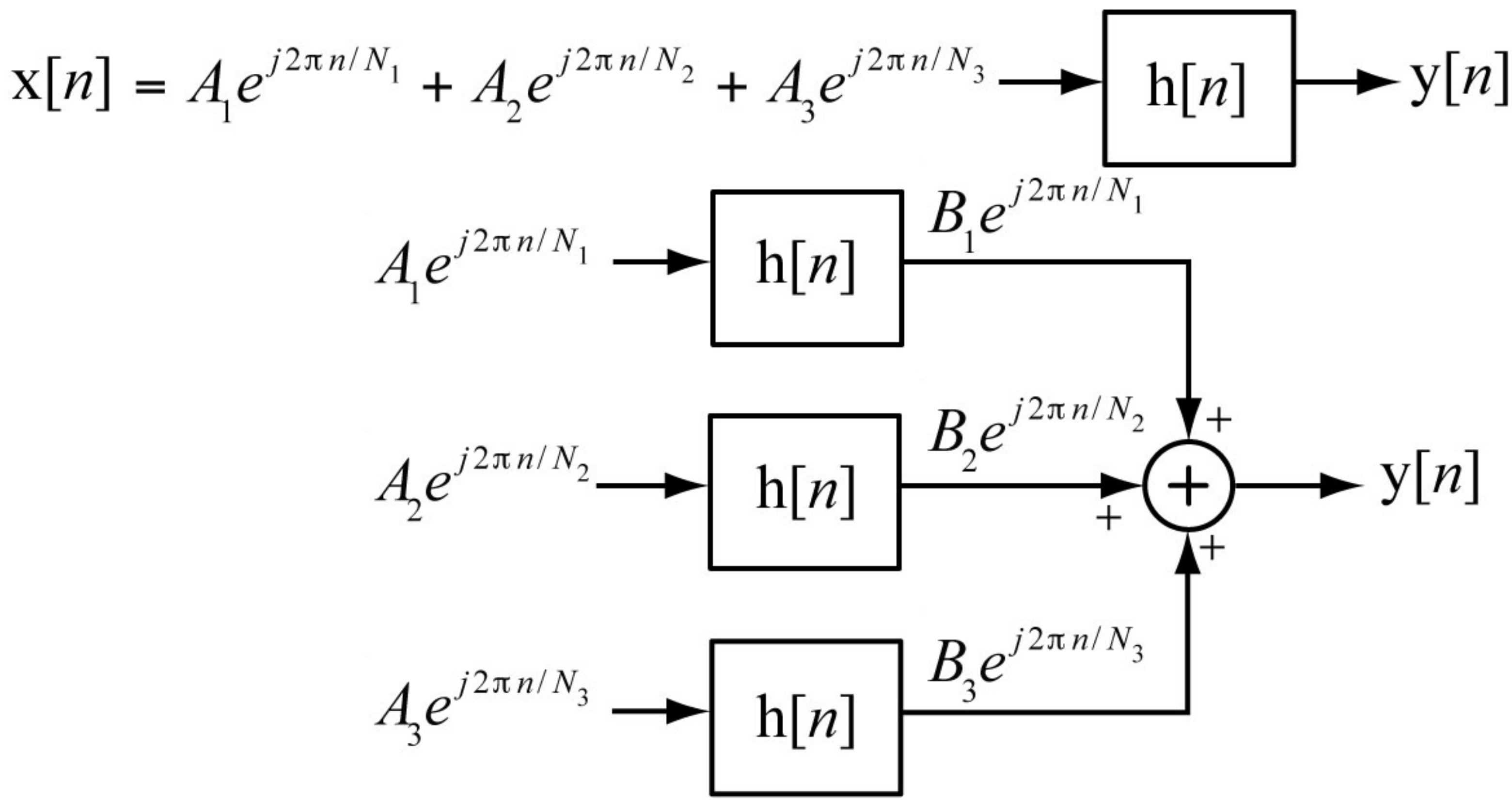
Discrete-Time Fourier Methods

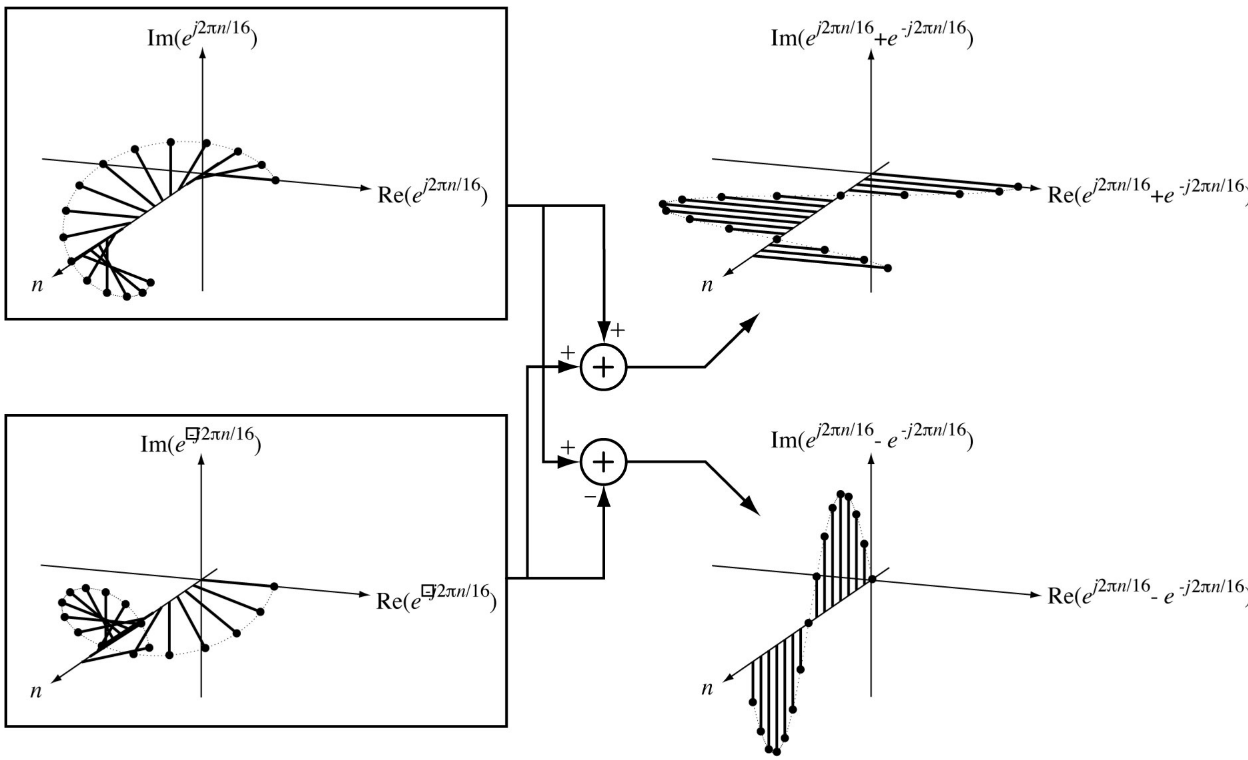
Discrete-Time Fourier Series Concept

A signal can be represented as a linear combination of sinusoids.

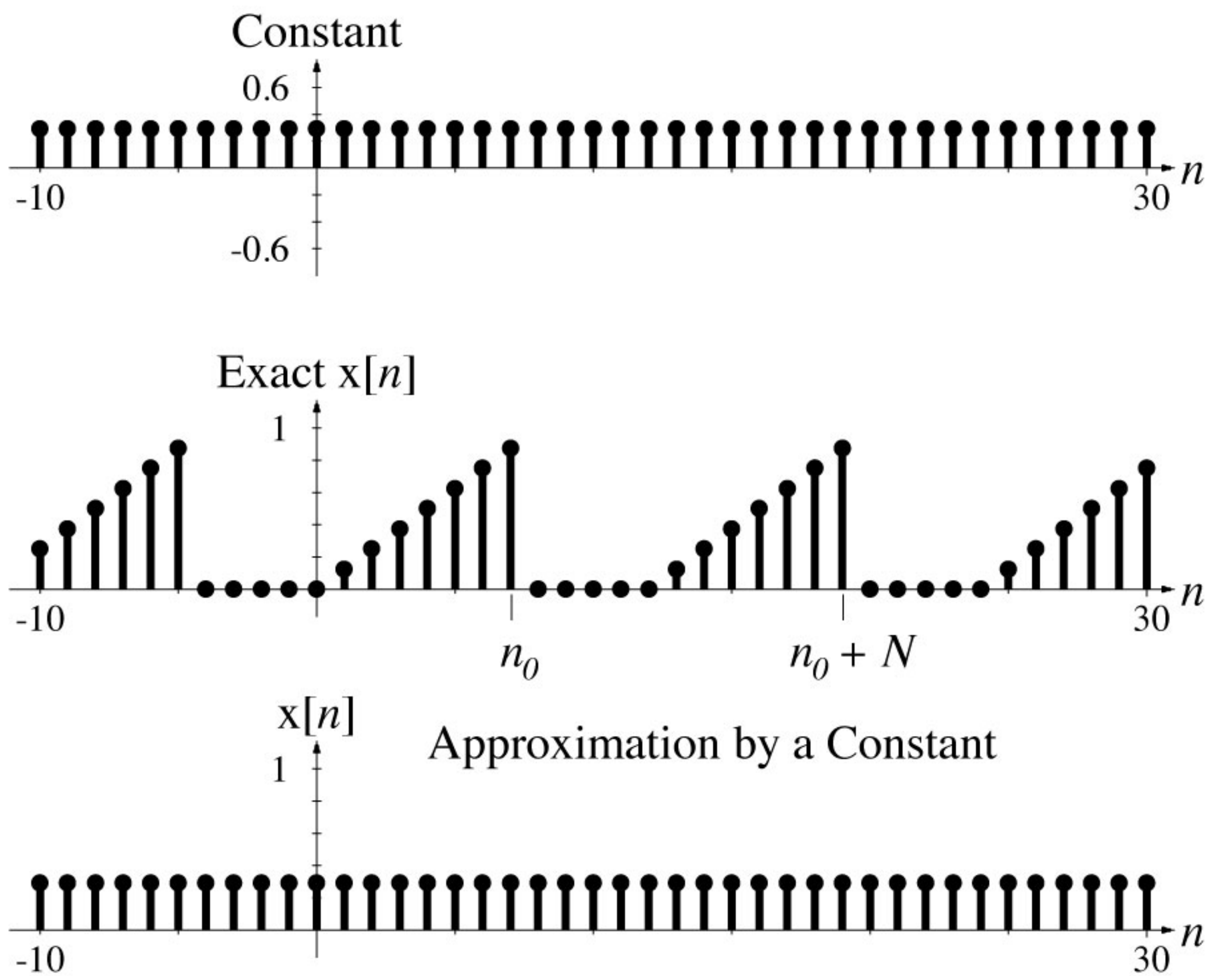


Discrete-Time Fourier Series Concept

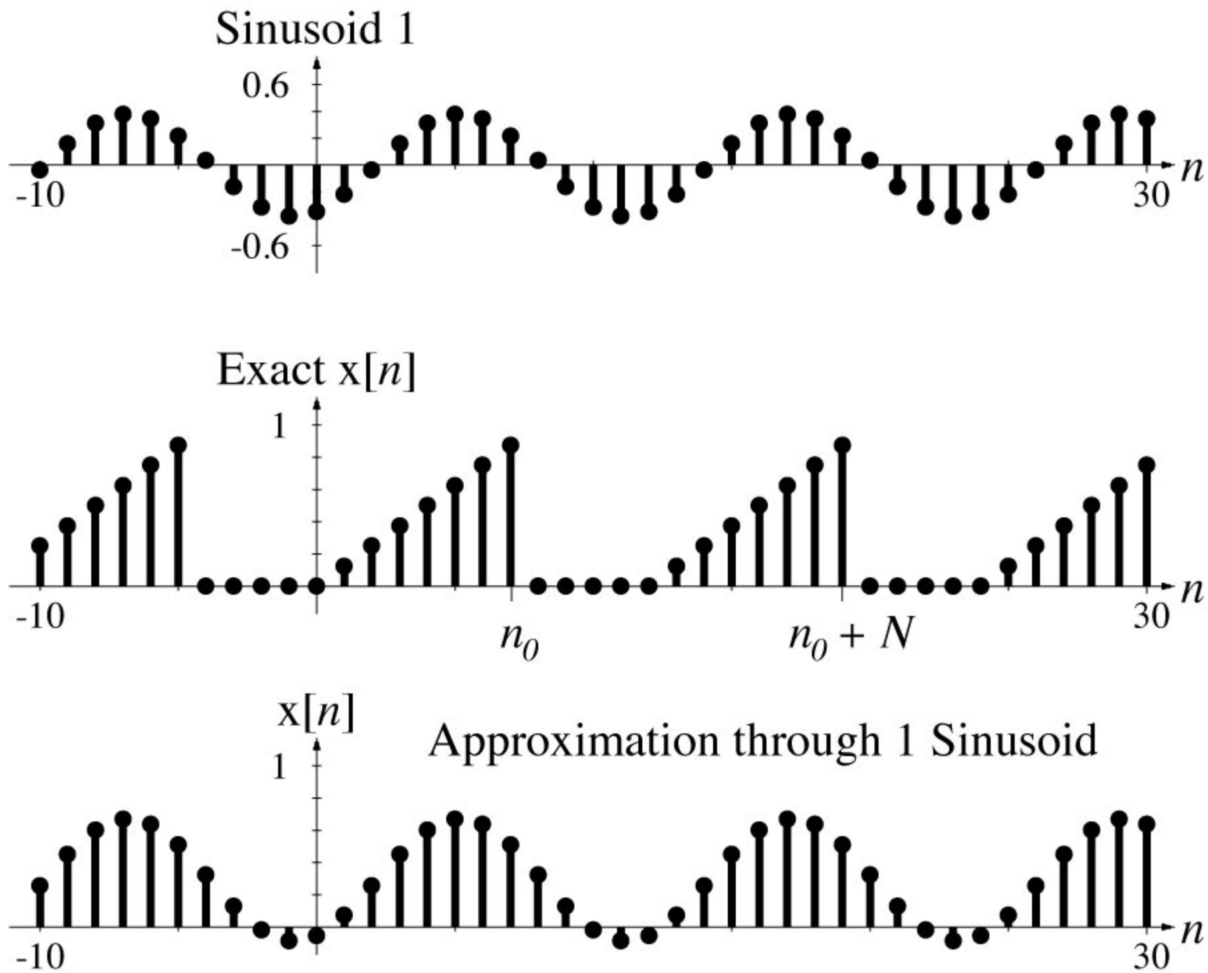
The relationship between complex and real sinusoids



