algebraic</u> equation.

LTI Systems with Periodic Excitation

Solving the kth-harmonic equation,

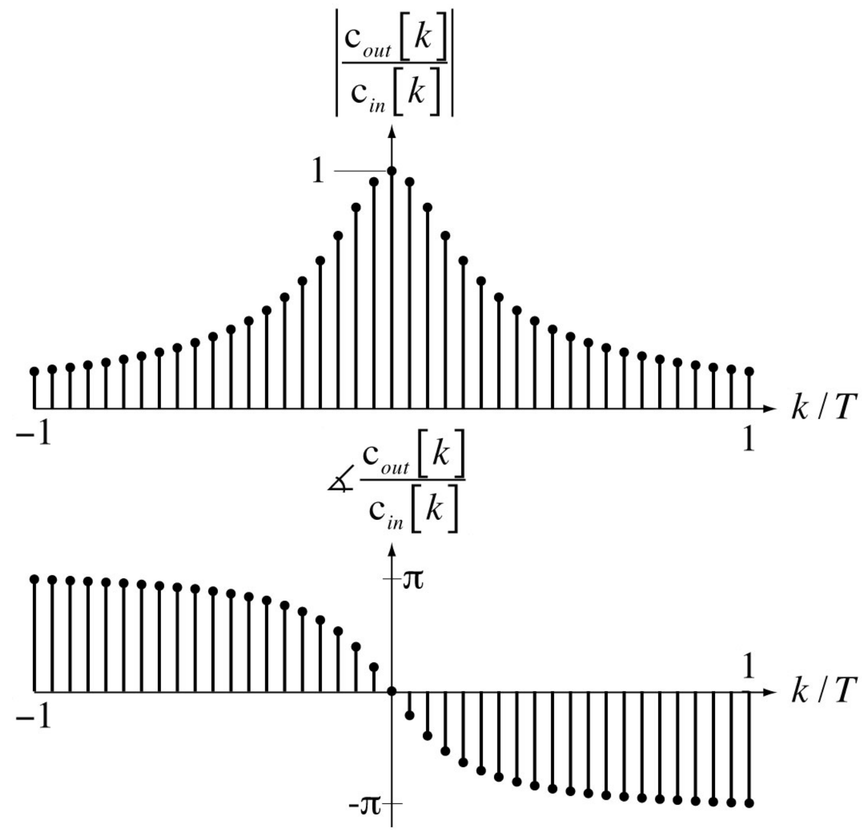
$$c_{out}[k] = \frac{c_{in}[k]}{j2k\pi RC/T + 1}$$

Then the response can be written as

$$v_{out}(t) = \sum_{k=-\infty}^{\infty} c_{out}[k] e^{j2\pi kt/T} = \sum_{k=-\infty}^{\infty} \frac{c_{in}[k]}{j2k\pi RC/T + 1} e^{j2\pi kt/T}$$

LTI Systems with Periodic Excitation

The ratio $\dfrac{c_{out}[k]}{c_{in}[k]}$ is the **harmonic response** of the system.



$R = 1\Omega$, $C = 1\text{F}$, $T = 20\text{s}$

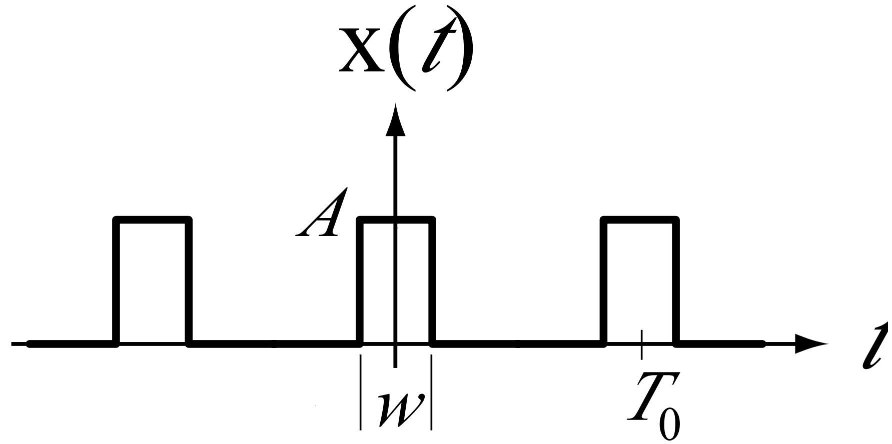
The Continuous-Time Fourier Transform

Extending the CTFS

- The CTFS is a good analysis tool for systems with periodic excitation but the CTFS cannot represent an **aperiodic** signal for all time
- The continuous-time **Fourier transform** (CTFT) <u>can</u> represent an aperiodic (and also a periodic) signal for all time

CTFS-to-CTFT Transition

Consider a periodic pulse-train signal $\mathrm{x}(t)$ with **duty cycle** w / T_0

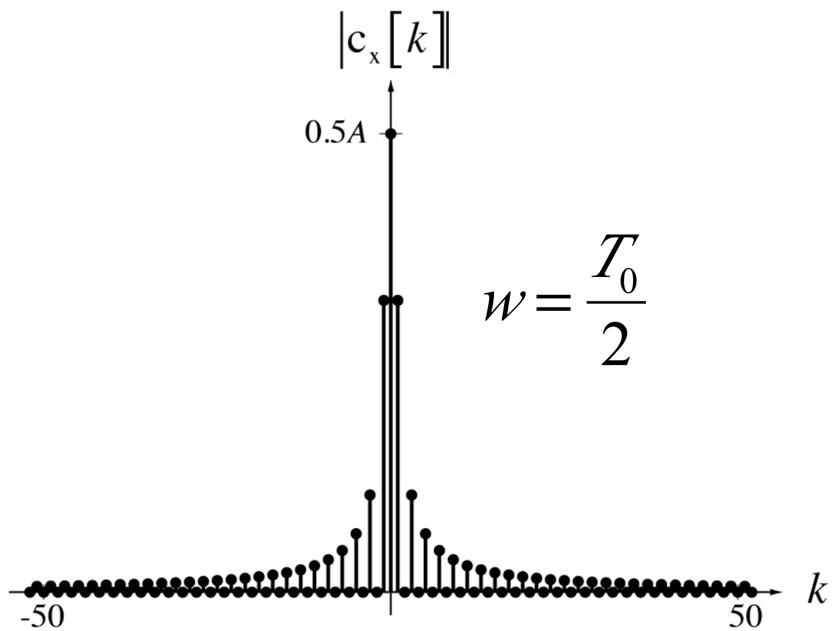


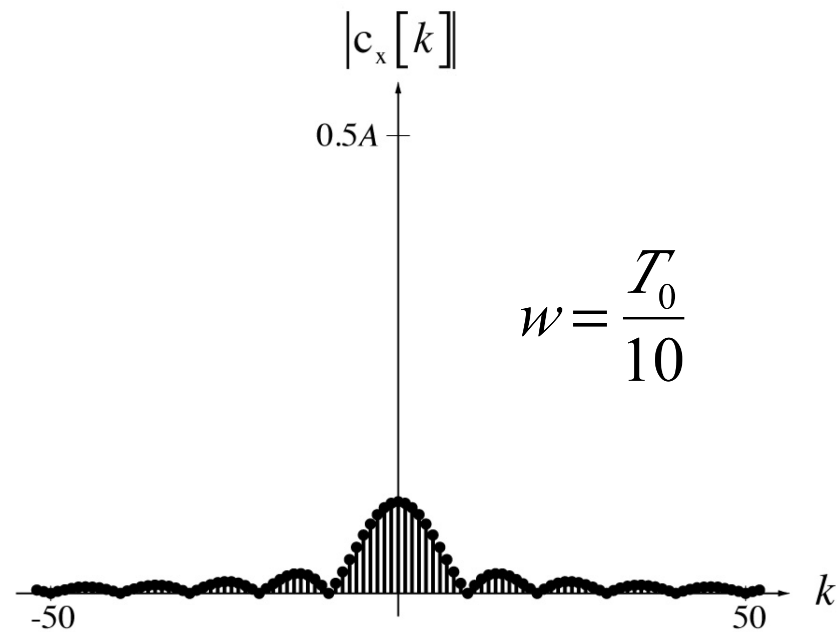
Its CTFS harmonic function is $\mathrm{c_x}[k] = \dfrac{Aw}{T_0}\mathrm{sinc}\left(\dfrac{kw}{T_0}\right)$

As the period T_0 is increased, holding w constant, the duty cycle is decreased. When the period becomes infinite (and the duty cycle becomes zero) $\mathrm{x}(t)$ is no longer periodic.

CTFS-to-CTFT Transition

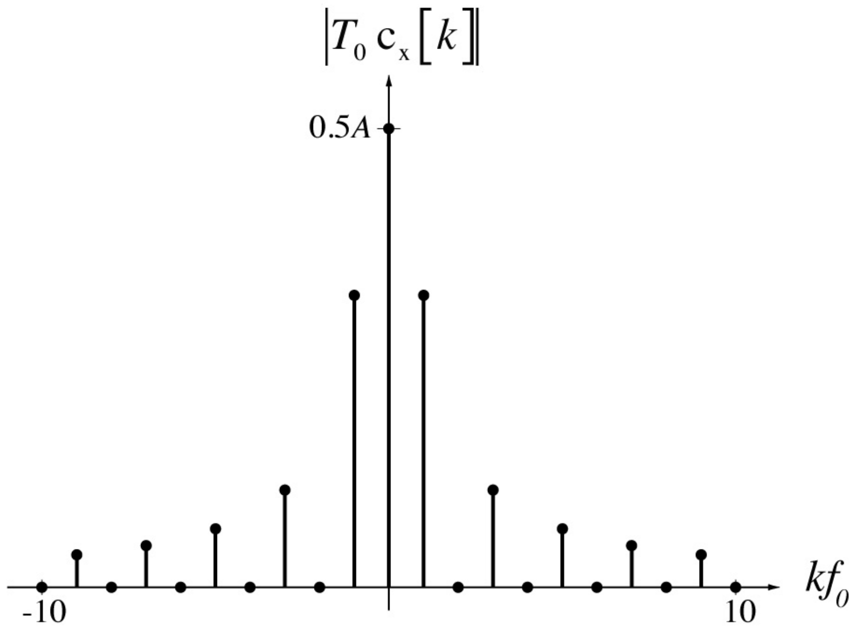
Below are graphs of the magnitude of $c_x[k]$ for 50% and 10% duty cycles. As the period increases the sinc function widens and its magnitude falls. As the period approaches infinity, the CTFS harmonic function approaches an infinitely-wide sinc function with zero amplitude.

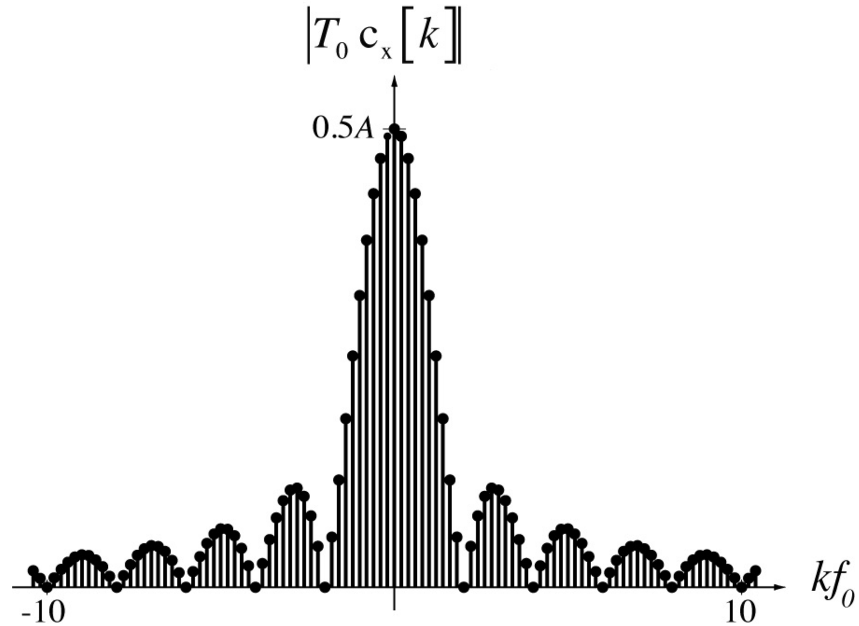




CTFS-to-CTFT Transition

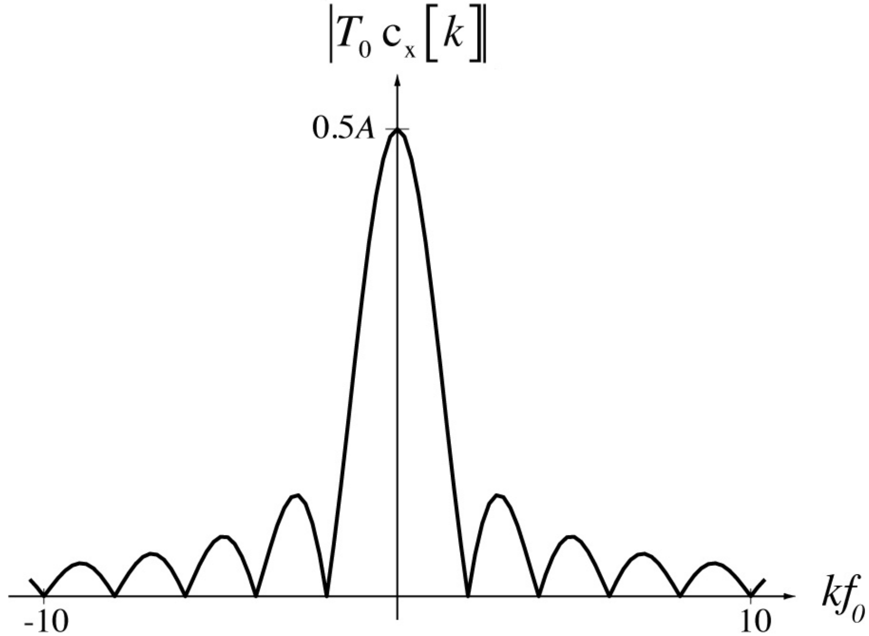
This infinity-and-zero problem can be solved by **normalizing** the CTFS harmonic function. Define a new "modified" CTFS harmonic function $T_0 \, \mathrm{c_x}[k] = Aw\,\mathrm{sinc}(wkf_0)$ and graph it versus kf_0 instead of versus k. $\left(f_0 = 1/T_0\right)$





CTFS-to-CTFT Transition

In the limit as the period approaches infinity, the modified CTFS harmonic function approaches a function of continuous frequency $f\ (kf_0)$. kf_0 is continuous because in the limit as T_0 approaches infinity, f_0 approaches zero and the "gaps" between the harmonics disappear.



CTFS-to-CTFT Transition

In $\mathrm{x}(t) = \sum_{k=-\infty}^{\infty} \mathrm{c_x}[k] e^{j2\pi kt/T}$ let $\Delta f = 1/T$. Then $\mathrm{x}(t) = \sum_{k=-\infty}^{\infty} \mathrm{c_x}[k] e^{j2\pi k\Delta ft}$.

Substituting the integral expression for $\mathrm{c_x}[k]$,

$$\mathrm{x}(t) = \sum_{k=-\infty}^{\infty} \left[\frac{1}{T} \int_{t_0}^{t_0+T} \mathrm{x}(\tau) e^{-j2\pi k\Delta f\tau} \, d\tau \right] e^{j2\pi k\Delta ft}.$$

Let $t_0 = -T/2$. Then $\mathrm{x}(t) = \sum_{k=-\infty}^{\infty} \left[\int_{-T/2}^{T/2} \mathrm{x}(\tau) e^{-j2\pi k\Delta f\tau} \, d\tau \right] e^{j2\pi k\Delta ft} \Delta f$

$$\mathrm{x}(t) = \lim_{T\to\infty} \left\{ \sum_{k=-\infty}^{\infty} \left[\int_{-T/2}^{T/2} \mathrm{x}(\tau) e^{-j2\pi k\Delta f\tau} \, d\tau \right] e^{j2\pi k\Delta ft} \Delta f \right\} = \int_{-\infty}^{\infty} \underbrace{\left[\int_{-\infty}^{\infty} \mathrm{x}(\tau) e^{-j2\pi f\tau} \, d\tau \right]}_{=\mathcal{F}(x(t))} e^{j2\pi ft} \, df$$

Definition of the CTFT

Forward — f form — Inverse

$$\mathrm{X}(f) = \mathcal{F}\left(\mathrm{x}(t)\right) = \int_{-\infty}^{\infty} \mathrm{x}(t) e^{-j2\pi f t}\, dt \qquad \mathrm{x}(t) = \mathcal{F}^{-1}\left(\mathrm{X}(f)\right) = \int_{-\infty}^{\infty} \mathrm{X}(f) e^{+j2\pi f t}\, df$$

Forward — ω form — Inverse

$$\mathrm{X}(j\omega) = \mathcal{F}\left(\mathrm{x}(t)\right) = \int_{-\infty}^{\infty} \mathrm{x}(t) e^{-j\omega t}\, dt \qquad \mathrm{x}(t) = \mathcal{F}^{-1}\left(\mathrm{X}(j\omega)\right) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \mathrm{X}(j\omega) e^{+j\omega t}\, d\omega$$

Commonly-used notation:

$$\mathrm{x}(t) \xleftrightarrow{\mathcal{F}} \mathrm{X}(f) \quad \text{or} \quad \mathrm{x}(t) \xleftrightarrow{\mathcal{F}} \mathrm{X}(j\omega)$$

Definition of the CTFT

Example

Use the definition of the forward CTFT

$$\mathrm{X}(f) = \mathscr{F}(\mathrm{x}(t)) = \int_{-\infty}^{\infty} \mathrm{x}(t) e^{-j2\pi ft}\, dt$$

to find the forward CTFT of $\mathrm{x}(t) = 4e^{-7t}\,\mathrm{u}(t)$.

$$\mathrm{X}(f) = \int_{-\infty}^{\infty} 4e^{-7t}\,\mathrm{u}(t) e^{-j2\pi ft}\, dt = 4\int_{0}^{\infty} e^{-(j2\pi f+7)t}\, dt$$

$$\mathrm{X}(f) = 4\left[\frac{e^{-(7+j2\pi f)t}}{-(j2\pi f + 7)}\right]_0^{\infty} = \frac{4}{j2\pi f + 7}$$

Definition of the CTFT

Example

Use the definition of the inverse CTFT

$$\mathrm{x}(t)=\frac{1}{2\pi}\int_{-\infty}^{\infty}\mathrm{X}(j\omega)e^{j\omega t}\,d\omega$$

to find the inverse CTFT of $\mathrm{X}(j\omega)=5\,\mathrm{rect}\left(\frac{\omega}{200}\right)$.

$$\mathrm{x}(t)=\frac{1}{2\pi}\int_{-\infty}^{\infty}5\,\mathrm{rect}\left(\frac{\omega}{200}\right)e^{j\omega t}\,d\omega=\frac{5}{2\pi}\int_{-100}^{100}e^{j\omega t}\,d\omega$$

$$\mathrm{x}(t)=\frac{5}{2\pi}\int_{-100}^{100}\left[\cos(\omega t)+j\sin(\omega t)\right]d\omega=\frac{5}{\pi}\int_{0}^{100}\cos(\omega t)\,d\omega=\frac{5}{\pi}\left[\frac{\sin(\omega t)}{t}\right]_{0}^{100}$$

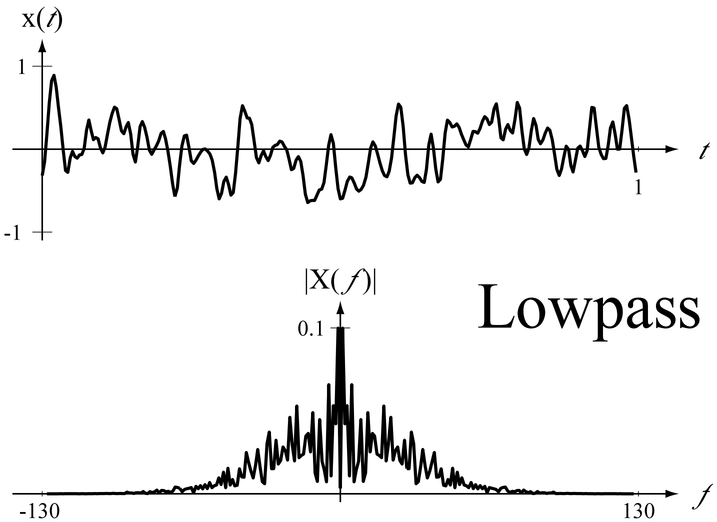
$$\mathrm{x}(t)=\frac{5}{\pi}\frac{\sin(100t)}{t}=\frac{500}{\pi}\frac{\sin((100/\pi)\pi t)}{(100/\pi)\pi t}=\frac{500}{\pi}\mathrm{sinc}\left(\frac{100t}{\pi}\right)$$

Some Remarkable Implications of the Fourier Transform

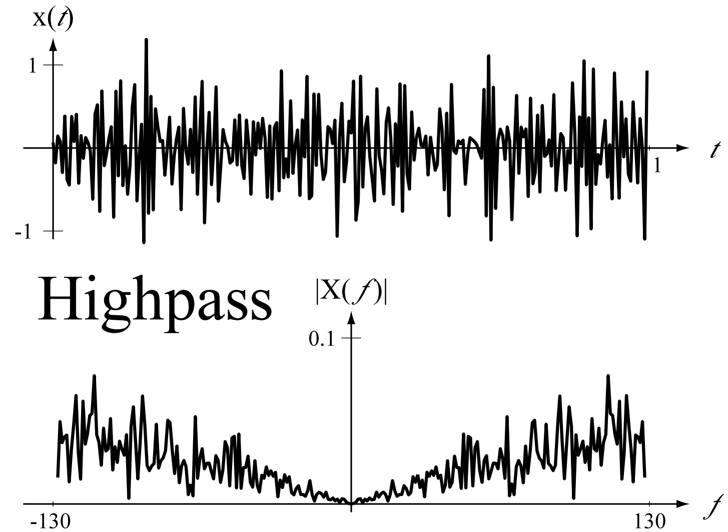
The CTFT can express a finite-amplitude, real-valued, aperiodic signal, which can also, in general, be time-limited, as a summation (an integral) of an infinite continuum of weighted, infinitesimal-amplitude, complex-valued sinusoids, each of which is unlimited in time.

(Time limited means "having non-zero values only for a finite time.")