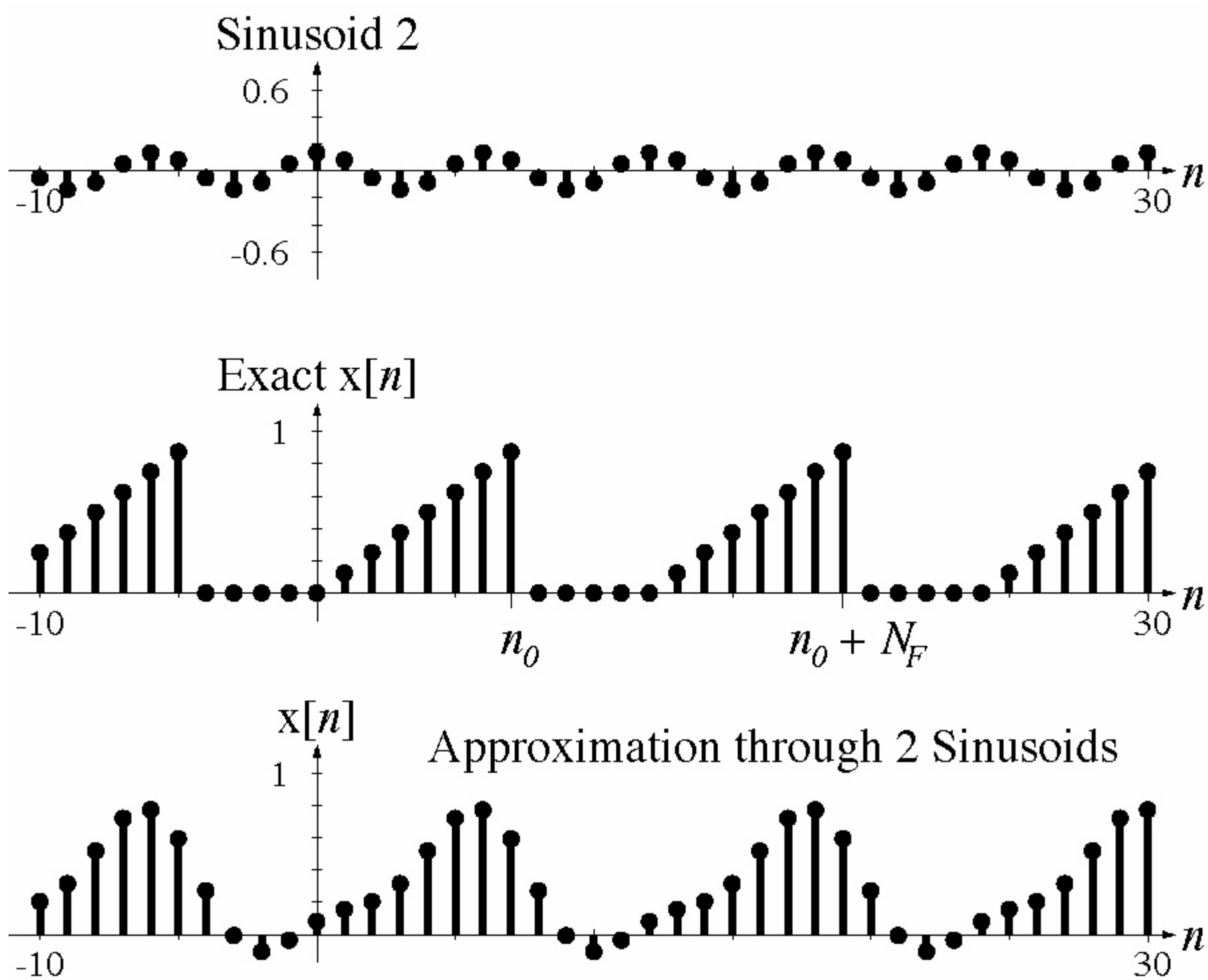
Discrete-Time Fourier Series Concept



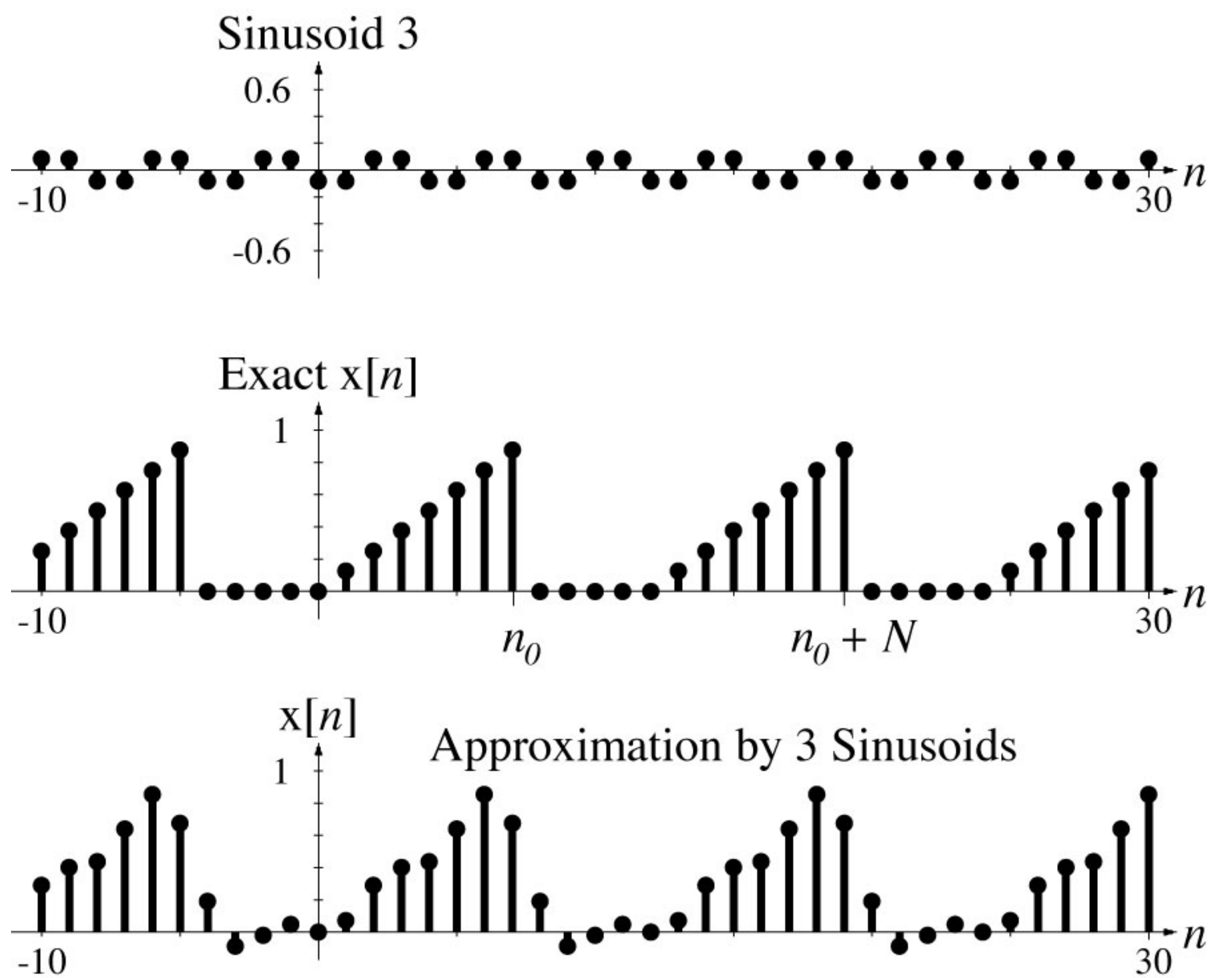
Discrete-Time Fourier Series Concept



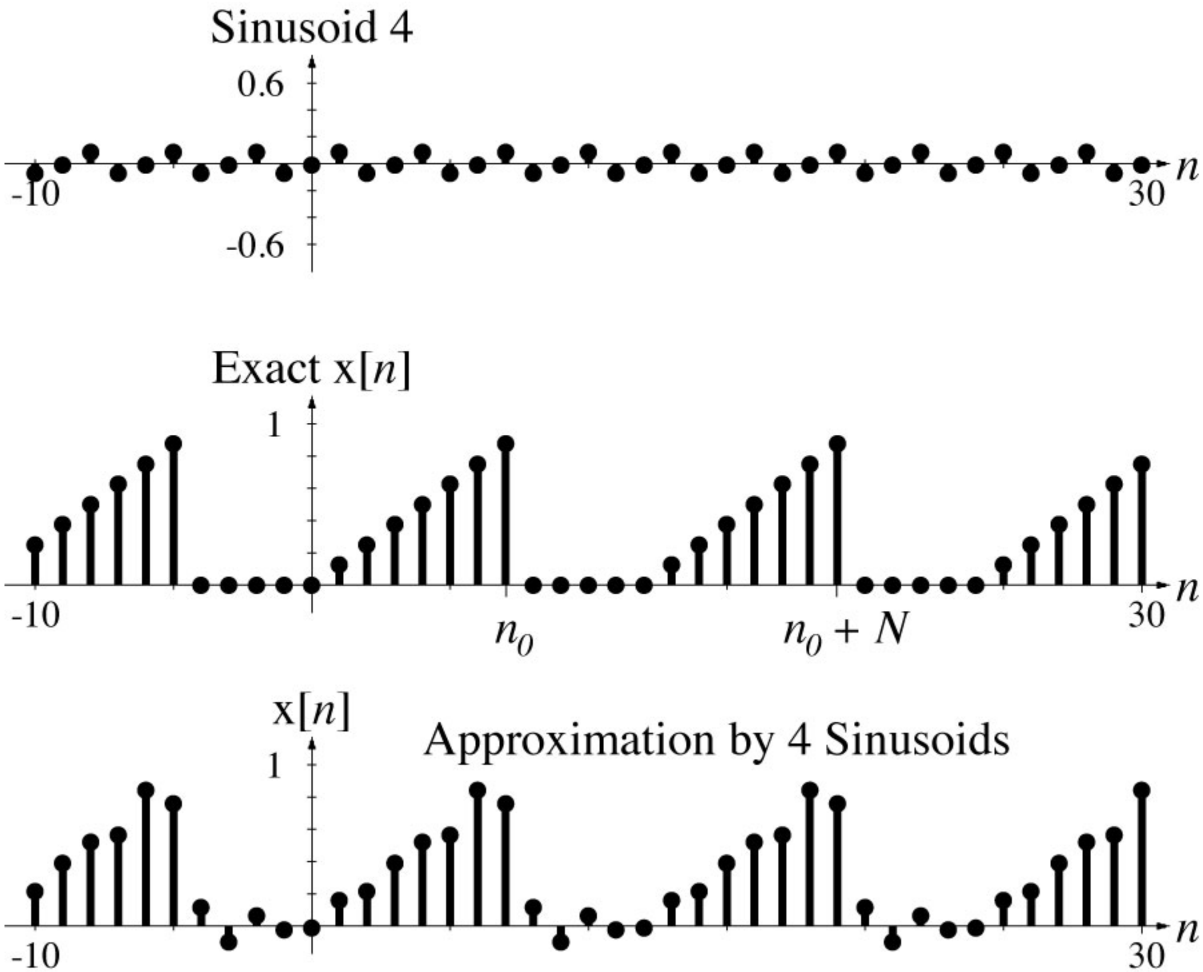
Discrete-Time Fourier Series Concept



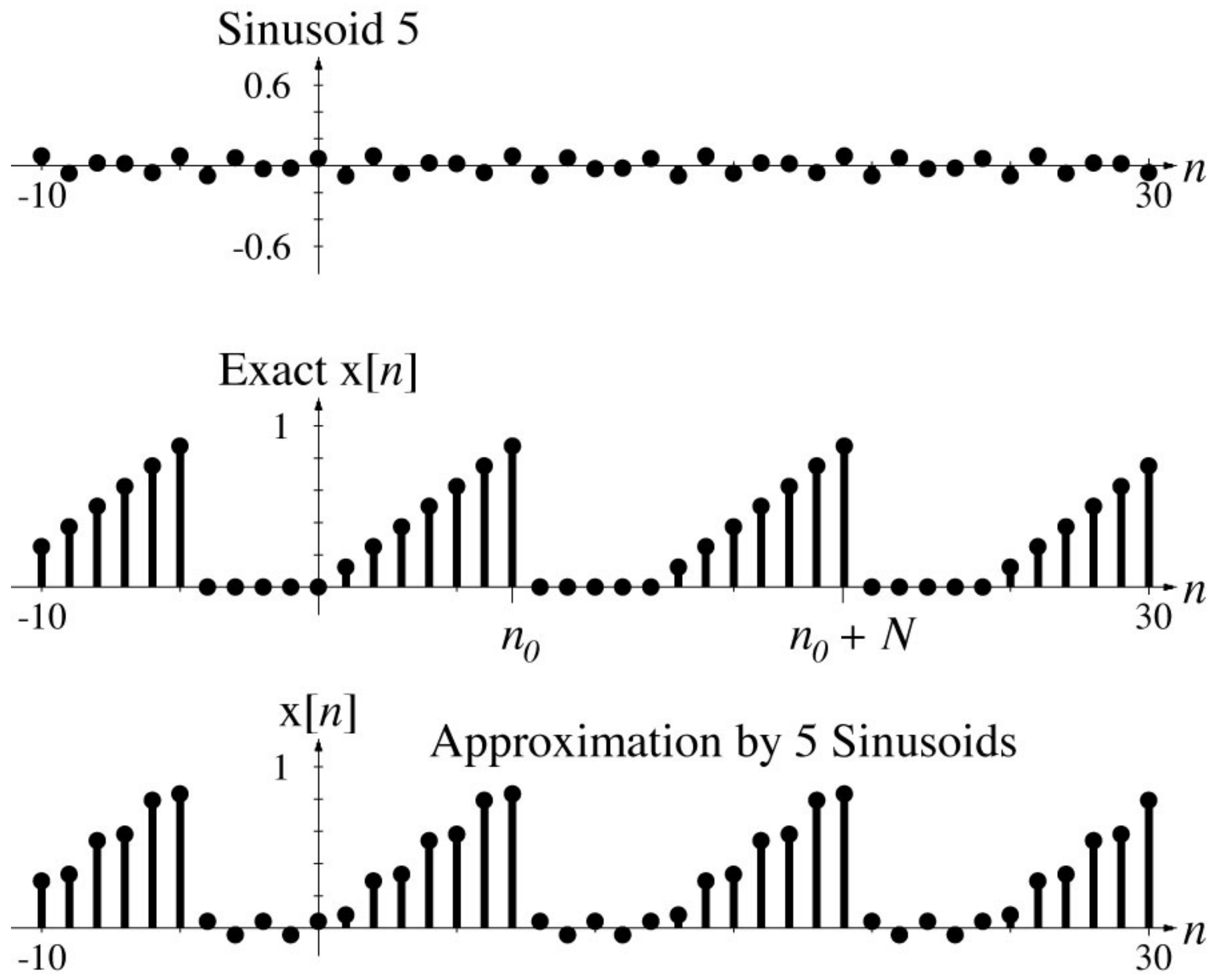
Discrete-Time Fourier Series Concept



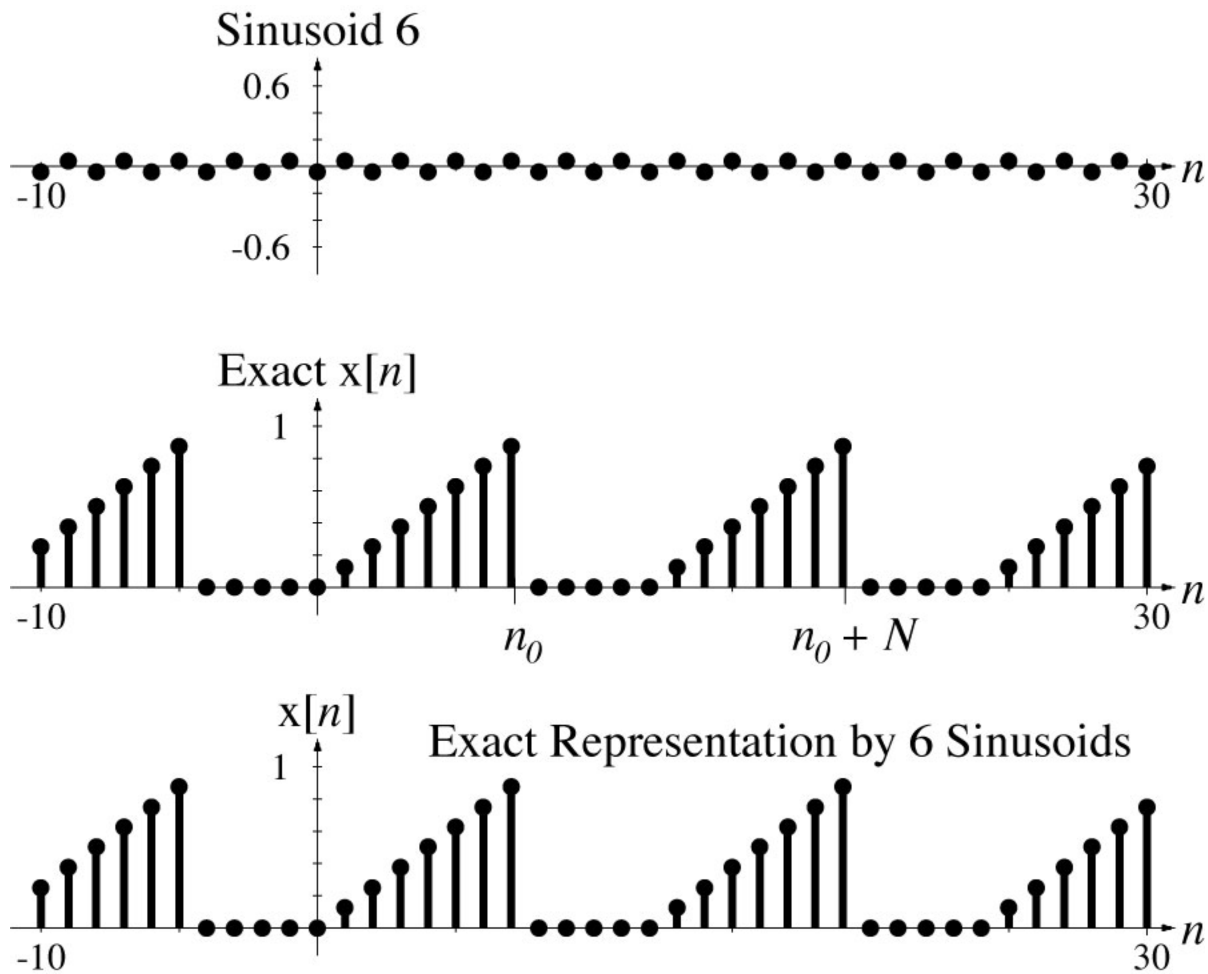
Discrete-Time Fourier Series Concept



Discrete-Time Fourier Series Concept



Discrete-Time Fourier Series Concept



The Discrete-Time Fourier Series

The discrete-time Fourier series (DTFS) is similar to the CTFS.

A periodic discrete-time signal can be expressed as

$$\mathrm{x}[n] = \sum_{k=\langle N \rangle} \mathrm{c_x}[k] e^{j2\pi kn/N} \qquad \mathrm{c_x}[k] = \frac{1}{N} \sum_{n=n_0}^{n_0+N-1} \mathrm{x}[n] e^{-j2\pi kn/N}$$

where $\mathrm{c_x}[k]$ is the harmonic function, N is any period of $\mathrm{x}[n]$ and the notation, $\sum_{k=\langle N \rangle}$ means a summation over any range of consecutive k's exactly N in length.

The Discrete Fourier Transform

The discrete Fourier transform (DFT) is almost identical to the DTFS.

A periodic discrete-time signal can be expressed as

$$\mathrm{x}[n] = \frac{1}{N} \sum_{k=\langle N \rangle} \mathrm{X}[k] e^{j2\pi kn/N} \qquad \mathrm{X}[k] = \sum_{n=n_0}^{n_0+N-1} \mathrm{x}[n] e^{-j2\pi kn/N}$$

where $\mathrm{X}[k]$ is the DFT harmonic function and N is any period of $\mathrm{x}[n]$.

The main difference between the DTFS and the DFT is the location of the $1/N$ term. So $\mathrm{X}[k] = N\mathrm{c}_{\mathrm{x}}[k]$.

The Discrete Fourier Transform

Because the DTFS and DFT are so similar, and because the DFT is so widely used in digital signal processing (DSP), we will concentrate on the DFT realizing we can always form the DTFS from $c_x[k] = X[k]/N$.

The Discrete Fourier Transform

Notice that in

$$\mathrm{x}[n]=\frac{1}{N}\sum_{k=\langle N\rangle}\mathrm{X}[k]e^{j2\pi kn/N}$$

the summation is over N values of k, a finite summation. This is because of the periodicity of the complex sinusoid, $e^{-j2\pi kn/N}$ in harmonic number k. If k is increased by any integer multiple of N the complex sinusoid does not change.

$$e^{-j2\pi kn/N}=e^{-j2\pi(k+mN)n/N}=e^{-j2\pi kn/N}\underbrace{e^{-j2\pi mn}}_{=1}\quad,\ m\text{ an integer}$$