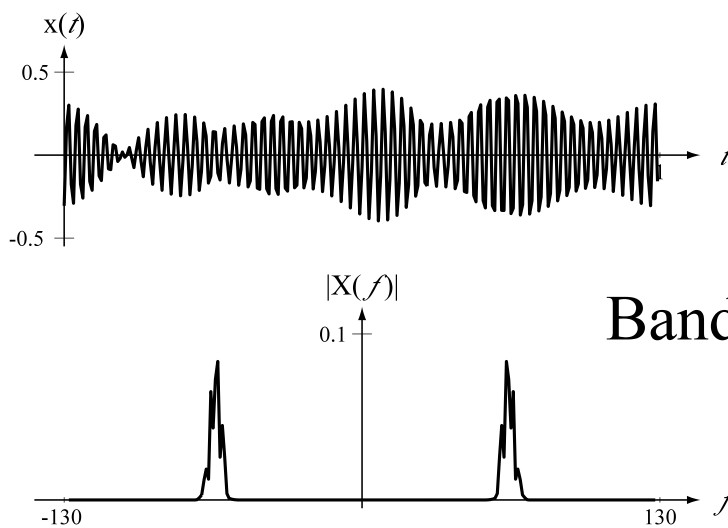
Frequency Content



Lowpass



Highpass



Bandpass

Some CTFT Pairs

$$\delta(t) \xleftrightarrow{\mathcal{F}} 1$$

$$e^{-\alpha t}\,\mathrm{u}(t) \xleftrightarrow{\mathcal{F}} 1/(j\omega+\alpha) \;\;,\;\; \alpha>0 \qquad\qquad -e^{-\alpha t}\,\mathrm{u}(-t) \xleftrightarrow{\mathcal{F}} 1/(j\omega+\alpha) \;\;,\;\; \alpha<0$$

$$te^{-\alpha t}\,\mathrm{u}(t) \xleftrightarrow{\mathcal{F}} 1/(j\omega+\alpha)^2 \;\;,\;\; \alpha>0 \qquad\qquad -te^{-\alpha t}\,\mathrm{u}(-t) \xleftrightarrow{\mathcal{F}} 1/(j\omega+\alpha)^2 \;\;,\;\; \alpha<0$$

$$t^n e^{-\alpha t}\,\mathrm{u}(t) \xleftrightarrow{\mathcal{F}} \frac{n!}{(j\omega+\alpha)^{n+1}} \;\;,\;\; \alpha>0 \qquad\qquad -t^n e^{-\alpha t}\,\mathrm{u}(-t) \xleftrightarrow{\mathcal{F}} \frac{n!}{(j\omega+\alpha)^{n+1}} \;\;,\;\; \alpha<0$$

$$e^{-\alpha t}\sin(\omega_0 t)\mathrm{u}(t) \xleftrightarrow{\mathcal{F}} \frac{\omega_0}{(j\omega+\alpha)^2+\omega_0^2} \;\;,\;\; \alpha>0 \qquad\qquad -e^{-\alpha t}\sin(\omega_0 t)\mathrm{u}(-t) \xleftrightarrow{\mathcal{F}} \frac{\omega_0}{(j\omega+\alpha)^2+\omega_0^2} \;\;,\;\; \alpha<0$$

$$e^{-\alpha t}\cos(\omega_0 t)\mathrm{u}(t) \xleftrightarrow{\mathcal{F}} \frac{j\omega+\alpha}{(j\omega+\alpha)^2+\omega_0^2} \;\;,\;\; \alpha>0 \qquad\qquad -e^{-\alpha t}\cos(\omega_0 t)\mathrm{u}(-t) \xleftrightarrow{\mathcal{F}} \frac{j\omega+\alpha}{(j\omega+\alpha)^2+\omega_0^2} \;\;,\;\; \alpha<0$$

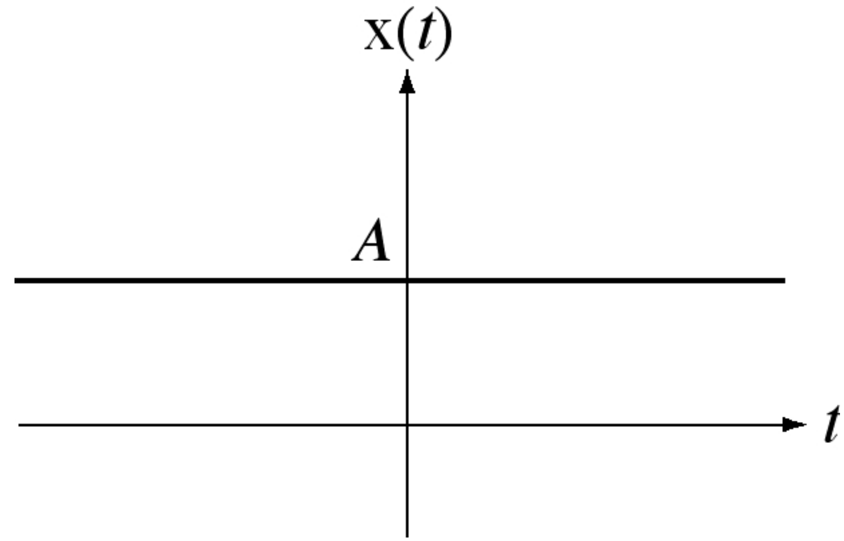
$$e^{-\alpha|t|} \xleftrightarrow{\mathcal{F}} \frac{2\alpha}{\omega^2+\alpha^2} \;\;,\;\; \alpha>0$$

Convergence and the Generalized Fourier Transform

Let $x(t) = A$. Then from the definition of the CTFT,

$$X(f) = \int_{-\infty}^{\infty} A e^{-j2\pi f t}\, dt = A \int_{-\infty}^{\infty} e^{-j2\pi f t}\, dt$$

This integral does not converge so, strictly speaking, the CTFT does not exist.



Convergence and the Generalized Fourier Transform

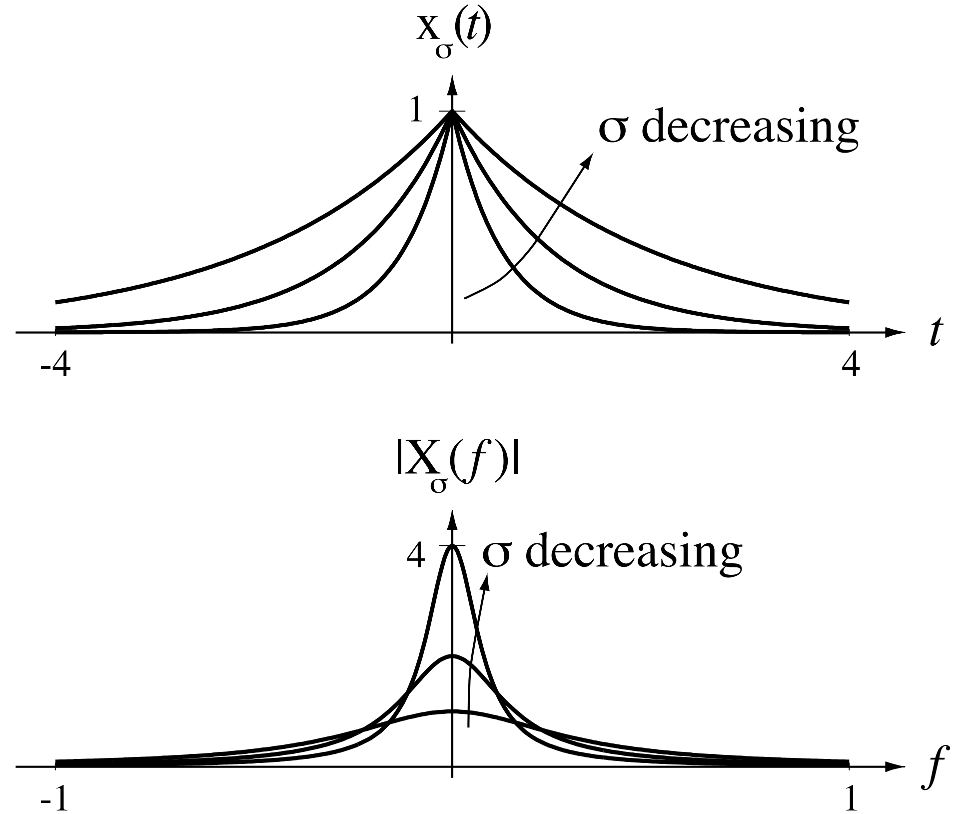
But consider a similar function,

$$x_\sigma(t) = Ae^{-\sigma|t|} \quad , \quad \sigma > 0$$

Its CTFT integral

$$X_\sigma(f) = \int_{-\infty}^{\infty} Ae^{-\sigma|t|} e^{-j2\pi ft}\, dt$$

does converge.



Convergence and the Generalized Fourier Transform

Carrying out the integral, $X_\sigma(f) = A\dfrac{2\sigma}{\sigma^2 + (2\pi f)^2}$.

Now let σ approach zero.

If $f \neq 0$ then $\lim_{\sigma \to 0} A\dfrac{2\sigma}{\sigma^2 + (2\pi f)^2} = 0$. The area under this

function is $A\int_{-\infty}^{\infty} \dfrac{2\sigma}{\sigma^2 + (2\pi f)^2} df$ which is A, independent of

the value of σ. So, in the limit as σ approaches zero, the CTFT has an area of A and is zero unless $f = 0$. This exactly defines an **impulse** of strength A. Therefore $A \xleftrightarrow{\mathcal{F}} A\delta(f)$.

Convergence and the Generalized Fourier Transform

By a similar process it can be shown that

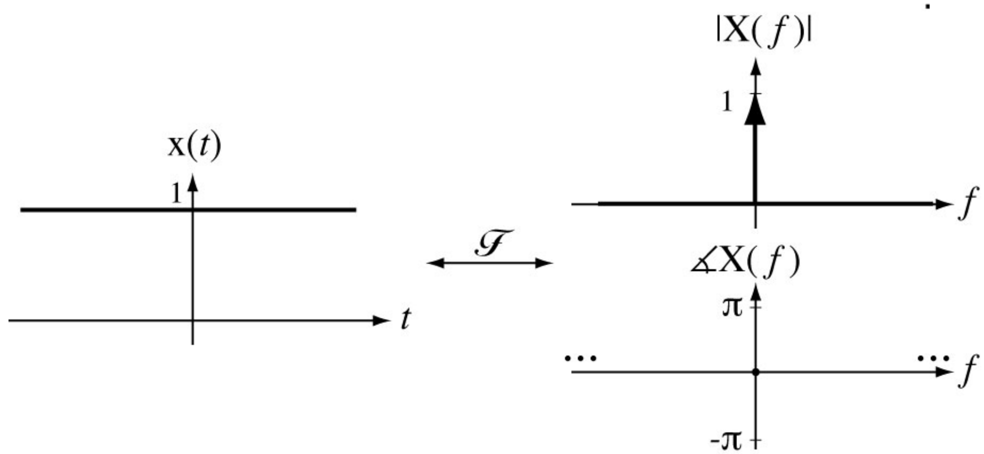
$$\cos(2\pi f_0 t) \xleftrightarrow{\mathcal{F}} \frac{1}{2}\left[\delta(f - f_0) + \delta(f + f_0)\right]$$

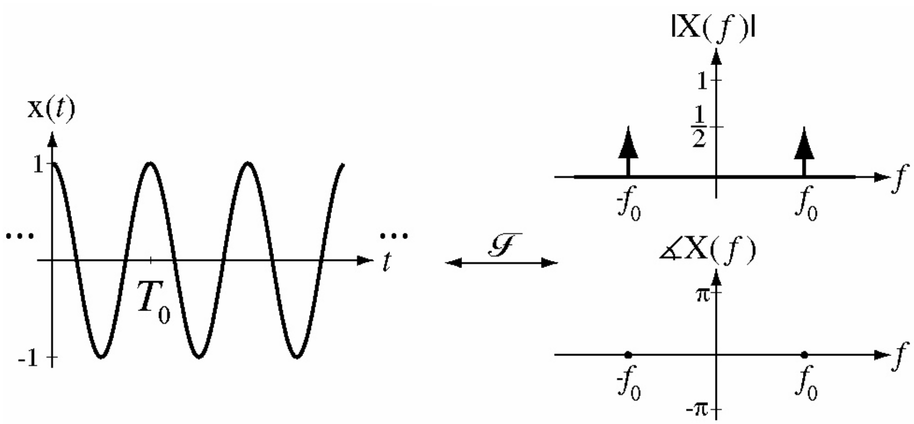
and

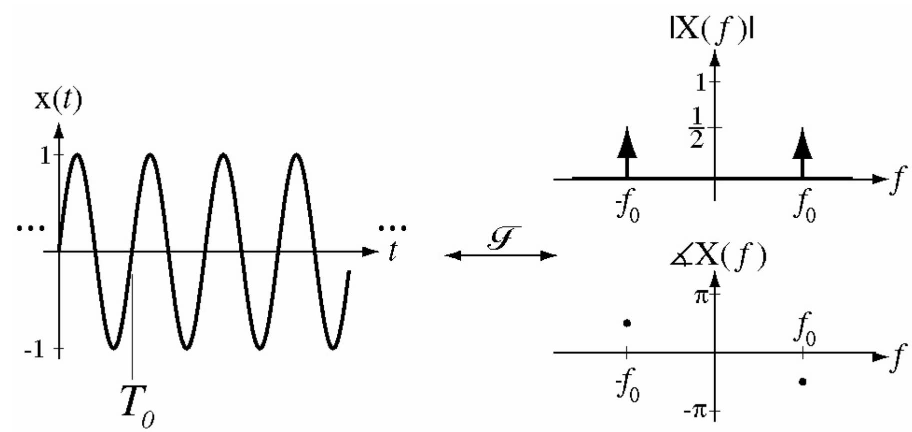
$$\sin(2\pi f_0 t) \xleftrightarrow{\mathcal{F}} \frac{j}{2}\left[\delta(f + f_0) - \delta(f - f_0)\right]$$

These CTFT's that involve impulses are called **generalized Fourier transforms** (probably because the impulse is sometimes called a generalized function).

Convergence and the Generalized Fourier Transform







More CTFT Pairs

The generalization of the CTFT allows us to extend the table of CTFT pairs to some very useful functions.

$$\delta(t) \xleftrightarrow{\mathcal{F}} 1 \qquad 1 \xleftrightarrow{\mathcal{F}} \delta(f)$$

$$\operatorname{sgn}(t) \xleftrightarrow{\mathcal{F}} 1/j\pi f \qquad \mathrm{u}(t) \xleftrightarrow{\mathcal{F}} (1/2)\delta(f) + 1/j2\pi f$$

$$\operatorname{rect}(t) \xleftrightarrow{\mathcal{F}} \operatorname{sinc}(f) \qquad \operatorname{sinc}(t) \xleftrightarrow{\mathcal{F}} \operatorname{rect}(f)$$

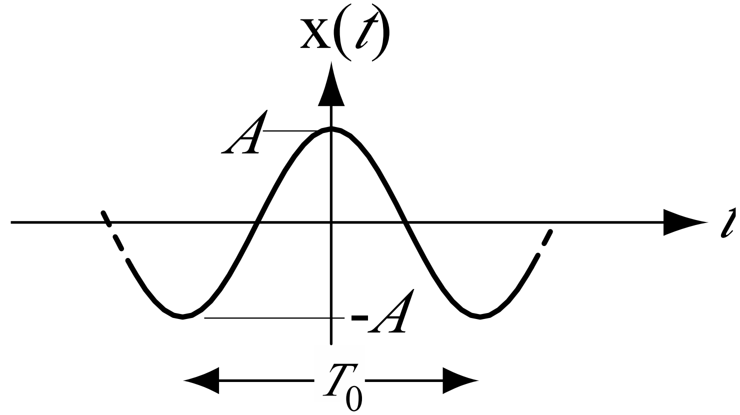
$$\operatorname{tri}(t) \xleftrightarrow{\mathcal{F}} \operatorname{sinc}^2(f) \qquad \operatorname{sinc}^2(t) \xleftrightarrow{\mathcal{F}} \operatorname{tri}(f)$$

$$\delta_{T_0}(t) \xleftrightarrow{\mathcal{F}} f_0 \delta_{f_0}(f)\ ,\ f_0 = 1/T_0 \qquad T_0 \delta_{T_0}(t) \xleftrightarrow{\mathcal{F}} \delta_{f_0}(f)\ ,\ T_0 = 1/f_0$$

$$\cos(2\pi f_0 t) \xleftrightarrow{\mathcal{F}} (1/2)\left[\delta(f - f_0) + \delta(f + f_0)\right] \qquad \sin(2\pi f_0 t) \xleftrightarrow{\mathcal{F}} (j/2)\left[\delta(f + f_0) - \delta(f - f_0)\right]$$

Negative Frequency

This signal is obviously a sinusoid. How is it described mathematically?

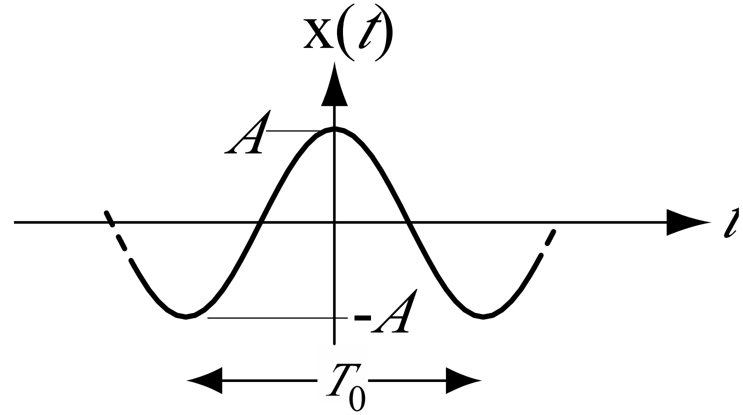


It could be described by $\mathrm{x}(t) = A\cos(2\pi t / T_0) = A\cos(2\pi f_0 t)$

But it could also be described by $\mathrm{x}(t) = A\cos(2\pi(-f_0)t)$

Negative Frequency

x(t) could also be described by



$$x(t) = A\frac{e^{j2\pi f_0 t} + e^{-j2\pi f_0 t}}{2}$$

or

$$x(t) = A_1 \cos(2\pi f_0 t) + A_2 \cos(2\pi(-f_0)t) \ , \ A_1 + A_2 = A$$

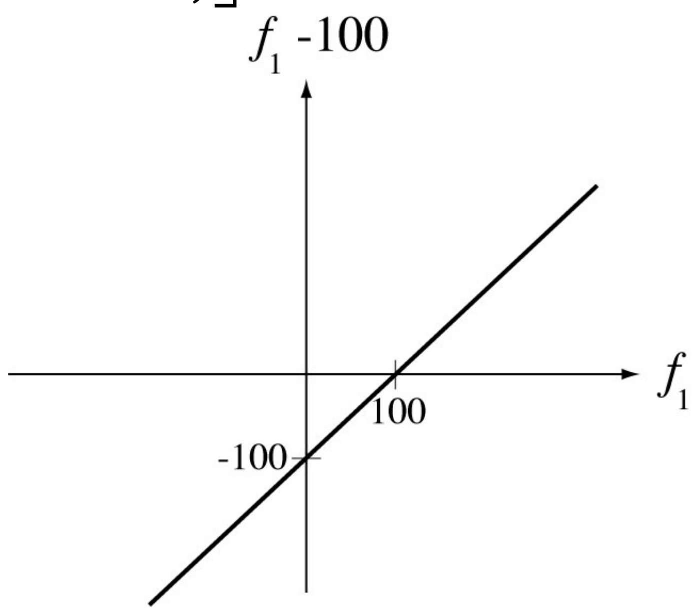
and probably in a few other different-looking ways. So who is to say whether the frequency is positive or negative? For the purposes of signal analysis, it does not matter.

Negative Frequency

Consider an experiment in which we multiply two sinusoidal signals $x_1(t) = \cos(2\pi f_1 t)$ and $x_2(t) = \cos(200\pi t)$ to form $x(t) = x_1(t) x_2(t)$. $x(t)$ can be expressed using a trigonometric identity as

$$x(t) = (1/2)\left[\cos\left(2\pi\left(f_1 - 100\right)t\right) + \cos\left(2\pi\left(f_1 + 100\right)t\right)\right]$$

Now imagine that we continuously change f_1 from a frequency above100 to a frequency below 100. $f_1 - 100$ becomes negative.



CTFT Properties

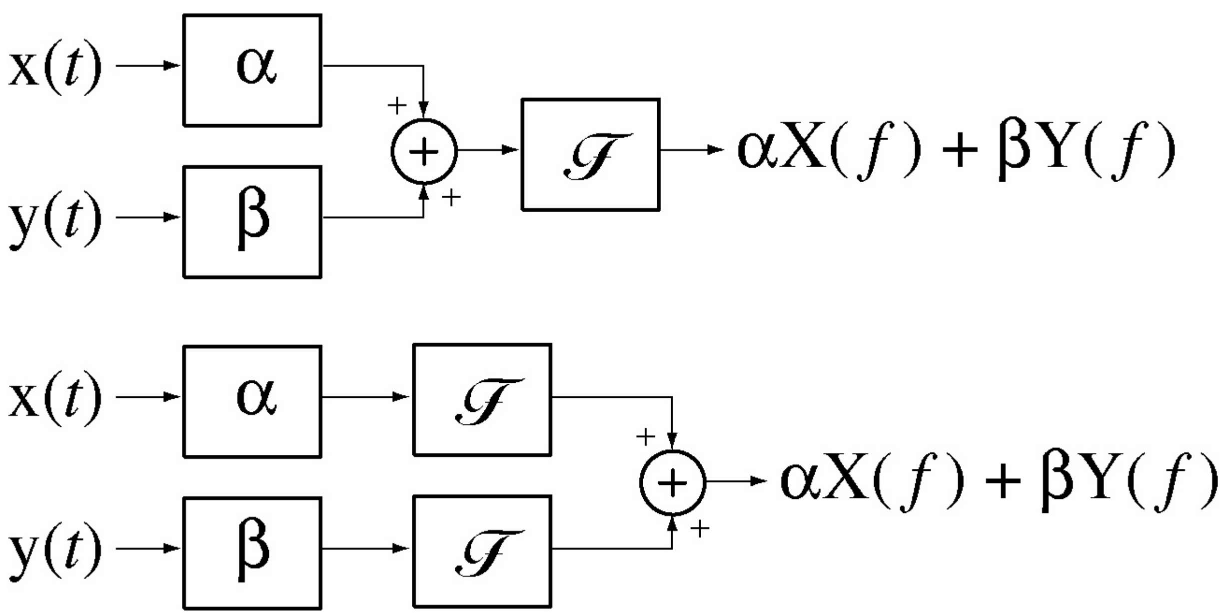
If $\mathcal{F}\left(\mathrm{x}(t)\right) = \mathrm{X}(f)$ or $\mathrm{X}(j\omega)$ and $\mathcal{F}\left(\mathrm{y}(t)\right) = \mathrm{Y}(f)$ or $\mathrm{Y}(j\omega)$

then the following properties can be proven.