This occurs because discrete time n is always an integer.

DFT Example

Find the DFT harmonic function for

$$x[n] = \left(u[n] - u[n-3]\right) * \delta_5[n]$$

using its fundamental period as the representation time.

$$X[k] = \sum_{n=\langle N \rangle} x[n] e^{-j2\pi kn/N}$$

$$X[k] = \sum_{n=0}^{4} \left(u[n] - u[n-3]\right) * \delta_5[n] e^{-j2\pi kn/5}$$

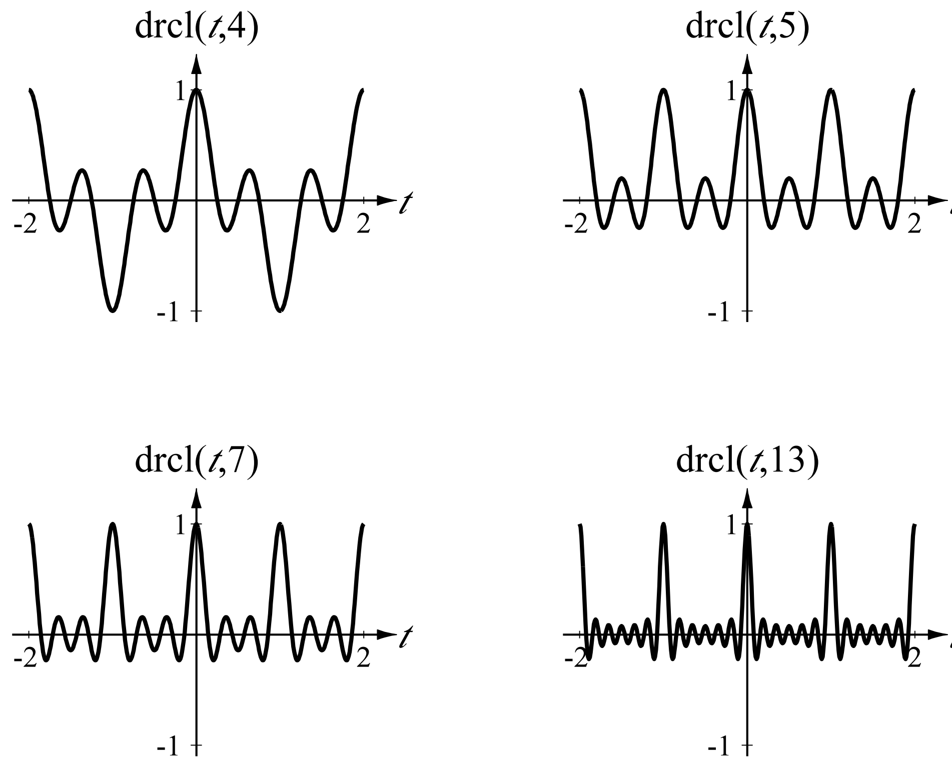
$$X[k] = \sum_{n=0}^{2} e^{-j2\pi kn/5} = \frac{1 - e^{-j6\pi k/5}}{1 - e^{-j2\pi k/5}} = \frac{e^{-j3\pi k/5}}{e^{-j\pi k/5}} \times \frac{e^{j3\pi k/5} - e^{-j3\pi k/5}}{e^{j\pi k/5} - e^{-j\pi k/5}}$$

$$X[k] = \sum_{n=0}^{2} e^{-j2\pi kn/5} = e^{-j2\pi k/5} \frac{\sin(3\pi k/5)}{\sin(\pi k/5)} = 3e^{-j2\pi k/5} \operatorname{drcl}(k/5, 3)$$

The Dirichlet Function

The functional form $\dfrac{\sin(\pi N t)}{N \sin(\pi t)}$ appears often in discrete-time signal analysis and is given the special name **Dirichlet** function. That is

$$\text{drcl}(t, N) = \frac{\sin(\pi N t)}{N \sin(\pi t)}$$



The DFT Harmonic Function

We know that $\mathrm{x}[n]=\frac{1}{N}\sum_{k=\langle N\rangle}\mathrm{X}[k]e^{j2\pi kn/N}$ so we can find $\mathrm{x}[n]$ from its harmonic function. But how do we find the harmonic function from $\mathrm{x}[n]$? We use the principle of orthogonality like we did with the CTFS except that now the orthogonality is in discrete time instead of continuous time.

The DFT Harmonic Function

Let $W_N = e^{j2\pi/N}$. Then $\mathrm{x}[n] = (1/N)\sum_{k=\langle N\rangle} \mathrm{X}[k]e^{j2\pi kn/N} = (1/N)\sum_{k=\langle N\rangle} \mathrm{X}[k]W_N^{kn}$.

Since the starting point of the summation is arbitrary let it be $k=0$ for convenience. Also let the range of $\mathrm{x}[n]$ be $n_0 \le n < n_0 + N$. Now we can express the inverse DFT in matrix form as

$$\underbrace{\begin{bmatrix} \mathrm{x}[n_0] \\ \mathrm{x}[n_0+1] \\ \vdots \\ \mathrm{x}[n_0+N-1] \end{bmatrix}}_{\mathbf{x}} = \frac{1}{N} \underbrace{\begin{bmatrix} W_N^0 & W_N^{n_0} & \cdots & W_N^{n_0(N-1)} \\ W_N^0 & W_N^{n_0+1} & \cdots & W_N^{(n_0+1)(N-1)} \\ \vdots & \vdots & \ddots & \vdots \\ W_N^0 & W_N^{n_0+N-1} & \cdots & W_N^{(n_0+N-1)(N-1)} \end{bmatrix}}_{\mathbf{W}} \underbrace{\begin{bmatrix} \mathrm{X}[0] \\ \mathrm{X}[1] \\ \vdots \\ \mathrm{X}[N-1] \end{bmatrix}}_{\mathbf{X}}$$

or as $N\mathbf{x} = \mathbf{WX}$.

The DFT Harmonic Function

We can solve $N\mathbf{x} = \mathbf{W}\mathbf{X}$ (if $\mathbf{W}$ is not singular) for $\mathbf{X}$ as $\mathbf{X} = \mathbf{W}^{-1}N\mathbf{x}$. The matrix equation can be written in the form

$$N\begin{bmatrix} \mathrm{x}[n_0] \\ \mathrm{x}[n_0+1] \\ \vdots \\ \mathrm{x}[n_0+N-1] \end{bmatrix} = \underbrace{\begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix}}_{k=0}\mathrm{X}[0] + \underbrace{\begin{bmatrix} W_N^{n_0} \\ W_N^{n_0+1} \\ \vdots \\ W_N^{n_0+N-1} \end{bmatrix}}_{k=1}\mathrm{X}[1] + \cdots + \underbrace{\begin{bmatrix} W_N^{n_0(N-1)} \\ W_N^{(n_0+1)(N-1)} \\ \vdots \\ W_N^{(n_0+N-1)(N-1)} \end{bmatrix}}_{k=N-1}\mathrm{X}[N-1]$$

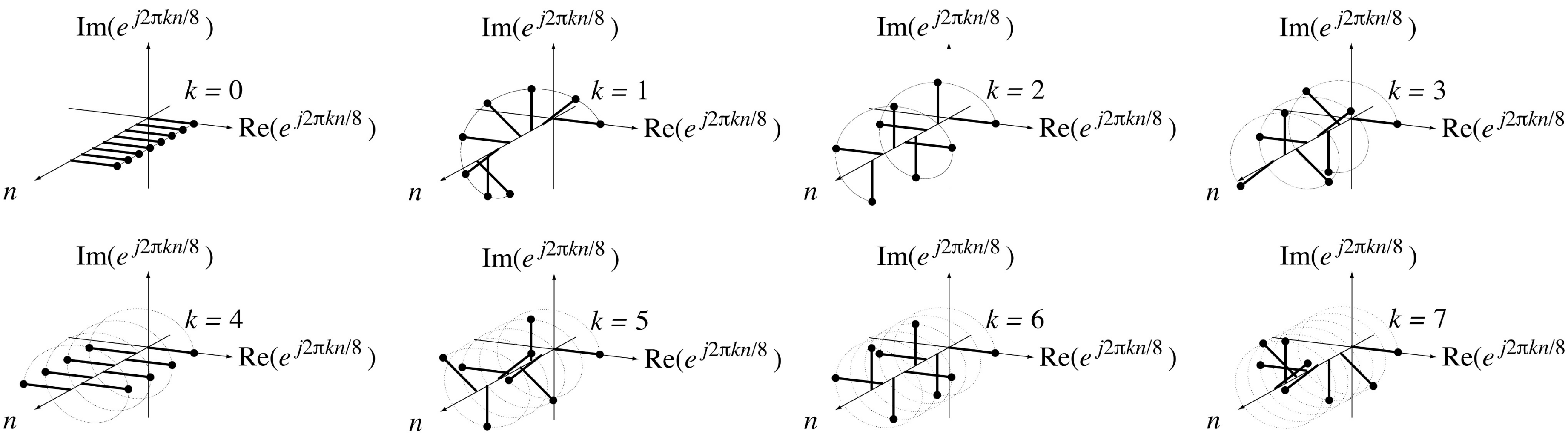
or

$$N\mathbf{x} = \mathbf{w}_0\,\mathrm{X}[0] + \mathbf{w}_1\,\mathrm{X}[1] + \cdots + \mathbf{w}_{N-1}\,\mathrm{X}[N-1]$$

where $\mathbf{W} = [\mathbf{w}_0\mathbf{w}_1\cdots\mathbf{w}_{N-1}]$. The first column $\mathbf{w}_0$ of $\mathbf{W}$ is the constant 1, which can be thought of as a complex sinusoid of zero frequency. The second column is one cycle of a complex sinusoid, the third is two cycles, etc. through $\mathbf{w}_{N-1}$ which is $N-1$ cycles of a complex sinusoid.

The DFT Harmonic Function

Below is a set of complex sinusoids for $N = 8$. They form a set of **basis** vectors. Notice that the $k = 7$ complex sinusoid rotates counterclockwise through 7 cycles but <u>appears to rotate</u> clockwise through one cycle. The $k = 7$ complex sinusoid is exactly the same as the $k = -1$ complex sinusoid. This must be true because the DFT is periodic with period N.



The DFT Harmonic Function

The projection of a real vector $\mathbf{x}$ in the direction of another real vector $\mathbf{y}$ is

$$\mathbf{p} = \frac{\mathbf{x}^T \mathbf{y}}{\mathbf{y}^T \mathbf{y}} \mathbf{y}$$

If $\mathbf{p} = 0$, $\mathbf{x}$ and $\mathbf{y}$ are orthogonal. If the vectors are complex-valued

$$\mathbf{p} = \frac{\mathbf{x}^H \mathbf{y}}{\mathbf{y}^H \mathbf{y}} \mathbf{y}$$

where the $\mathbf{x}^H$ is the complex-conjugate transpose of $\mathbf{x}$. $\mathbf{x}^T \mathbf{y}$ and $\mathbf{x}^H \mathbf{y}$ are both the **dot product** of $\mathbf{x}$ and $\mathbf{y}$.

The DFT Harmonic Function

The dot product of the first two columns of $\mathbf{W}$ is

$$\mathbf{w}_0^H \mathbf{w}_1 = \begin{bmatrix} 1 & 1 & \cdots & 1 \end{bmatrix} \begin{bmatrix} W_N^{n_0} \\ W_N^{n_0+1} \\ \vdots \\ W_N^{n_0+N-1} \end{bmatrix} = W_N^{n_0} \left(1 + W_N + \cdots + W_N^{N-1} \right)$$

The factor $\left(1 + W_N + \cdots + W_N^{N-1} \right)$ is a finite-length geometric series which can be summed using

$$\sum_{n=0}^{N-1} r^n = \begin{cases} N & , \ r = 1 \\ \dfrac{1 - r^N}{1 - r} & , \ r \neq 1 \end{cases}$$

The DFT Harmonic Function

$$\left(1 + W_{N_0} + \cdots + W_N^{N-1}\right) = \frac{1 - W_N^N}{1 - W_N}$$

and

$$\mathbf{w}_0^H \mathbf{w}_1 = W_{N_0}^{n_0} \frac{1 - W_N^N}{1 - W_N} = W_{N_0}^{n_0} \frac{1 - e^{j2\pi}}{1 - e^{j2\pi/N}} = 0$$

This proves that those two columns of **W** are orthogonal. Using similar logic it can be shown that

$$\mathbf{w}_{k_1}^H \mathbf{w}_{k_2} = \begin{cases} 0 \ , \ k_1 \neq k_2 \\ N \ , \ k_1 = k_2 \end{cases} = N\delta\left[k_1 - k_2\right]$$

Any two distinct columns of **W** are orthogonal.

The DFT Harmonic Function

Starting with

$$N\mathbf{x} = \mathbf{w}_0 \, \mathrm{X}[0] + \mathbf{w}_1 \, \mathrm{X}[1] + \cdots + \mathbf{w}_{N-1} \, \mathrm{X}[N-1]$$

premultiply all the terms by $\mathbf{w}_0^H$ to get

$$\mathbf{w}_0^H N\mathbf{x} = \underbrace{\mathbf{w}_0^H \mathbf{w}_0}_{=N} \mathrm{X}[0] + \underbrace{\mathbf{w}_0^H \mathbf{w}_1}_{=0} \mathrm{X}[1] + \cdots + \underbrace{\mathbf{w}_0^H \mathbf{w}_{N-1}}_{=0} \mathrm{X}[N-1] = N\mathrm{X}[0]$$

and solve for $\mathrm{X}[0]$

$$\mathrm{X}[0] = \frac{\mathbf{w}_0^H N\mathbf{x}}{\underbrace{\mathbf{w}_0^H \mathbf{w}_0}_{=N}} = \mathbf{w}_0^H \mathbf{x}$$

The vector $\mathrm{X}[0]\mathbf{w}_0$ is the projection of the vector $N\mathbf{x}$ in the direction of the basis vector $\mathbf{w}_0$. Similarly, each $\mathrm{X}[k]\mathbf{w}_k$ is the projection of the vector $N\mathbf{x}$ in the direction of the basis vector $\mathbf{w}_k$.

The DFT Harmonic Function

The entire vector $\mathbf{X}$ can be found from

$$\mathbf{X} = \begin{bmatrix} \mathbf{w}_0^H \\ \mathbf{w}_1^H \\ \vdots \\ \mathbf{w}_{N-1}^H \end{bmatrix} \mathbf{x} = \mathbf{W}^H \mathbf{x}.$$

This can be written in summation form as

$$\mathrm{X}[k] = \sum_{n=n_0}^{n_0+N-1} \mathrm{x}[n] e^{-j2\pi kn/N}$$

This defines the forward DFT.

The DFT Harmonic Function

The most common definition of the DFT is

$$\mathrm{X}[k] = \sum_{n=0}^{N-1} \mathrm{x}[n] e^{-j2\pi kn/N} \quad , \quad \mathrm{x}[n] = \frac{1}{N} \sum_{k=\langle N \rangle} \mathrm{X}[k] e^{j2\pi kn/N}$$

Here the beginning point for $\mathrm{x}[n]$ is taken as $n_0 = 0$. This is the form of the DFT that is implemented in practically all computer languages.

Convergence of the DFT

- The DFT converges exactly with a finite number of terms. It does not have a “Gibbs phenomenon” in the same sense that the CTFS does

The Discrete Fourier Transform

$\mathrm{X}[k]$ is called the DFT **harmonic function** of $\mathrm{x}[n]$ and k is the harmonic number just as we have seen in the CTFS. $\mathrm{x}[n]$ and $\mathrm{X}[k]$ form a DFT pair based on N points.

$$\mathrm{x}[n] \xleftrightarrow[N]{\mathcal{DFT}} \mathrm{X}[k]$$

From $\mathrm{x}[n] = \frac{1}{N} \sum_{k=\langle N \rangle} \mathrm{X}[k] e^{j2\pi kn/N}$ we see that $\mathrm{x}[n]$ is formed by a linear combination of functions of the form $e^{j2\pi kn/N}$ each of which has a period N. Therefore $\mathrm{x}[n]$ must also be periodic with period (but not necessarily fundamental period) N.

DFT Properties

Linearity $\alpha\,x[n] + \beta\,y[n] \xleftrightarrow[N]{\mathcal{DFT}} \alpha\,X[k] + \beta\,Y[k]$

Time Shifting $x[n - n_0] \xleftrightarrow[N]{\mathcal{DFT}} X[k]e^{-j2\pi k n_0/N}$

Frequency Shifting $x[n]e^{j2\pi k_0 n/N} \xleftrightarrow[N]{\mathcal{DFT}} X[k - k_0]$

Time Reversal $x[-n] = x[N - n] \xleftrightarrow[N]{\mathcal{DFT}} X[-k] = X[N - k]$

Conjugation $x^*[n] \xleftrightarrow[N]{\mathcal{DFT}} X^*[-k] = X^*[N - k]$

⋮ $x^*[-n] = x^*[N - n] \xleftrightarrow[N]{\mathcal{DFT}} X^*[k]$

Time Scaling $z[n] = \begin{cases} x[n/m] & , \ n/m \text{ an integer} \\ 0 & , \ \text{otherwise} \end{cases}$

⋮ $N \to mN \ , \ Z[k] = (1/m)X[k]$

DFT Properties

Change of Period $N \to qN$, q a positive integer

$\vdots$

$$X_q[k] = \begin{cases} X[k/q] & , \ k/q \text{ an integer} \\ 0 & , \ \text{otherwise} \end{cases}$$

Multiplication - Convolution Duality

$$x[n]y[n] \xleftrightarrow[N]{\mathcal{DFT}} (1/N)Y[k] \circledast X[k]$$

$\vdots$

$$x[n] \circledast y[n] \xleftrightarrow[N]{\mathcal{DFT}} Y[k]X[k]$$

$\vdots$

$$\text{where } x[n] \circledast y[n] = \sum_{m=\langle N \rangle} x[m]y[n-m]$$

Parseval's Theorem

$$\sum_{n=\langle N \rangle} |x[n]|^2 = \frac{1}{N} \sum_{k=\langle N \rangle} |X[k]|^2$$

DFT Properties

It can be shown (and is in the text) that if $\mathrm{x}[n]$ is an even function, $\mathrm{X}[k]$ is purely real and if $\mathrm{x}[n]$ is an odd function $\mathrm{X}[k]$ is purely imaginary.

DFT Pairs

$$e^{j2\pi n/N} \xleftrightarrow[mN]{\mathcal{DFT}} mN\delta_{mN}[k-m]$$

$$\cos(2\pi qn/N) \xleftrightarrow[mN]{\mathcal{DFT}} (mN/2)(\delta_{mN}[k-mq]+\delta_{mN}[k+mq])$$

$$\sin(2\pi qn/N) \xleftrightarrow[mN]{\mathcal{DFT}} (jmN/2)(\delta_{mN}[k+mq]-\delta_{mN}[k-mq])$$

$$\delta_N[n] \xleftrightarrow[mN]{\mathcal{DFT}} m\delta_{mN}[k]$$

$$1 \xleftrightarrow[N]{\mathcal{DFT}} N\delta_N[k]$$

$$(\mathrm{u}[n-n_0]-\mathrm{u}[n-n_1]) * \delta_N[n] \xleftrightarrow[N]{\mathcal{DFT}} \frac{e^{-j\pi k(n_1+n_0)/N}}{e^{-j\pi k/N}}(n_1-n_0)\,\mathrm{drcl}(k/N, n_1-n_0)$$

$$\mathrm{tri}(n/N_w) * \delta_N[n] \xleftrightarrow[N]{\mathcal{DFT}} N_w\,\mathrm{drcl}^2(k/N, N_w) \quad , \; N_w \text{ an integer}$$

$$\mathrm{sinc}(n/w) * \delta_N[n] \xleftrightarrow[N]{\mathcal{DFT}} w\mathrm{rect}(wk/N) * \delta_N[k]$$

The Fast Fourier Transform

One could write a MATLAB program to implement the DFT.

```
.
%   (Acquire the input data in an array x with N elements.)
.
%   Initialize the DFT array to a column vector of zeros.
X = zeros(N,1) ;
%   Compute the X(k)'s in a nested, double for loop.
for k = 0:N-1
    for n = 0:N-1
        X(k+1) = X(k+1)+x(n+1)*exp(-j*2*pi*n*k/N) ;
    end
end
```

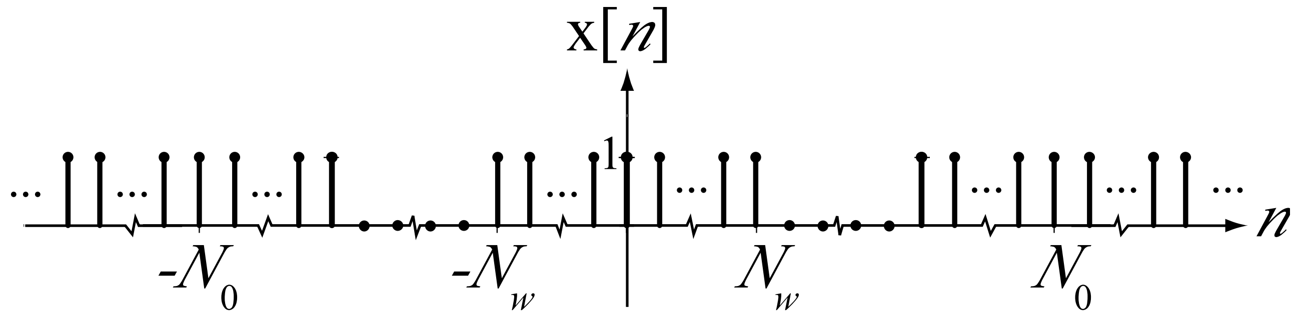
The Fast Fourier Transform

There is a function in MATLAB `fft` that accomplishes the same goal and is typically much faster. This table compares the speeds of the two methods. M stands for computer multiplies and A stands for computer additions.