Linearity

$$\alpha\,\mathrm{x}(t) + \beta\,\mathrm{y}(t) \xleftrightarrow{\mathcal{F}} \alpha\,\mathrm{X}(f) + \beta\,\mathrm{Y}(f)$$

$$\alpha\,\mathrm{x}(t) + \beta\,\mathrm{y}(t) \xleftrightarrow{\mathcal{F}} \alpha\,\mathrm{X}(j\omega) + \beta\,\mathrm{Y}(j\omega)$$

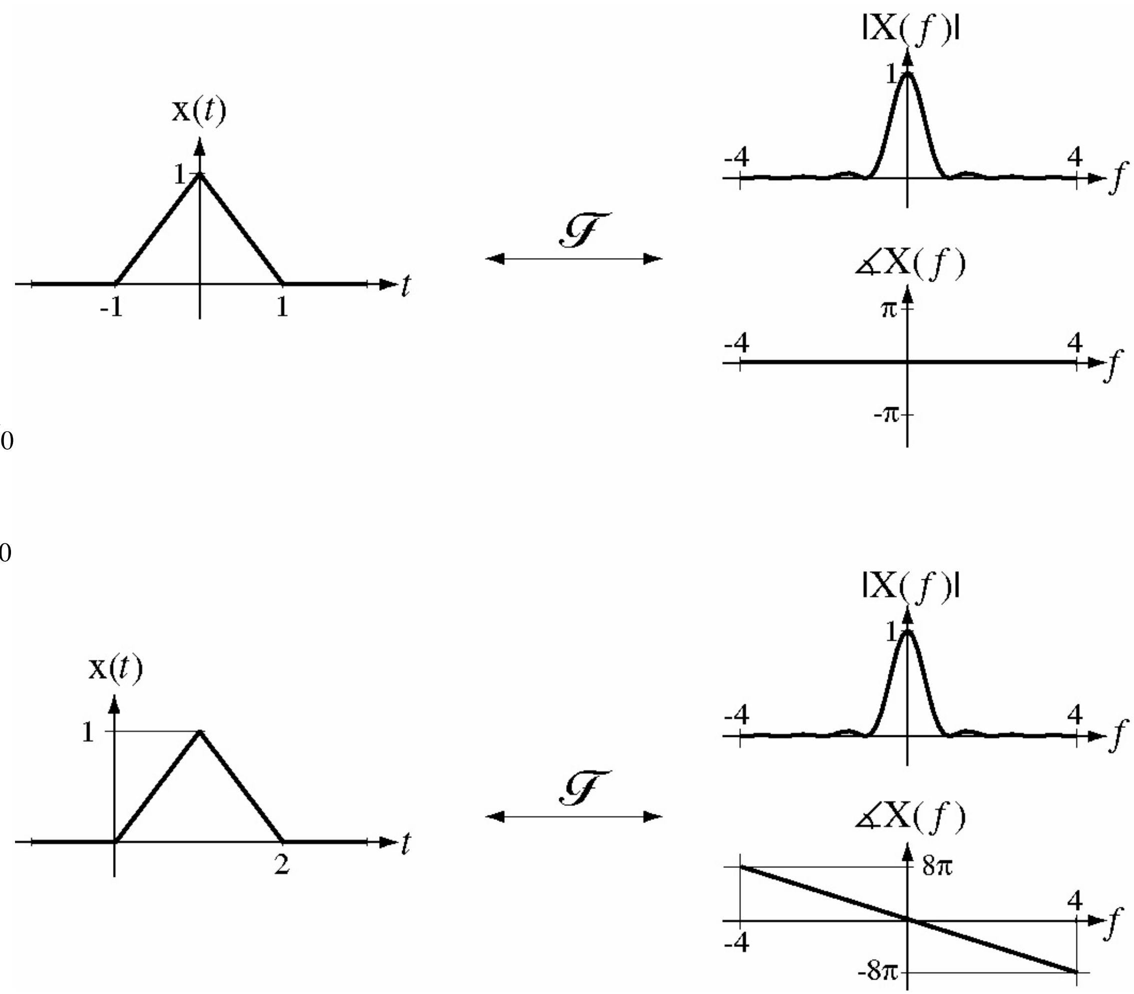


CTFT Properties

Time Shifting

$$x(t - t_0) \xleftrightarrow{\mathcal{F}} X(f)e^{-j2\pi f t_0}$$

$$x(t - t_0) \xleftrightarrow{\mathcal{F}} X(j\omega)e^{-j\omega t_0}$$

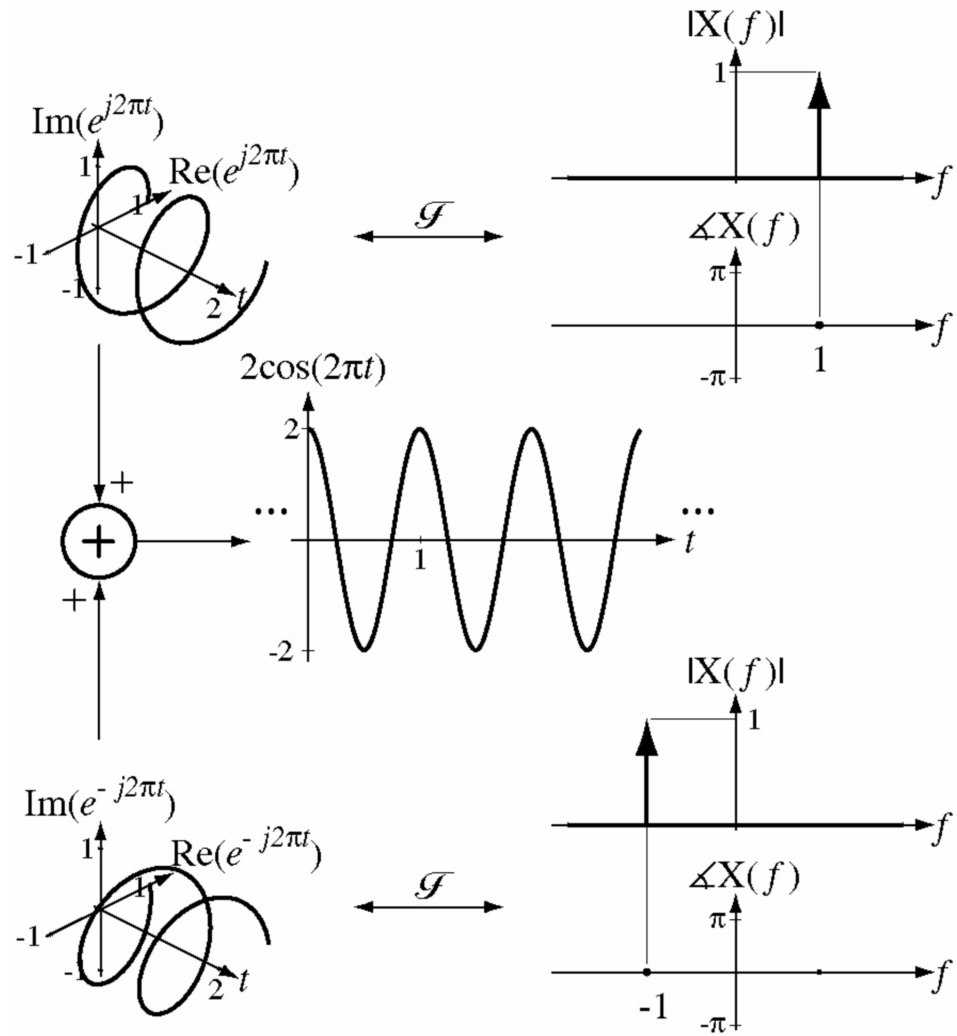


CTFT Properties

Frequency Shifting

$$\mathrm{x}(t)e^{+j2\pi f_0 t} \xleftrightarrow{\mathcal{F}} \mathrm{X}(f - f_0)$$

$$\mathrm{x}(t)e^{+j\omega_0 t} \xleftrightarrow{\mathcal{F}} \mathrm{X}(\omega - \omega_0)$$



CTFT Properties

Time Scaling

$$x(at) \xleftrightarrow{\mathcal{F}} \frac{1}{|a|} X\left(\frac{f}{a}\right)$$

$$x(at) \xleftrightarrow{\mathcal{F}} \frac{1}{|a|} X\left(j\frac{\omega}{a}\right)$$

Frequency Scaling

$$\frac{1}{|a|} x\left(\frac{t}{a}\right) \xleftrightarrow{\mathcal{F}} X(af)$$

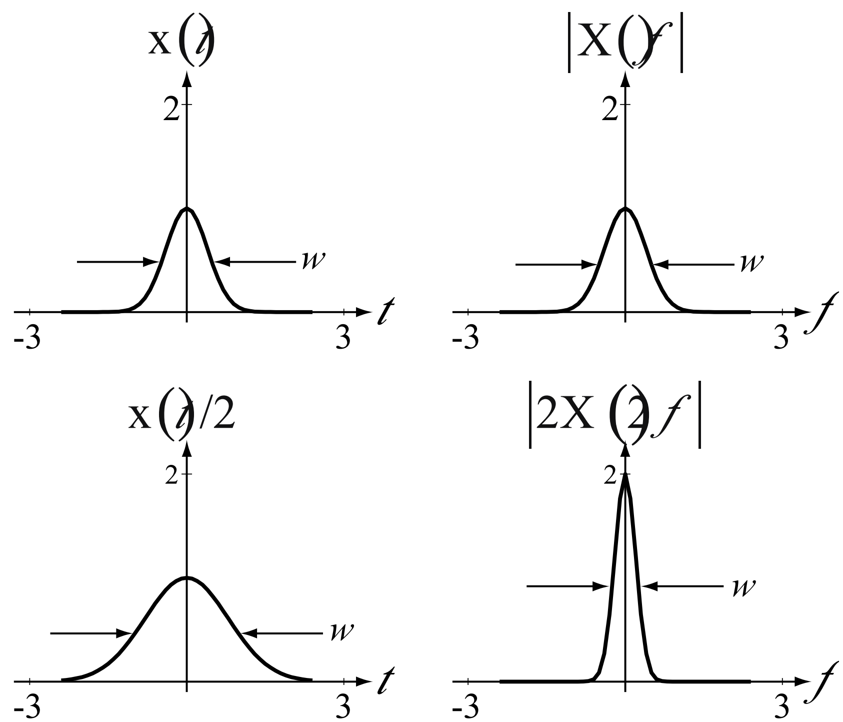
$$\frac{1}{|a|} x\left(\frac{t}{a}\right) \xleftrightarrow{\mathcal{F}} X(ja\omega)$$

The "Uncertainty" Principle

The time and frequency scaling properties indicate that if a signal is expanded in one domain it is compressed in the other domain. This is called the "uncertainty principle" of Fourier analysis.