γ	$N = 2^\gamma$	A_{DFT}	M_{DFT}	A_{FFT}	M_{FFT}	A_{DFT} / A_{FFT}	M_{DFT} / M_{FFT}
1	2	2	4	2	1	1	4
2	4	12	16	8	4	1.5	4
3	8	56	64	24	12	2.33	5.33
4	16	240	256	64	32	3.75	8
5	32	992	1024	160	80	6.2	12.8
6	64	4032	4096	384	192	10.5	21.3
7	128	16256	16384	896	448	18.1	36.6
8	256	65280	65536	2048	1024	31.9	64
9	512	261632	262144	4608	2304	56.8	113.8
10	1024	1047552	1048576	10240	5120	102.3	204.8

Generalizing the DFT for Aperiodic Signals

Pulse Train

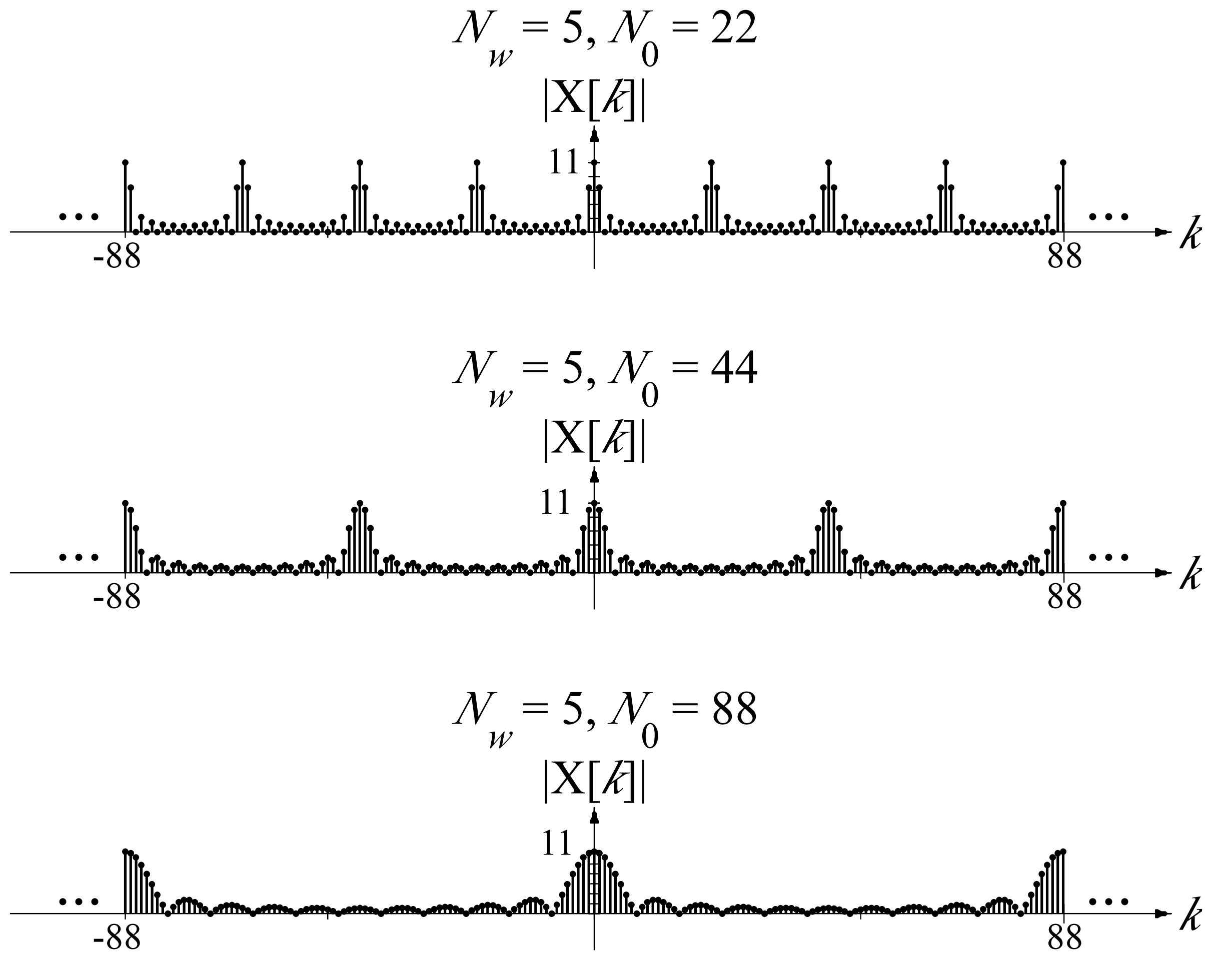


This periodic rectangular-wave signal is analogous to the continuous-time periodic rectangular-wave signal used to illustrate the transition from the CTFS to the CTFT.

Generalizing the DFT for Aperiodic Signals

DFT of Pulse Train

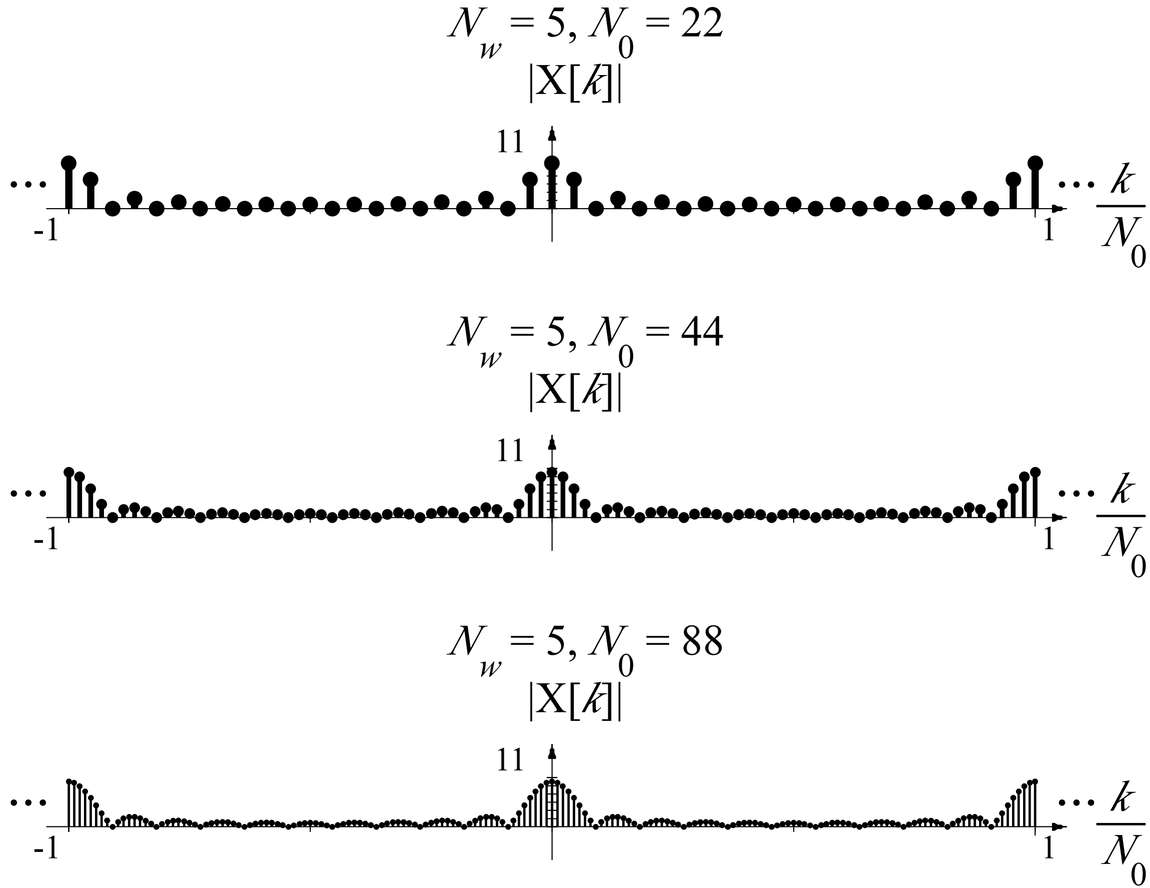
As the period of the rectangular wave increases, the period of the DFT increases



Generalizing the DFT for Aperiodic Signals

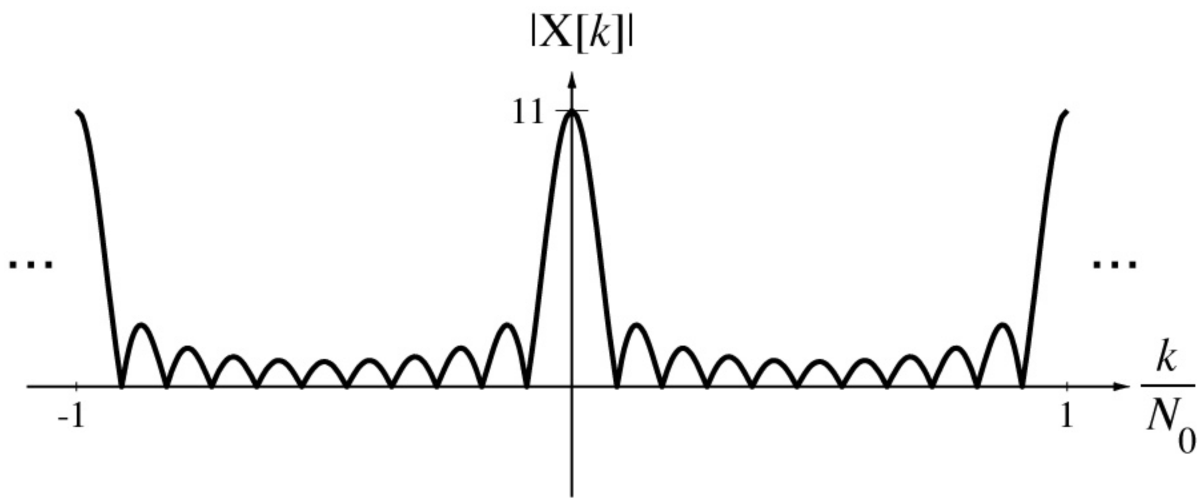
Normalized
DFT of
Pulse Train

By plotting versus k / N_0 instead of k, the period of the normalized DFT stays at one.



Generalizing the DFT for Aperiodic Signals

The normalized DFT approaches this limit as the period approaches infinity.



Definition of the Discrete-Time Fourier Transform (DTFT)

Inverse — **F Form** — Forward

$$\mathrm{x}[n]=\int_1 \mathrm{X}(F)e^{j2\pi Fn}\,dF \xleftrightarrow{\mathscr{F}} \mathrm{X}(F)=\sum_{n=-\infty}^{\infty}\mathrm{x}[n]e^{-j2\pi Fn}$$

Inverse — **Ω Form** — Forward

$$\mathrm{x}[n]=\frac{1}{2\pi}\int_{2\pi}\mathrm{X}(e^{j\Omega})e^{j\Omega n}\,d\Omega \xleftrightarrow{\mathscr{F}} \mathrm{X}(e^{j\Omega})=\sum_{n=-\infty}^{\infty}\mathrm{x}[n]e^{-j\Omega n}$$

The Discrete-Time Fourier Transform

The function $e^{-j\Omega}$ appears in the forward DTFT raised to the nth power. It is periodic in Ω with fundamental period 2π. n is an integer. Therefore $e^{-j\Omega n}$ is periodic with fundamental period $2\pi / n$ and 2π is also a period of $e^{-j\Omega n}$. The forward DTFT is

$$\mathrm{X}\left(e^{j\Omega}\right)=\sum_{n=-\infty}^{\infty}\mathrm{x}[n]e^{-j\Omega n}$$

a weighted summation of functions of the form $e^{-j\Omega n}$, all of which repeat with every 2π change in Ω. Therefore $\mathrm{X}\left(e^{j\Omega}\right)$ is always periodic in Ω with period 2π. This also implies that $\mathrm{X}(F)$ is always periodic in F with period 1.

DTFT Pairs

We can begin a table of DTFT pairs directly from the definition.
(There is a more extensive table in the text.)

$$\delta[n] \xleftrightarrow{\mathcal{F}} 1$$

$$\alpha^n \mathrm{u}[n] \xleftrightarrow{\mathcal{F}} \frac{e^{j\Omega}}{e^{j\Omega} - \alpha} = \frac{1}{1 - \alpha e^{-j\Omega}} \ , \ |\alpha| < 1 \quad , \quad -\alpha^n \mathrm{u}[-n-1] \xleftrightarrow{\mathcal{F}} \frac{e^{j\Omega}}{e^{j\Omega} - \alpha} = \frac{1}{1 - \alpha e^{-j\Omega}} \ , \ |\alpha| > 1$$

$$\alpha^n \sin(\Omega_0 n)\mathrm{u}[n] \xleftrightarrow{\mathcal{F}} \frac{e^{j\Omega}\alpha \sin(\Omega_0)}{e^{j2\Omega} - 2\alpha e^{j\Omega}\cos(\Omega_0) + \alpha^2} \ , \ |\alpha| < 1 \ , \ -\alpha^n \sin(\Omega_0 n)\mathrm{u}[-n-1] \xleftrightarrow{\mathcal{F}} \frac{e^{j\Omega}\alpha \sin(\Omega_0)}{e^{j2\Omega} - 2\alpha e^{j\Omega}\cos(\Omega_0) + \alpha^2} \ , \ |\alpha| > 1$$

$$\alpha^n \cos(\Omega_0 n)\mathrm{u}[n] \xleftrightarrow{\mathcal{F}} \frac{e^{j\Omega}\left[e^{j\Omega} - \alpha\cos(\Omega_0)\right]}{e^{j2\Omega} - 2\alpha e^{j\Omega}\cos(\Omega_0) + \alpha^2} \ , \ |\alpha| < 1 \ , \ -\alpha^n \cos(\Omega_0 n)\mathrm{u}[-n-1] \xleftrightarrow{\mathcal{F}} \frac{e^{j\Omega}\left[e^{j\Omega} - \alpha\cos(\Omega_0)\right]}{e^{j2\Omega} - 2\alpha e^{j\Omega}\cos(\Omega_0) + \alpha^2} \ , \ |\alpha| > 1$$

The Generalized DTFT

By generalizing the CTFT to include transforms that have impulses we were able to find CTFT's of some important practical functions. The same is true of the DTFT. The DTFT of a constant

$$X(F) = \sum_{n=-\infty}^{\infty} A e^{-j2\pi Fn} = A \sum_{n=-\infty}^{\infty} e^{-j2\pi Fn}$$

does not converge. The CTFT of a constant turned out to be an impulse. Since the DTFT must be periodic, that cannot be the transform of a constant in discrete time. Instead the transform must be a periodic impulse.