$$e^{-\pi t^2} \xleftrightarrow{\mathcal{F}} e^{-\pi f^2}$$

$$e^{-\pi (t/2)^2} \xleftrightarrow{\mathcal{F}} 2e^{-\pi (2f)^2}$$



CTFT Properties

Transform of a Conjugate

$$\mathrm{x}^*(t) \xleftrightarrow{\mathcal{F}} \mathrm{X}^*(-f)$$

$$\mathrm{x}^*(t) \xleftrightarrow{\mathcal{F}} \mathrm{X}^*(-j\omega)$$

Multiplication Convolution Duality

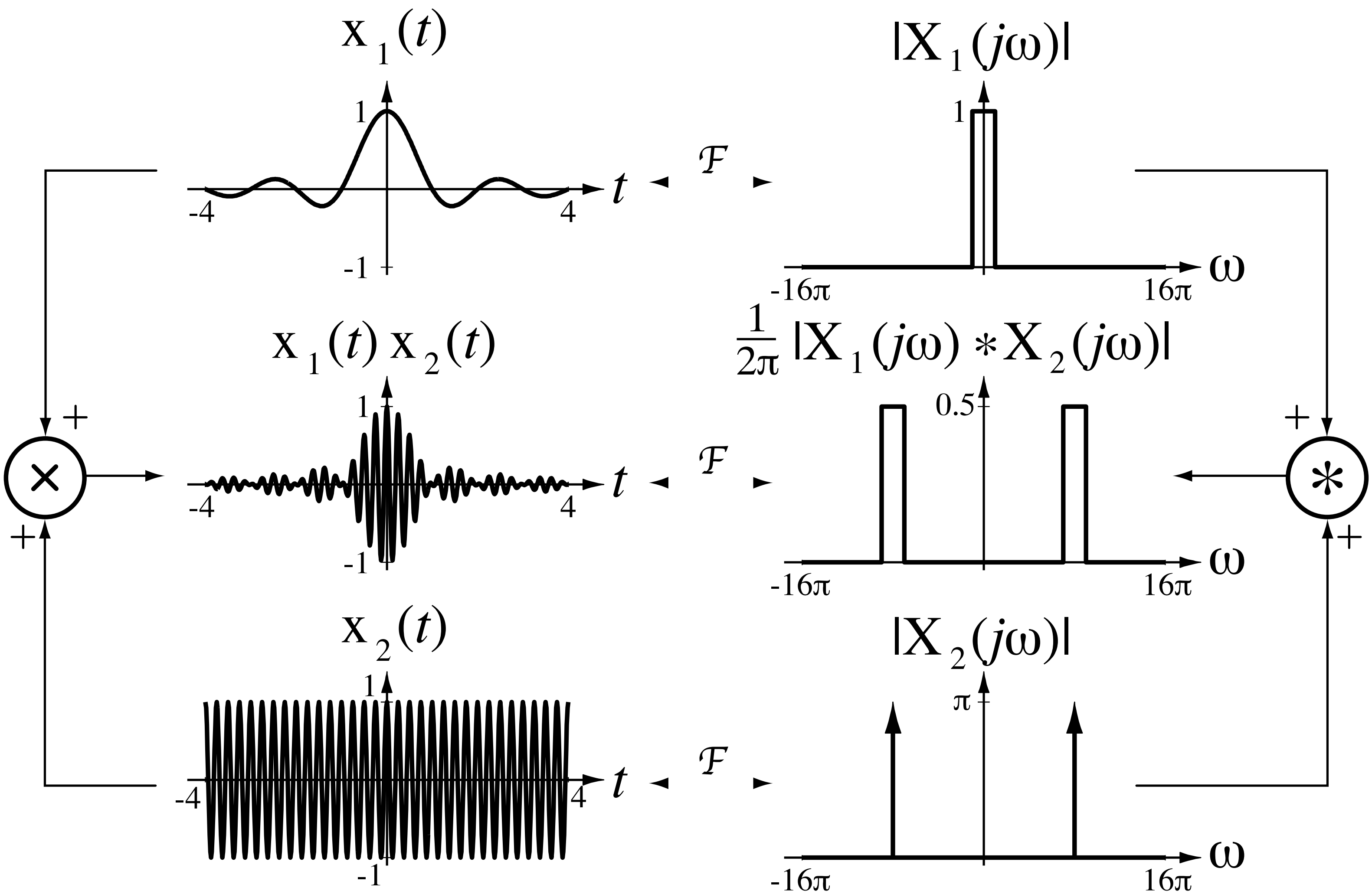
$$\mathrm{x}(t) * \mathrm{y}(t) \xleftrightarrow{\mathcal{F}} \mathrm{X}(f)\mathrm{Y}(f)$$

$$\mathrm{x}(t) * \mathrm{y}(t) \xleftrightarrow{\mathcal{F}} \mathrm{X}(j\omega)\mathrm{Y}(j\omega)$$

$$\mathrm{x}(t)\mathrm{y}(t) \xleftrightarrow{\mathcal{F}} \mathrm{X}(f) * \mathrm{Y}(f)$$

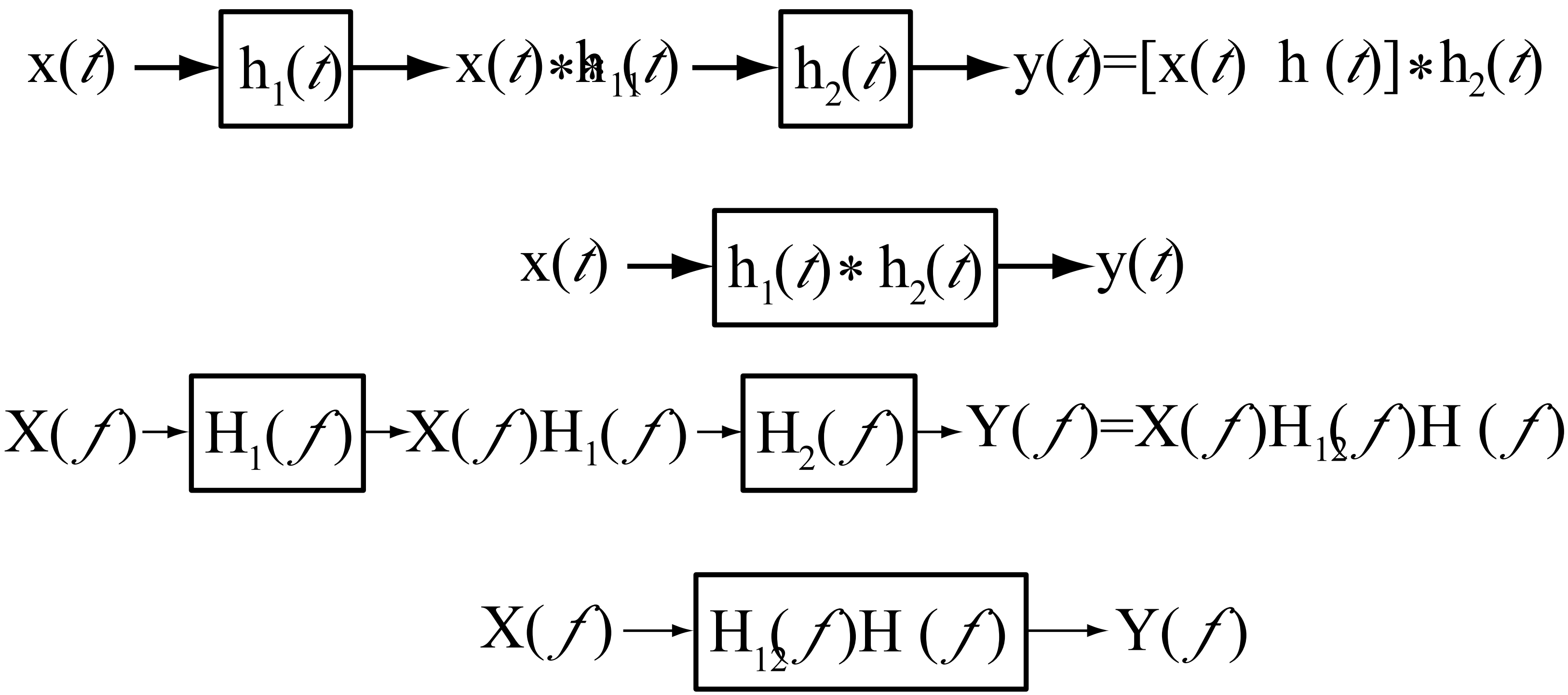
$$\mathrm{x}(t)\mathrm{y}(t) \xleftrightarrow{\mathcal{F}} (1/2\pi)\mathrm{X}(j\omega) * \mathrm{Y}(j\omega)$$

CTFT Properties



CTFT Properties

In the frequency domain, the **cascade connection** multiplies the frequency responses instead of convolving the impulse responses.



CTFT Properties

Time Differentiation

$$\frac{d}{dt}(\mathrm{x}(t)) \xleftrightarrow{\mathcal{F}} j2\pi f\,\mathrm{X}(f)$$

$$\frac{d}{dt}(\mathrm{x}(t)) \xleftrightarrow{\mathcal{F}} j\omega\,\mathrm{X}(j\omega)$$

Modulation

$$\mathrm{x}(t)\cos(2\pi f_0 t) \xleftrightarrow{\mathcal{F}} \frac{1}{2}\left[\mathrm{X}(f-f_0)+\mathrm{X}(f+f_0)\right]$$

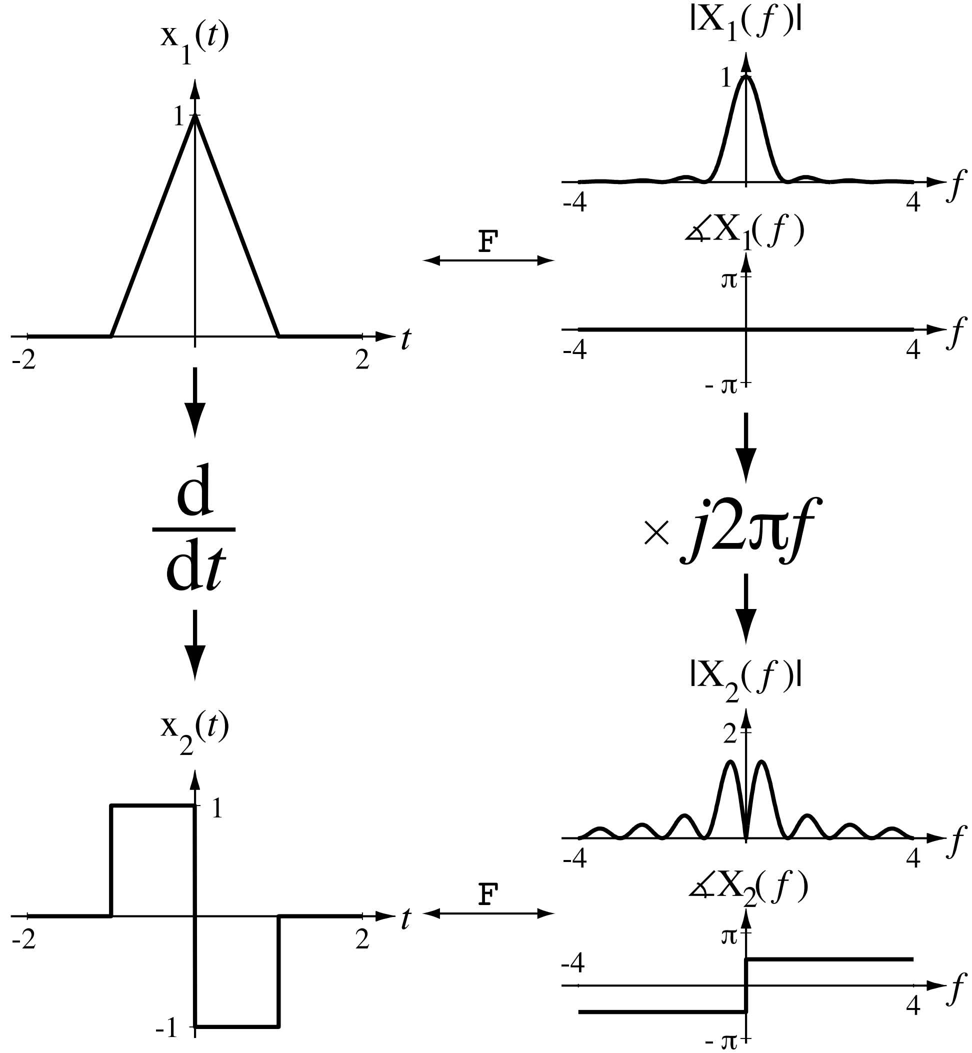
$$\mathrm{x}(t)\cos(\omega_0 t) \xleftrightarrow{\mathcal{F}} \frac{1}{2}\left[\mathrm{X}(j(\omega-\omega_0))+\mathrm{X}(j(\omega+\omega_0))\right]$$

Transforms of Periodic Signals

$$\mathrm{x}(t)=\sum_{k=-\infty}^{\infty}\mathrm{X}[k]e^{-j2\pi k f_F t} \xleftrightarrow{\mathcal{F}} \mathrm{X}(f)=\sum_{k=-\infty}^{\infty}\mathrm{X}[k]\delta(f-kf_0)$$

$$\mathrm{x}(t)=\sum_{k=-\infty}^{\infty}\mathrm{X}[k]e^{-jk\omega_F t} \xleftrightarrow{\mathcal{F}} \mathrm{X}(j\omega)=2\pi\sum_{k=-\infty}^{\infty}\mathrm{X}[k]\delta(\omega-k\omega_0)$$

CTFT Properties



CTFT Properties

Parseval's Theorem

$$\int_{-\infty}^{\infty} \left|\mathrm{x}(t)\right|^2 dt = \int_{-\infty}^{\infty} \left|\mathrm{X}(f)\right|^2 df$$

$$\int_{-\infty}^{\infty} \left|\mathrm{x}(t)\right|^2 dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} \left|\mathrm{X}(j\omega)\right|^2 d\omega$$

Integral Definition of an Impulse

$$\int_{-\infty}^{\infty} e^{-j2\pi xy} dy = \delta(x)$$

Duality

$$\mathrm{X}(t) \xleftrightarrow{\mathcal{F}} \mathrm{x}(-f) \text{ and } \mathrm{X}(-t) \xleftrightarrow{\mathcal{F}} \mathrm{x}(f)$$

$$\mathrm{X}(jt) \xleftrightarrow{\mathcal{F}} 2\pi\,\mathrm{x}(-\omega) \text{ and } \mathrm{X}(-jt) \xleftrightarrow{\mathcal{F}} 2\pi\,\mathrm{x}(\omega)$$

CTFT Properties

Total - Area Integral

$$\mathrm{X}(0)=\left[\int_{-\infty}^{\infty}\mathrm{x}(t)e^{-j2\pi ft}\,dt\right]_{f\to 0}=\int_{-\infty}^{\infty}\mathrm{x}(t)\,dt$$

$$\mathrm{x}(0)=\left[\int_{-\infty}^{\infty}\mathrm{X}(f)e^{+j2\pi ft}\,df\right]_{t\to 0}=\int_{-\infty}^{\infty}\mathrm{X}(f)\,df$$

$$\mathrm{X}(0)=\left[\int_{-\infty}^{\infty}\mathrm{x}(t)e^{-j\omega t}\,dt\right]_{\omega\to 0}=\int_{-\infty}^{\infty}\mathrm{x}(t)\,dt$$

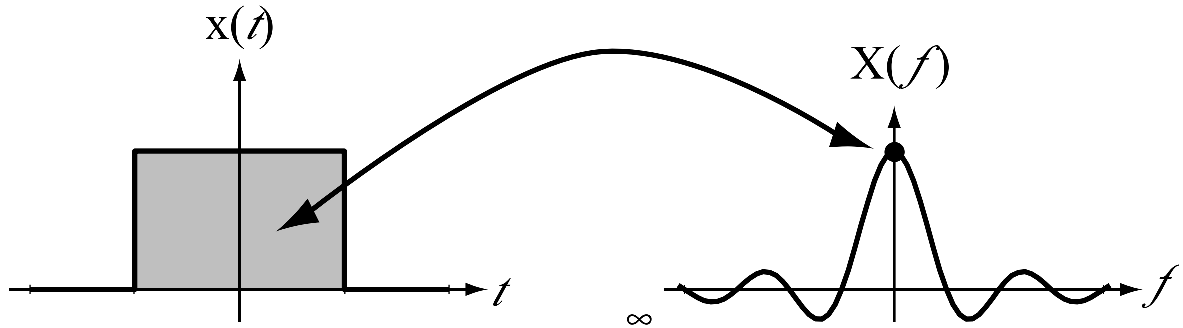
$$\mathrm{x}(0)=\left[\frac{1}{2\pi}\int_{-\infty}^{\infty}\mathrm{X}(j\omega)e^{+j\omega t}\,d\omega\right]_{t\to 0}=\frac{1}{2\pi}\int_{-\infty}^{\infty}\mathrm{X}(j\omega)\,d\omega$$

Integration

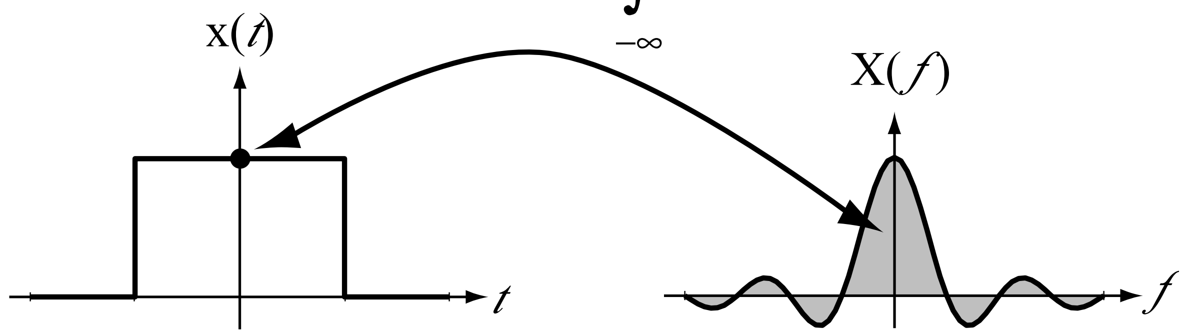
$$\int_{-\infty}^{t}\mathrm{x}(\lambda)\,d\lambda \xleftrightarrow{\mathcal{F}} \frac{\mathrm{X}(f)}{j2\pi f}+\frac{1}{2}\mathrm{X}(0)\delta(f)$$

$$\int_{-\infty}^{t}\mathrm{x}(\lambda)\,d\lambda \xleftrightarrow{\mathcal{F}} \frac{\mathrm{X}(j\omega)}{j\omega}+\pi\,\mathrm{X}(0)\delta(\omega)$$

CTFT Properties

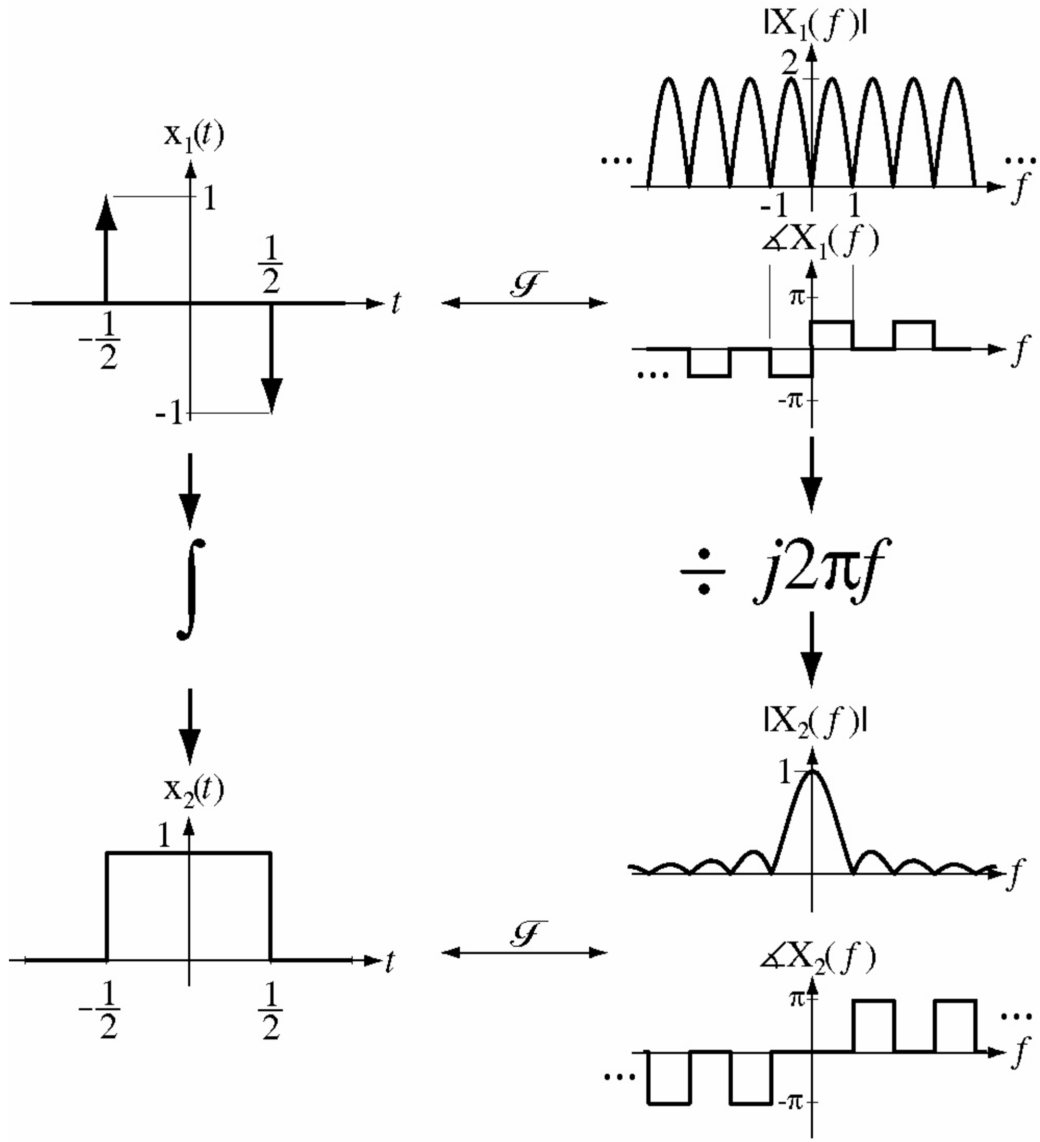


$$X(0) = \int_{-\infty}^{\infty} x(t)\,dt$$



$$x(0) = \int_{-\infty}^{\infty} X(f)\,df$$

CTFT Properties



Numerical Computation of the CTFT

It can be shown that the **DFT** can be used to approximate samples from the CTFT. If the signal $\mathrm{x}(t)$ is a **causal energy signal** and N samples are taken from it over a finite time beginning at $t = 0$, at a rate f_s then the relationship between the CTFT of $\mathrm{x}(t)$ and the DFT of the samples taken from it is

$$\mathrm{X}(kf_s / N) \cong T_s e^{-j\pi k/N} \operatorname{sinc}(k / N) \mathrm{X}_{DFT}[k]$$

For those harmonic numbers k for which $k << N$

$$\mathrm{X}(kf_s / N) \cong T_s \mathrm{X}_{DFT}[k]$$

As the sampling rate and number of samples are increased, this approximation is improved.