The Generalized DTFT

Find the inverse DTFT of a periodic impulse of the form $A\delta_1(F)$. Using the formula

$$\mathrm{x}[n] = \int_1 A\delta_1(F) e^{j2\pi Fn}\, dF = A \int_{-1/2}^{1/2} \delta(F) e^{j2\pi Fn}\, dF = A$$

proving that the DTFT of a constant A is $A\delta_1(F)$ or, in radian-frequency form $A \xleftrightarrow{\mathcal{F}} 2\pi A\delta_{2\pi}(\Omega)$.

The Generalized DTFT

Now consider the function $\mathrm{X}(F) = A\delta_1(F - F_0)$, $-1/2 < F_0 < 1/2$.

Its inverse DTFT is

$$\mathrm{x}[n] = \int_1 A\delta_1(F - F_0) e^{j2\pi Fn}\, dF = A \int_{-1/2}^{1/2} \delta(F - F_0) e^{j2\pi Fn}\, dF = Ae^{j2\pi F_0 n}$$

Now change $\mathrm{x}[n]$ to $A\cos(2\pi F_0 n) = (A/2)\left(e^{j2\pi F_0 n} + e^{-j2\pi F_0 n}\right)$.

Then

$$A\cos(2\pi F_0 n) \xleftrightarrow{\mathscr{F}} (A/2)\left[\delta_1(F - F_0) + \delta_1(F + F_0)\right]$$

or

$$A\cos(\Omega_0 n) \xleftrightarrow{\mathscr{F}} \pi A\left[\delta_{2\pi}(\Omega - \Omega_0) + \delta_{2\pi}(\Omega + \Omega_0)\right]$$

Forward DTFT Example

Find the forward DTFT of $\mathrm{x}[n] = \mathrm{u}[n-n_0] - \mathrm{u}[n-n_1]$.

$$\mathrm{u}[n-n_0] - \mathrm{u}[n-n_1] \xleftrightarrow{\mathcal{F}} \sum_{n=-\infty}^{\infty} \left(\mathrm{u}[n-n_0] - \mathrm{u}[n-n_1]\right) e^{-j2\pi Fn} = \sum_{n=n_0}^{n_1-1} e^{-j2\pi Fn}$$

Let $m = n - n_0$. Then

$$\mathrm{u}[n-n_0] - \mathrm{u}[n-n_1] \xleftrightarrow{\mathcal{F}} \sum_{m=0}^{n_1-n_0-1} e^{-j2\pi F(m+n_0)} = e^{-j2\pi Fn_0} \sum_{m=0}^{n_1-n_0-1} e^{-j2\pi Fm}$$

Summing this geometric series

$$\mathrm{u}[n-n_0] - \mathrm{u}[n-n_1] \xleftrightarrow{\mathcal{F}} e^{-j2\pi Fn_0} \frac{1-e^{-j2\pi F(n_1-n_0)}}{1-e^{-j2\pi F}}$$

Forward DTFT Example

$$\mathrm{u}[n-n_0]-\mathrm{u}[n-n_1] \xleftrightarrow{\mathcal{F}} e^{-j2\pi F n_0}\frac{1-e^{-j2\pi F(n_1-n_0)}}{1-e^{-j2\pi F}}$$

Factor out $e^{-j\pi F(n_1-n_0)}$ from the numerator and $e^{-j\pi F}$ from the denominator

$$\mathrm{u}[n-n_0]-\mathrm{u}[n-n_1] \xleftrightarrow{\mathcal{F}} e^{-j2\pi F n_0}\frac{e^{-j\pi F(n_1-n_0)}}{e^{-j\pi F}}\frac{e^{j\pi F(n_1-n_0)}-e^{-j\pi F(n_1-n_0)}}{e^{j\pi F}-e^{-j\pi F}}$$

By the definition of the sine function in terms of complex exponentials

$$\mathrm{u}[n-n_0]-\mathrm{u}[n-n_1] \xleftrightarrow{\mathcal{F}} \frac{e^{-j\pi F(n_0+n_1)}}{e^{-j\pi F}}\frac{\sin(\pi F(n_1-n_0))}{\sin(\pi F)}$$

$$\mathrm{u}[n-n_0]-\mathrm{u}[n-n_1] \xleftrightarrow{\mathcal{F}} \frac{e^{-j\pi F(n_0+n_1)}}{e^{-j\pi F}}(n_1-n_0)\,\mathrm{drcl}(F,n_1-n_0)$$

Forward DTFT Example

Consider the special case of $n_0 + n_1 = 1 \Rightarrow n_0 = 1 - n_1$ (making the function a periodic repetition of a discrete-time rectangular pulse of width $2n_0+1$ centered at $n = 0$).

$$\mathrm{u}[n-n_0]-\mathrm{u}[n-n_1] \xleftrightarrow{\mathcal{F}} \frac{\sin\left(\pi F\left(n_1-n_0\right)\right)}{\sin(\pi F)}$$

$$\mathrm{u}[n-n_0]-\mathrm{u}[n-n_1] \xleftrightarrow{\mathcal{F}} \left(n_1-n_0\right)\mathrm{drcl}\left(F, n_1-n_0\right) \quad , \quad n_0+n_1=1$$

Compare this to the CTFT of a rectangular pulse of width w centered at $t = 0$.

$$\mathrm{rect}(t/w) \xleftrightarrow{\mathcal{F}} w\,\mathrm{sinc}(wf) = \frac{\sin(\pi w f)}{\pi f}$$

The DTFT is a periodically-repeated sinc function and also a Dirichlet function.

More DTFT Pairs

We can now extend the table of DTFT pairs.

$$\delta[n] \xleftrightarrow{\mathcal{F}} 1$$

$$\mathrm{u}[n] \xleftrightarrow{\mathcal{F}} \frac{1}{1-e^{-j2\pi F}} + (1/2)\delta_1(F) \quad , \quad \mathrm{u}[n] \xleftrightarrow{\mathcal{F}} \frac{1}{1-e^{-j\Omega}} + \pi\delta_1(\Omega)$$

$$\mathrm{sinc}(n/w) \xleftrightarrow{\mathcal{F}} w\,\mathrm{rect}(wF) * \delta_1(F) \quad , \quad \mathrm{sinc}(n/w) \xleftrightarrow{\mathcal{F}} w\,\mathrm{rect}(w\Omega/2\pi) * \delta_{2\pi}(\Omega)$$

$$\mathrm{tri}(n/w) \xleftrightarrow{\mathcal{F}} w\,\mathrm{drcl}^2(F,w) \quad , \quad \mathrm{tri}(n/w) \xleftrightarrow{\mathcal{F}} w\,\mathrm{drcl}^2(\Omega/2\pi,w)$$

$$1 \xleftrightarrow{\mathcal{F}} \delta_1(F) \quad , \quad 1 \xleftrightarrow{\mathcal{F}} 2\pi\delta_{2\pi}(\Omega)$$

$$\delta_{N_0}[n] \xleftrightarrow{\mathcal{F}} (1/N_0)\delta_{1/N_0}(F) \quad , \quad \delta_{N_0}[n] \xleftrightarrow{\mathcal{F}} (2\pi/N_0)\delta_{2\pi/N_0}(\Omega)$$

$$\cos(2\pi F_0 n) \xleftrightarrow{\mathcal{F}} (1/2)\left[\delta_1(F-F_0) + \delta_1(F+F_0)\right] \quad , \quad \cos(\Omega_0 n) \xleftrightarrow{\mathcal{F}} \pi\left[\delta_{2\pi}(\Omega-\Omega_0) + \delta_{2\pi}(\Omega+\Omega_0)\right]$$

$$\sin(2\pi F_0 n) \xleftrightarrow{\mathcal{F}} (j/2)\left[\delta_1(F+F_0) - \delta_1(F-F_0)\right] \quad , \quad \sin(\Omega_0 n) \xleftrightarrow{\mathcal{F}} j\pi\left[\delta_{2\pi}(\Omega+\Omega_0) - \delta_{2\pi}(\Omega-\Omega_0)\right]$$

$$\mathrm{u}[n-n_0] - \mathrm{u}[n-n_1] \xleftrightarrow{\mathcal{F}} \frac{e^{j2\pi F}}{e^{j2\pi F}-1}\left(e^{-j2\pi n_0 F} - e^{-j2\pi n_1 F}\right) = \frac{e^{-j\pi F(n_0+n_1)}}{e^{-j\pi F}}(n_1-n_0)\,\mathrm{drcl}(F, n_1-n_0)$$

$$\mathrm{u}[n-n_0] - \mathrm{u}[n-n_1] \xleftrightarrow{\mathcal{F}} \frac{e^{j\Omega}}{e^{j\Omega}-1}\left(e^{-jn_0\Omega} - e^{-jn_1\Omega}\right) = \frac{e^{-j\Omega(n_0+n_1)/2}}{e^{-j\Omega/2}}(n_1-n_0)\,\mathrm{drcl}(\Omega/2\pi, n_1-n_0)$$

DTFT Properties

$$\alpha\,\mathrm{x}[n]+\beta\,\mathrm{y}[n] \xleftrightarrow{\mathcal{F}} \alpha\,\mathrm{X}(F)+\beta\,\mathrm{Y}(F) \quad , \quad \alpha\,\mathrm{x}[n]+\beta\,\mathrm{y}[n] \xleftrightarrow{\mathcal{F}} \alpha\,\mathrm{X}(e^{j\Omega})+\beta\,\mathrm{Y}(e^{j\Omega})$$

$$\mathrm{x}[n-n_0] \xleftrightarrow{\mathcal{F}} e^{-j2\pi F n_0}\,\mathrm{X}(F) \quad , \quad \mathrm{x}[n-n_0] \xleftrightarrow{\mathcal{F}} e^{-j\Omega n_0}\,\mathrm{X}(e^{j\Omega})$$

$$e^{j2\pi F_0 n}\,\mathrm{x}[n] \xleftrightarrow{\mathcal{F}} \mathrm{X}(F-F_0) \quad , \quad e^{j\Omega_0 n}\,\mathrm{x}[n] \xleftrightarrow{\mathcal{F}} \mathrm{X}\left(e^{j(\Omega-\Omega_0)}\right)$$

$$\text{If } \mathrm{y}[n]=\begin{cases}\mathrm{x}[n/m] & , \; n/m \text{ an integer} \\ 0 & , \text{ otherwise}\end{cases} \text{ then } \mathrm{y}[n] \xleftrightarrow{\mathcal{F}} \mathrm{X}(mF) \text{ or } \mathrm{y}[n] \xleftrightarrow{\mathcal{F}} \mathrm{X}(e^{jm\Omega})$$

$$\mathrm{x}^*[n] \xleftrightarrow{\mathcal{F}} \mathrm{X}^*(-F) \quad , \quad \mathrm{x}^*[n] \xleftrightarrow{\mathcal{F}} \mathrm{X}^*(e^{-j\Omega})$$

$$\mathrm{x}[n]-\mathrm{x}[n-1] \xleftrightarrow{\mathcal{F}} \left(1-e^{-j2\pi F}\right)\mathrm{X}(F) \quad , \quad \mathrm{x}[n]-\mathrm{x}[n-1] \xleftrightarrow{\mathcal{F}} \left(1-e^{-j\Omega}\right)\mathrm{X}(e^{j\Omega})$$

DTFT Properties

$$\sum_{m=-\infty}^{n} \mathrm{x}[m] \xleftrightarrow{\mathcal{F}} \frac{\mathrm{X}(F)}{1-e^{-j2\pi F}} + \frac{1}{2}\mathrm{X}(0)\delta_1(F) \quad , \qquad \sum_{m=-\infty}^{n} \mathrm{x}[m] \xleftrightarrow{\mathcal{F}} \frac{\mathrm{X}(e^{j\Omega})}{1-e^{-j\Omega}} + \pi\,\mathrm{X}(e^{j0})\delta_{2\pi}(\Omega)$$

$$\mathrm{x}[-n] \xleftrightarrow{\mathcal{F}} \mathrm{X}(-F) \quad , \qquad \mathrm{x}[-n] \xleftrightarrow{\mathcal{F}} \mathrm{X}(e^{-j\Omega})$$

$$\mathrm{x}[n] * \mathrm{y}[n] \xleftrightarrow{\mathcal{F}} \mathrm{X}(F)\mathrm{Y}(F) \quad , \qquad \mathrm{x}[n] * \mathrm{y}[n] \xleftrightarrow{\mathcal{F}} \mathrm{X}(e^{j\Omega})\mathrm{Y}(e^{j\Omega})$$

$$\mathrm{x}[n]\mathrm{y}[n] \xleftrightarrow{\mathcal{F}} \mathrm{X}(F) \circledast \mathrm{Y}(F)^{\text{see note}} \quad , \qquad \mathrm{x}[n]\mathrm{y}[n] \xleftrightarrow{\mathcal{F}} (1/2\pi)\mathrm{X}(e^{j\Omega}) \circledast \mathrm{Y}(e^{j\Omega})^{\text{see note}}$$

$$\sum_{n=-\infty}^{\infty} e^{j2\pi Fn} = \delta_1(F) \quad , \qquad \sum_{n=-\infty}^{\infty} e^{j\Omega n} = 2\pi\delta_{2\pi}(\Omega)$$

$$\sum_{n=-\infty}^{\infty} \left|\mathrm{x}[n]\right|^2 = \int_1 \left|\mathrm{X}(F)\right|^2 dF \quad , \qquad \sum_{n=-\infty}^{\infty} \left|\mathrm{x}[n]\right|^2 = (1/2\pi)\int_{2\pi} \left|\mathrm{X}(e^{j\Omega})\right|^2 d\Omega$$