If $\mathrm{x}(t) \xleftrightarrow{\mathcal{F}} \mathrm{X}(f) = \delta(f-8) + \delta(f+8)$ and $\mathrm{x}(t) \xleftrightarrow[1]{\mathcal{FS}} \mathrm{c_x}[k]$, find $\mathrm{c_x}[k]$.

The relationship between the CTFT and the CTFS harmonic function is

$$\mathrm{x}(t) = \sum_{k=-\infty}^{\infty} \mathrm{c_x}[k] e^{j2\pi kt/T} \xleftrightarrow{\mathcal{F}} \mathrm{X}(f) = \sum_{k=-\infty}^{\infty} \mathrm{c_x}[k] \delta(f - k/T)$$

In this case, setting $T = 1$,

$$\mathrm{x}(t) = \sum_{k=-\infty}^{\infty} \mathrm{c_x}[k] e^{j2\pi kt} \xleftrightarrow{\mathcal{F}} \mathrm{X}(f) = \sum_{k=-\infty}^{\infty} (\delta[k-8] + \delta[k+8]) \delta(f-k)$$

Therefore $\mathrm{c_x}[k] = \delta[k-8] + \delta[k+8]$.

If we instead set $T = 1/8$,

$$\mathrm{x}(t) = \sum_{k=-\infty}^{\infty} \mathrm{c_x}[k] e^{j16\pi kt} \xleftrightarrow{\mathcal{F}} \mathrm{X}(f) = \sum_{k=-\infty}^{\infty} (\delta[k-1] + \delta[k+1]) \delta(f-8k)$$

and $\mathrm{x}(t) \xleftrightarrow[1/8]{\mathcal{FS}} \delta[k-1] + \delta[k+1]$. Then, using the CTFS property

$$\begin{array}{r} \mathrm{x}(t) \xleftrightarrow[T]{\mathcal{FS}} \mathrm{c}_x[k] \\ \text{and } \mathrm{x}(t) \xleftrightarrow[mT]{\mathcal{FS}} c_{xm}[k] \end{array} \Rightarrow \mathrm{c}_{xm}[k] = \begin{cases} \mathrm{c}_x[k/m] & , k/m \text{ an integer} \\ 0 & , \text{otherwise} \end{cases}$$

$$\mathrm{x}(t) \xleftrightarrow[1]{\mathcal{FS}} \left\{ \begin{array}{ll} \delta[k/8-1] + \delta[k/8+1] & , k/m \text{ an integer} \\ 0 & , \text{otherwise} \end{array} \right\} = \delta[k-8] + \delta[k+8].$$

A continuous-time system has a transfer function

$$\mathrm{H}(s) = \frac{2\times10^6}{s^2 + 2000s + 2\times10^6}$$

and therefore a frequency response

$$\mathrm{H}(j\omega) = \frac{2\times10^6}{(j\omega)^2 + j2000\omega + 2\times10^6} = \frac{2\times10^6}{(j\omega+1000)^2 + 10^6}.$$

Find its impulse response.

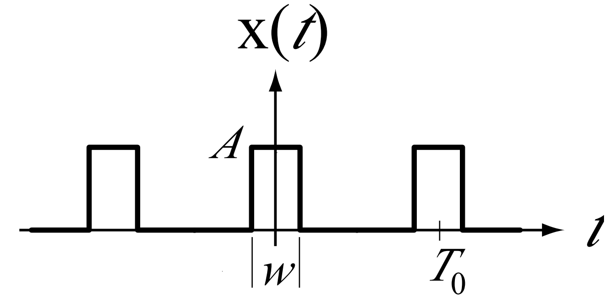
$$e^{-\alpha t}\sin(\omega_n t)\mathrm{u}(t) \xleftrightarrow{\mathscr{F}} \frac{\omega_n}{(j\omega+\alpha)^2 + \omega_n^2}\ ,\ \mathrm{Re}(\alpha) > 0$$

$$e^{-1000t}\sin(1000t)\mathrm{u}(t) \xleftrightarrow{\mathscr{F}} \frac{1000}{(j\omega+1000)^2 + (1000)^2}$$

$$2000e^{-1000t}\sin(1000t)\mathrm{u}(t) \xleftrightarrow{\mathscr{F}} \frac{2\times10^6}{(j\omega+1000)^2 + (1000)^2}$$

Therefore $\mathrm{h}(t) = 2000e^{-1000t}\sin(1000t)\mathrm{u}(t)$

This rectangular pulse train excites an amplifier with transfer function $\mathrm{H}(s) = \dfrac{500{,}000}{s + 50{,}000}$ and therefore frequency response $\mathrm{H}(f) = \dfrac{500{,}000}{j2\pi f + 50{,}000}$.



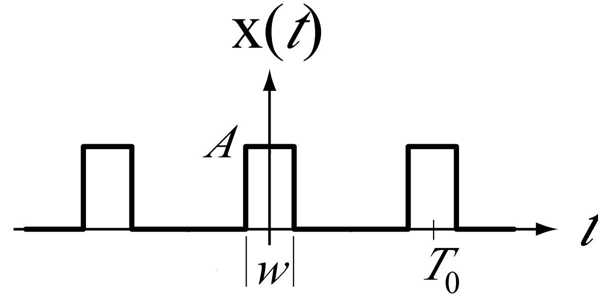
If $T_0 = 0.5$ ms, $w = 0.1$ ms and $A = 100$ mV find the average signal power of the amplifier response (using MATLAB where necessary).

$$\mathrm{rect}(t) \xleftrightarrow{\mathcal{F}} \mathrm{sinc}(f)$$

$$0.1\,\mathrm{rect}(10{,}000t) \xleftrightarrow{\mathcal{F}} \frac{0.1\,\mathrm{sinc}(f/10{,}000)}{10{,}000}$$

$$0.1\,\mathrm{rect}(10{,}000t) * \delta_{0.5\,\mathrm{ms}}(t) \xleftrightarrow{\mathcal{F}} \frac{0.1\,\mathrm{sinc}(f/10{,}000)}{10{,}000} 2000\delta_{2000}(f)$$

$$0.1\,\mathrm{rect}(10{,}000t) * \delta_{0.5\,\mathrm{ms}}(t) \xleftrightarrow{\mathcal{F}} \frac{\mathrm{sinc}(f/10{,}000)}{50} \delta_{2000}(f)$$



$$\mathrm{Y}(f) = \mathrm{H}(f)\mathrm{X}(f)$$

$$\mathrm{Y}(f) = \frac{500{,}000}{j2\pi f + 50{,}000} \frac{\mathrm{sinc}(f/10{,}000)}{50} \delta_{2000}(f)$$

Using the definition of the periodic impulse,

$$\mathrm{Y}(f) = (1/50) \sum_{k=-\infty}^{\infty} \frac{500{,}000\,\mathrm{sinc}(f/10{,}000)}{j2\pi f + 50{,}000} \delta(f - 2000k)$$

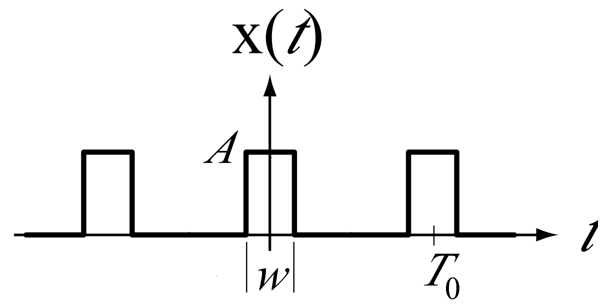
Using the equivalence property of the impulse

$$\mathrm{Y}(f) = (1/50) \sum_{k=-\infty}^{\infty} \frac{500{,}000\,\mathrm{sinc}(k/5)}{j4000\pi k + 50{,}000} \delta(f - 2000k)$$

From Parseval's theorem, $P_{\mathrm{y}} = \sum_{k=-\infty}^{\infty} \left|\mathrm{c}_{\mathrm{y}}[k]\right|^2$.

$$\mathrm{c}_{\mathrm{y}}[k] = \frac{1}{50} \frac{500{,}000\,\mathrm{sinc}(k/5)}{j4000\pi k + 50{,}000} \Rightarrow P_{\mathrm{y}} = (1/2500) \sum_{k=-\infty}^{\infty} \left| \frac{500{,}000\,\mathrm{sinc}(k/5)}{j4000\pi k + 50{,}000} \right|^2$$

$$P_y = (1/2500)\sum_{k=-\infty}^{\infty}\left|\frac{500{,}000\,\text{sinc}(k/5)}{j4000\pi k + 50{,}000}\right|^2$$



We can find this quantity using MATLAB.

…

```
k = [-kmax:kmax]' ;
Py = sum(abs(5e5*sinc(k/5)./(j*4000*k+5e4)).^2)/2500 ;
```

For `kmax = 10, Py = 0.1845`

For `kmax = 20, Py = 0.1867`

For `kmax = 50, Py = 0.1872`

For `kmax = 100, Py = 0.1873`

For `kmax = 200, Py = 0.1873`

If the amplifier had infinite bandwidth the response signal power would be 0.2.

The signal from a pressure sensor in an industrial plant is interfered by radiated EMI (electromagnetic interference) from a periodic rectangular pulse train of fundamental frequency 15 kHz. What would be the impulse response of a filter that would reject this EMI, including all its harmonics?

The source of the EMI is of the form $\mathrm{e}(t) = A\,\mathrm{rect}(t/w) * \delta_{T_0}(t)$.

The mechanism of interference through radiation depends on the first derivative of the EMI. So the received EMI is of the form $\mathrm{e}(t) = A\left[\delta(t+w/2) - \delta(t-w/2)\right] * \delta_{T_0}(t)$. Its CTFT is

$\mathrm{E}(f) = A\left[e^{j2\pi fw/2} - e^{-j2\pi fw/2}\right] f_0 \delta_{f_0}(f) = j2A\sin(2\pi fw/2) f_0 \delta_{f_0}(f)$.

where $f_0 = 1/T_0$. So it has impulses at integer multiples of f_0. An impulse response that averages the signal over exactly one fundamental period T_0 of the EMI would be $\mathrm{h}(t) = B\,\mathrm{rect}(t/T_0)$. Its CTFT is $\mathrm{H}(f) = BT_0\,\mathrm{sinc}(T_0 f) = BT_0\,\mathrm{sinc}(f/f_0)$. This frequency response has nulls at integer multiples of f_0. So it would reject the EMI, including all its harmonics.