(Note: $\mathrm{x}(t) \circledast \mathrm{y}(t) = \int_T \mathrm{x}(\tau)\mathrm{y}(t-\tau)\,d\tau$ where T is a period of both x and y)

DTFT Properties

Find the inverse DTFT of

$$X(F) = \left[\text{rect}\left(50\left(F - 1/4\right)\right) + \text{rect}\left(50\left(F + 1/4\right)\right)\right] * \delta_1(F)$$

Start with

$$\text{sinc}(n/w) \xleftrightarrow{\mathcal{F}} w\,\text{rect}(wF) * \delta_1(F)$$

In this case $w = 50$.

$$(1/50)\,\text{sinc}(n/50) \xleftrightarrow{\mathcal{F}} \text{rect}(50F) * \delta_1(F)$$

Then, using the frequency-shifting property

$$e^{j2\pi F_0 n}\,x[n] \xleftrightarrow{\mathcal{F}} X(F - F_0)$$

$$e^{j\pi n/2}(1/50)\,\text{sinc}(n/50) \xleftrightarrow{\mathcal{F}} \text{rect}\left(50\left(F - 1/4\right)\right) * \delta_1(F)$$

DTFT Properties

Frequency shifting the other direction

$$e^{-j\pi n/2}(1/50)\operatorname{sinc}(n/50) \xleftrightarrow{\mathcal{F}} \operatorname{rect}(50(F+1/4)) * \delta_1(F)$$

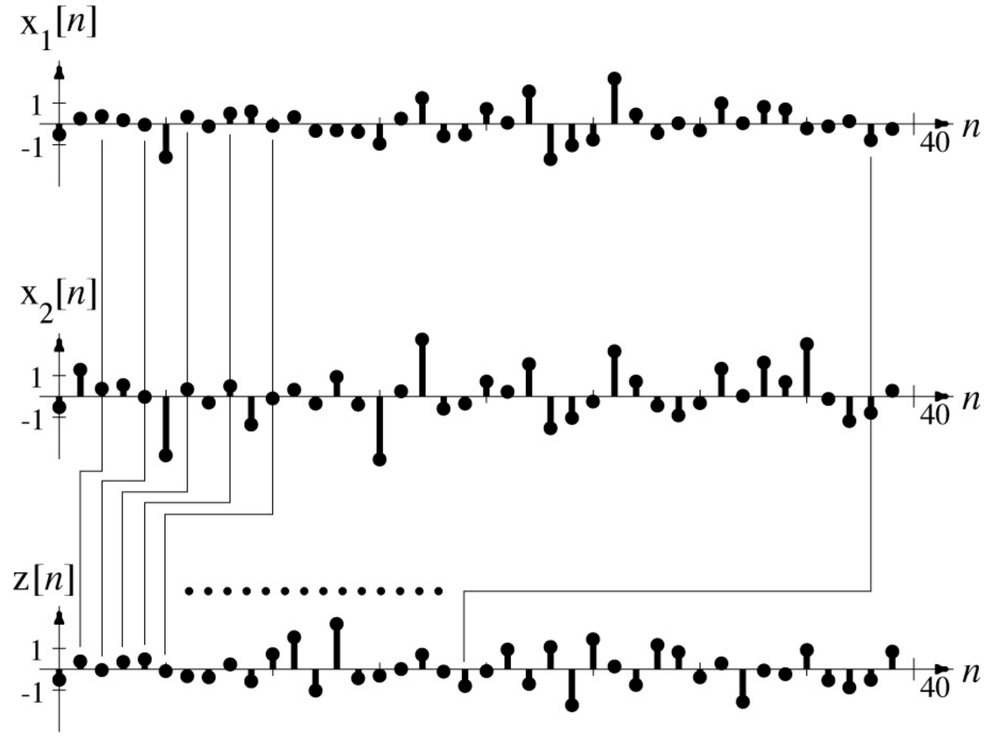
Combining the last two results and using $\cos(x) = \dfrac{e^{jx}+e^{-jx}}{2}$

$$(1/25)\operatorname{sinc}(n/50)\cos(\pi n/2) \xleftrightarrow{\mathcal{F}}$$

$$\left[\operatorname{rect}(50(F-1/4)) + \operatorname{rect}(50(F+1/4))\right] * \delta_1(F)$$

DTFT Properties

Time scaling in discrete time is quite different from time scaling in continuous-time. Let $\mathrm{z}[n] = \mathrm{x}[an]$. If a is not an integer, some values of $\mathrm{z}[n]$ are undefined and a DTFT cannot be found for it. If a is an integer greater than one, some values of $\mathrm{x}[n]$ will not appear in $\mathrm{z}[n]$ because of decimation and there cannot be a unique relationship between their DTFT's



DTFT Properties

Time scaling does not work for time compression because of decimation. But it does work for a special type of time expansion.

Let $z[n] = \begin{cases} x[n/m] & , \ n/m \text{ an integer} \\ 0 & , \ \text{otherwise} \end{cases}$. Then $Z(F) = X(mF)$.

So the time-scaling property of the DTFT is

$$z[n] = \begin{cases} x[n/m] & , \ n/m \text{ an integer} \\ 0 & , \ \text{otherwise} \end{cases} \quad , \quad \begin{array}{c} z[n] \xleftrightarrow{\mathcal{F}} X(mF) \\ \text{or } z[n] \xleftrightarrow{\mathcal{F}} X(e^{jm\Omega}) \end{array}$$

DTFT Properties

In the time domain, the response of a system is the convolution of the excitation with the impulse response of the system

$$y[n] = x[n] * h[n]$$

In the frequency domain the response of a system is the product of the excitation and the frequency response of the system

$$Y(e^{j\Omega}) = X(e^{j\Omega}) H(e^{j\Omega})$$

DTFT Properties

Find the signal energy of $\mathrm{x}[n] = (1/5)\mathrm{sinc}(n/100)$. The straightforward way of finding signal energy is directly from the definition $E_x = \sum_{n=-\infty}^{\infty} |\mathrm{x}[n]|^2$.

$$E_x = \sum_{n=-\infty}^{\infty} |(1/5)\mathrm{sinc}(n/100)|^2 = (1/25) \sum_{n=-\infty}^{\infty} \mathrm{sinc}^2(n/100)$$

In this case we run into difficulty because we don't know how to sum this series.

DTFT Properties

We can use Parseval's theorem to find the signal energy from the DTFT of the signal.

$$(1/5)\text{sinc}(n/100) \xleftrightarrow{\mathcal{F}} 20\text{rect}(100F) * \delta_1(F)$$

Parseval's theorem is

$$\sum_{n=-\infty}^{\infty} |\text{x}[n]|^2 = \int_1 |\text{X}(F)|^2 \, dF$$

For this case

$$E_x = \int_1 |20\text{rect}(100F) * \delta_1(F)|^2 \, dF = \int_{-1/2}^{1/2} |20\text{rect}(100F)|^2 \, dF$$

$$E_x = 400 \int_{-1/200}^{1/200} dF = 4$$

Transform Method Comparisons

A system with transfer function

$$\mathrm{H}(z)=\frac{z}{(z-0.3)(z+0.8)} \quad , \quad |z|>0.8$$

is excited by a unit sequence. Find the total response.

Using the DTFT

$$\mathrm{Y}(e^{j\Omega})=\mathrm{H}(e^{j\Omega})\mathrm{X}(e^{j\Omega})=\underbrace{\frac{e^{j\Omega}}{(e^{j\Omega}-0.3)(e^{j\Omega}+0.8)}}_{\text{DTFT of Impulse Response}}\times\underbrace{\left(\frac{1}{1-e^{-j\Omega}}+\pi\delta_{2\pi}(\Omega)\right)}_{\text{DTFT of Unit Sequence}}$$

$$\mathrm{Y}(e^{j\Omega})=\frac{e^{j2\Omega}}{(e^{j\Omega}-0.3)(e^{j\Omega}+0.8)(e^{j\Omega}-1)}+\pi\frac{e^{j\Omega}}{(e^{j\Omega}-0.3)(e^{j\Omega}+0.8)}\delta_{2\pi}(\Omega)$$

$$\mathrm{Y}(e^{j\Omega})=\frac{-0.1169}{e^{j\Omega}-0.3}+\frac{0.3232}{e^{j\Omega}+0.8}+\frac{0.7937}{e^{j\Omega}-1}+\frac{\pi}{(1-0.3)(1+0.8)}\delta_{2\pi}(\Omega)$$

Transform Method Comparisons

Using the equivalence property of the impulse and the periodicity of both $\delta_{2\pi}(\Omega)$ and $e^{j\Omega}$

$$\mathrm{Y}\left(e^{j\Omega}\right)=\frac{-0.1169\,e^{-j\Omega}}{1-0.3e^{-j\Omega}}+\frac{0.3232\,e^{-j\Omega}}{1+0.8\,e^{-j\Omega}}+\frac{0.7937\,e^{-j\Omega}}{1-e^{-j\Omega}}+2.4933\delta_{2\pi}(\Omega)$$

$$\mathrm{Y}\left(e^{j\Omega}\right)=\frac{-0.1169\,e^{-j\Omega}}{1-0.3e^{-j\Omega}}+\frac{0.3232\,e^{-j\Omega}}{1+0.8\,e^{-j\Omega}}+0.7937\left(\frac{e^{-j\Omega}}{1-e^{-j\Omega}}+\pi\delta_{2\pi}(\Omega)\right)\underbrace{-0.7937\pi\delta_{2\pi}(\Omega)+2.4933\delta_{2\pi}(\Omega)}_{=0}$$

$$\mathrm{Y}\left(e^{j\Omega}\right)=\frac{-0.1169\,e^{-j\Omega}}{1-0.3e^{-j\Omega}}+\frac{0.3232\,e^{-j\Omega}}{1+0.8\,e^{-j\Omega}}+0.7937\left(\frac{e^{-j\Omega}}{1-e^{-j\Omega}}+\pi\delta_{2\pi}(\Omega)\right)$$

$$\mathrm{y}[n]=\left[-0.1169(0.3)^{n-1}+0.3232(-0.8)^{n-1}+0.7937\right]\mathrm{u}[n-1]$$

Transform Method Comparisons

A system with transfer function

$$\mathrm{H}(z) = \frac{z}{z - 0.9} \quad , \quad |z| > 0.9$$

is excited by the sinusoid $\mathrm{x}[n] = \cos(2\pi n / 12)$. Find the response.

$$\cos(2\pi F_0 n) \xleftrightarrow{\mathcal{F}} (1/2)\left[\delta_1(F - F_0) + \delta_1(F + F_0)\right]$$

$$\mathrm{Y}(F) = \underbrace{\frac{e^{j2\pi F}}{e^{j2\pi F} - 0.9}}_{\substack{\text{DTFT of Impulse} \\ \text{Response}}} \times \underbrace{(1/2)\left[\delta_1(F - 1/12) + \delta_1(F + 1/12)\right]}_{\text{DTFT of Excitation}}$$

$$\mathrm{Y}(F) = (1/2)\left[e^{j2\pi F}\frac{\delta_1(F - 1/12)}{e^{j2\pi F} - 0.9} + e^{j2\pi F}\frac{\delta_1(F + 1/12)}{e^{j2\pi F} - 0.9}\right]$$

Transform Method Comparisons

Using the equivalence property of the impulse and fact that both $e^{j2\pi F}$ and $\delta_1(F)$ have a fundamental period of one,

$$\mathrm{Y}(F) = (1/2)\left[e^{j\pi/6} \frac{\delta_1(F-1/12)}{e^{j\pi/6} - 0.9} + e^{-j\pi/6} \frac{\delta_1(F+1/12)}{e^{-j\pi/6} - 0.9} \right]$$

Finding a common denominator and simplifying,

$$\mathrm{Y}(F) = (1/2) \frac{\delta_1(F-1/12)\left(1-0.9e^{j\pi/6}\right) + \delta_1(F+1/12)\left(1-0.9e^{-j\pi/6}\right)}{1.81 - 1.8\cos(\pi/6)}$$

$$\mathrm{Y}(F) = 0.4391\left[\delta_1(F-1/12) + \delta_1(F+1/12)\right] + j0.8957\left[\delta_1(F+1/12) - \delta_1(F-1/12)\right]$$

$$\mathrm{y}[n] = 0.8782\cos(2\pi n/12) + 1.7914\sin(2\pi n/12)$$

$$\mathrm{y}[n] = 1.995\cos(2\pi n/12 - 1.115)$$

Transform Method Comparisons

The DFT can often be used to find the DTFT of a signal. The DTFT is defined by $\mathrm{X}(F)=\sum_{n=-\infty}^{\infty}\mathrm{x}[n]e^{-j2\pi Fn}$ and the DFT is defined by $\mathrm{X}[k]=\sum_{n=0}^{N-1}\mathrm{x}[n]e^{-j2\pi kn/N}$. If the signal $\mathrm{x}[n]$ is causal and time limited, the summation in the DTFT is over a finite range of n values beginning with 0 and we can set the value of N by letting $N-1$ be the last value of n needed to cover that finite range. Then $\mathrm{X}(F)=\sum_{n=0}^{N-1}\mathrm{x}[n]e^{-j2\pi Fn}$. Now let $F\rightarrow k/N$ yielding

$$\mathrm{X}(k/N)=\sum_{n=0}^{N-1}\mathrm{x}[n]e^{-j2\pi kn/N}=\mathrm{X}[k]$$

Transform Method Comparisons

The result

$$X(k/N) = \sum_{n=0}^{N-1} x[n] e^{-j2\pi kn/N} = X[k]$$

is the DTFT of $x[n]$ at a discrete set of frequencies $F = k/N$ or $\Omega = 2\pi k/N$. If that resolution in frequency is not sufficient, N can be made larger by augmenting the previous set of $x[n]$ values with zeros. That reduces the space between frequency points thereby increasing the resolution. This technique is called **zero padding**.

Transform Method Comparisons

We can also use the inverse DFT to approximate the inverse DTFT.

The inverse DTFT is defined by $\mathrm{x}[n] = \int_1 \mathrm{X}(F) e^{j2\pi Fn}\, dF$

and the inverse DFT is defined by $\mathrm{x}[n] = \frac{1}{N}\sum_{k=0}^{N-1} \mathrm{X}[k] e^{j2\pi kn/N}$.

We can approximate the inverse DTFT by

$$\mathrm{x}[n] \cong \sum_{k=0}^{N-1} \int_{k/N}^{(k+1)/N} \mathrm{X}(k/N) e^{j2\pi Fn}\, dF = \sum_{k=0}^{N-1} \mathrm{X}(k/N) \int_{k/N}^{(k+1)/N} e^{j2\pi Fn}\, dF$$

$$\mathrm{x}[n] \cong \sum_{k=0}^{N-1} \mathrm{X}(k/N) \frac{e^{j2\pi(k+1)n/N} - e^{j2\pi k/N}}{j2\pi n} = \frac{e^{j2\pi n/N} - 1}{j2\pi n} \sum_{k=0}^{N-1} \mathrm{X}(k/N) e^{j2\pi kn/N}$$

$$\mathrm{x}[n] \cong e^{j\pi n/N} \operatorname{sinc}(n/N) \frac{1}{N} \sum_{k=0}^{N-1} \mathrm{X}(k/N) e^{j2\pi kn/N}$$

Transform Method Comparisons

For $n << N$,

$$x[n] \cong \frac{1}{N}\sum_{k=0}^{N-1} X(k/N) e^{j2\pi kn/N}$$

This is the inverse DFT with $X[k] = X(k/N)$.

Use this result to find the inverse DTFT of

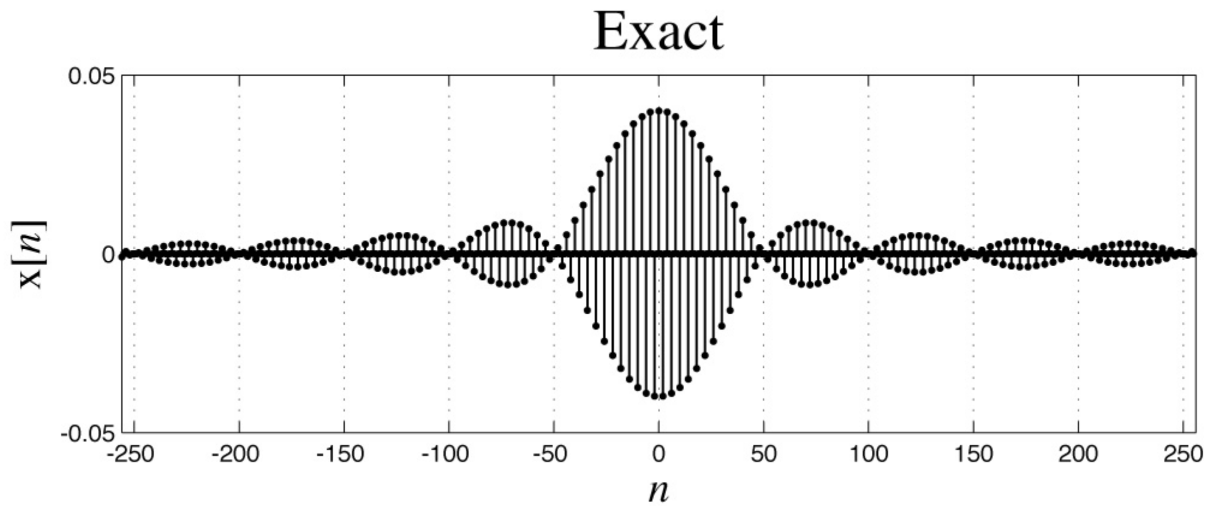
$$X(F) = \left[\text{rect}(50(F-1/4)) + \text{rect}(50(F+1/4))\right] * \delta_1(F)$$

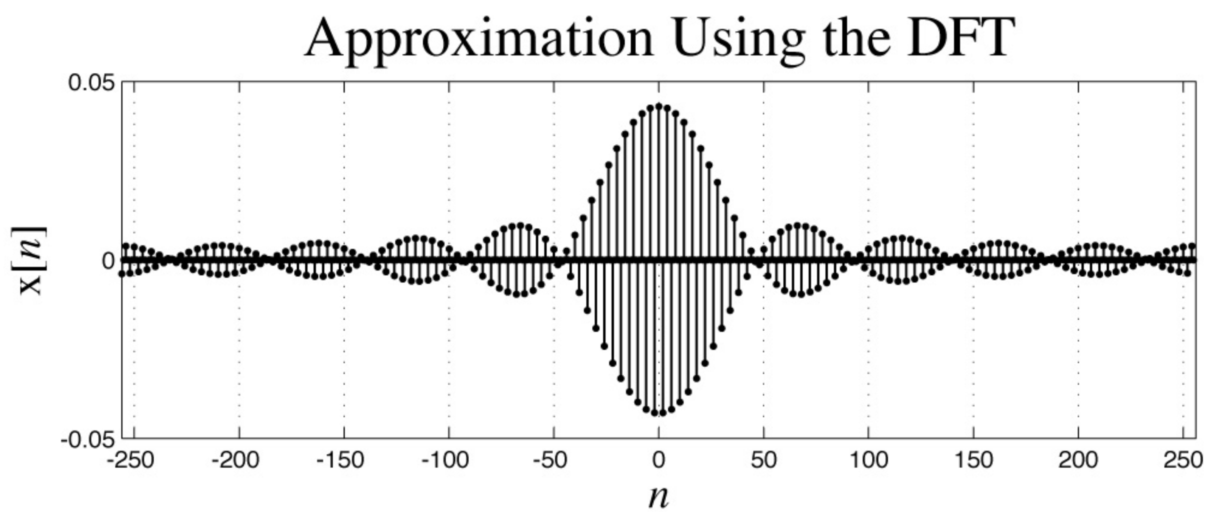
with the inverse DFT.

Transform Method Comparisons

```
N = 512 ;         %   Number of pts to approximate X(F)
k = [0:N-1]' ;   %   Harmonic numbers
%   Compute samples from X(F) between 0 and 1 assuming
%   periodic repetition with period 1
X = rect(50*(k/N - 1/4)) + rect(50*(k/N - 3/4)) ;
%   Compute the approximate inverse DTFT and
%   center the function on n = 0
xa = real(fftshift(ifft(X))) ;
n = [-N/2:N/2-1]' ;    %   Vector of discrete times for plotting
%   Compute exact x[n] from exact inverse DTFT
xe = sinc(n/50).*cos(pi*n/2)/25 ;
```

Transform Method Comparisons





The Four Fourier Methods

	Continuous Frequency	Discrete Frequency
Continuous Time	CTFT	CTFS
Discrete Time	DTFT	DFT

Relations Among Fourier Methods

Multiplication-Convolution Duality

	Discrete Frequency	Continuous Frequency
Continuous Time	$\mathrm{x}(t)\mathrm{y}(t) \xleftrightarrow{\mathcal{FS}} \mathrm{X}[k] * \mathrm{Y}[k]$	$\mathrm{x}(t)\mathrm{y}(t) \xleftrightarrow{\mathcal{F}} \mathrm{X}(f) * \mathrm{Y}(f)$
Discrete Time	$\mathrm{x}[n]\mathrm{y}[n] \xleftrightarrow{\mathcal{FS}} \mathrm{Y}[k] \circledast \mathrm{X}[k]$	$\mathrm{x}[n]\mathrm{y}[n] \xleftrightarrow{\mathcal{F}} \mathrm{X}(F) \circledast \mathrm{Y}(F)$

	Discrete Frequency	Continuous Frequency
Continuous Time	$\mathrm{x}(t) \circledast \mathrm{y}(t) \xleftrightarrow{\mathcal{FS}} T_0\, \mathrm{X}[k]\mathrm{Y}[k]$	$\mathrm{x}(t) * \mathrm{y}(t) \xleftrightarrow{\mathcal{F}} \mathrm{X}(f)\mathrm{Y}(f)$
Discrete Time	$\mathrm{x}[n] \circledast \mathrm{y}[n] \xleftrightarrow{\mathcal{FS}} N_0\, \mathrm{Y}[k]\mathrm{X}[k]$	$\mathrm{x}[n] * \mathrm{y}[n] \xleftrightarrow{\mathcal{F}} \mathrm{X}(F)\mathrm{Y}(F)$

Relations Among Fourier Methods

Parseval's Theorem

	Discrete Frequency	Continuous Frequency
Continuous Time	$\frac{1}{T_0}\int_{T_0} \left\|x(t)\right\|^2 dt = \sum_{k=-\infty}^{\infty} \left\|X[k]\right\|^2$	$\int_{-\infty}^{\infty} \left\|x(t)\right\|^2 dt = \int_{-\infty}^{\infty} \left\|X(f)\right\|^2 df$
Discrete Time	$\sum_{n=\langle N \rangle} \left\|x[n]\right\|^2 = \frac{1}{N}\sum_{k=\langle N \rangle} \left\|X[k]\right\|^2$	$\sum_{n=-\infty}^{\infty} \left\|x[n]\right\|^2 = \int_{1} \left\|X(F)\right\|^2 dF$

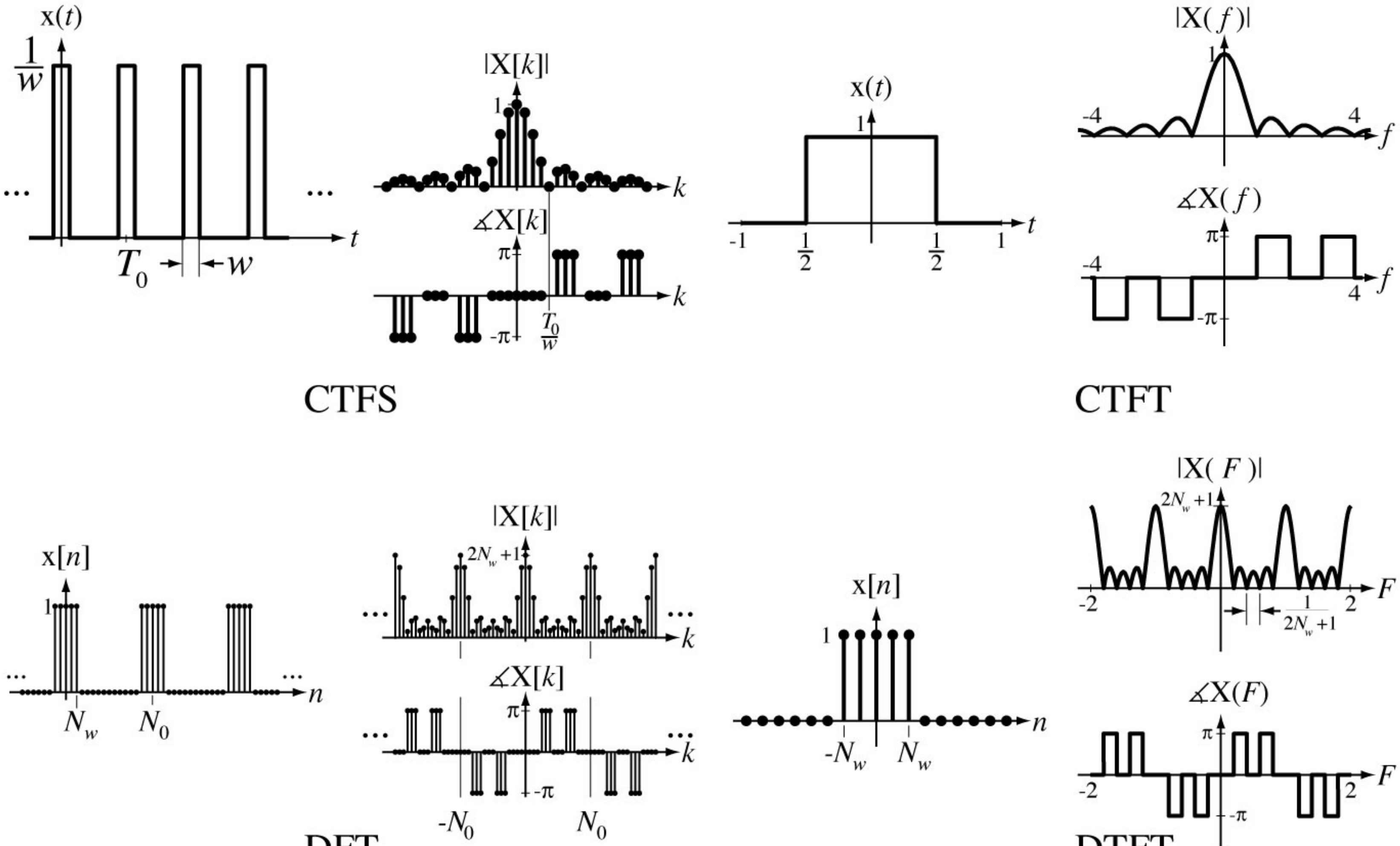
Relations Among Fourier Methods

Time and Frequency Shifting

	Discrete Frequency	Continuous Frequency
Continuous Time	$\mathrm{x}(t-t_0) \xleftrightarrow{\mathcal{FS}} \mathrm{c}_\mathrm{x}[k]e^{-j2\pi k t_0/N}$	$\mathrm{x}(t-t_0) \xleftrightarrow{\mathcal{F}} \mathrm{X}(j\omega)e^{-j\omega t_0}$
Discrete Time	$\mathrm{x}[n-n_0] \xleftrightarrow{\mathcal{FS}} \mathrm{X}[k]e^{-j2\pi k n_0/N}$	$\mathrm{x}[n-n_0] \xleftrightarrow{\mathcal{F}} \mathrm{X}(e^{j\Omega})e^{-j\Omega n_0}$

	Discrete Frequency	Continuous Frequency
Continuous Time	$\mathrm{x}(t)e^{+j2\pi k t/T} \xleftrightarrow[T]{\mathcal{FS}} \mathrm{c}_\mathrm{x}[k-k_0]$	$\mathrm{x}(t)e^{+j\omega_0 t} \xleftrightarrow{\mathcal{F}} \mathrm{X}(j(\omega-\omega_0))$
Discrete Time	$\mathrm{x}[n]e^{+j2\pi k n/N} \xleftrightarrow[N]{\mathcal{FS}} \mathrm{X}[k-k_0]$	$\mathrm{x}[n]e^{+j\Omega_0 n} \xleftrightarrow{\mathcal{F}} \mathrm{X}(e^{j(\Omega-\Omega_0)})$

Relations Among Fourier Methods



CTFS

CTFT

DFT

DTFT