Comparison Between the CTFT of One Cycle of a Sine Wave and the DFT of Samples Taken From It

Let $\mathrm{x}(t) = A\sin(2\pi t/T)\mathrm{rect}\left(\frac{t-T/2}{T}\right)$ and let the time between samples be $T_s = T/N$ where N is the number of points used in the DFT. The CTFT of $\mathrm{x}(t)$ is

$$\mathrm{X}(f) = (jA/2)\left[\delta(f+1/T) - \delta(f-1/T)\right] * T\,\mathrm{sinc}(Tf)e^{-j\pi fT}$$

$$\mathrm{X}(f) = (jAT/2)\left[\mathrm{sinc}\left(T(f+1/T)\right)e^{-j\pi(f+1/T)T} - \mathrm{sinc}\left(T(f-1/T)\right)e^{-j\pi(f-1/T)T}\right]$$

$$\mathrm{X}(f) = (jAT/2)\left[\mathrm{sinc}(Tf+1)e^{-j\pi(Tf+1)} - \mathrm{sinc}(Tf-1)e^{-j\pi(Tf-1)}\right]$$

The DFT of samples taken from $\mathrm{x}(t)$ is

$$\mathrm{X}[k] = \sum_{n=0}^{N-1}\mathrm{x}[n]e^{-j2\pi kn/N} = A\sum_{n=0}^{N-1}\sin(2\pi nT_s/T)e^{-j2\pi kn/N}$$

$$\mathrm{X}[k] = \frac{A}{j2}\sum_{n=0}^{N-1}\left(e^{j2\pi n/N} - e^{-j2\pi n/N}\right)e^{-j2\pi kn/N} = \frac{A}{j2}\sum_{n=0}^{N-1}\left(e^{j2\pi n(1-k)/N} - e^{j2\pi n(-1-k)/N}\right)$$

$$\mathrm{X}[k] = \frac{A}{j2}\left[\frac{1-e^{j2\pi(1-k)}}{1-e^{j2\pi(1-k)/N}} - \frac{1-e^{j2\pi(-1-k)}}{1-e^{j2\pi(-1-k)/N}}\right]$$

$$= \frac{A}{j2}\left[\frac{e^{-j\pi(k-1)}}{e^{-j\pi(k-1)/N}}\frac{e^{j\pi(k-1)} - e^{-j\pi(k-1)}}{e^{j\pi(k-1)/N} - e^{-j\pi(k-1)/N}} - \frac{e^{-j\pi(k+1)}}{e^{-j\pi(k+1)/N}}\frac{e^{j\pi(k+1)} - e^{-j\pi(k+1)}}{e^{j\pi(k+1)/N} - e^{j\pi(k+1)/N}}\right]$$

$$\mathrm{X}[k] = \frac{A}{j2}\left[\frac{e^{-j\pi(k-1)}}{e^{-j\pi(k-1)/N}}\frac{\sin(\pi(k-1))}{\sin(\pi(k-1)/N)} - \frac{e^{-j\pi(k+1)}}{e^{-j\pi(k+1)/N}}\frac{\sin(\pi(k+1))}{\sin(\pi(k+1)/N)}\right]$$

$$\mathrm{X}[k] = \frac{A}{j2}\left[\frac{e^{-j\pi(k-1)}}{e^{-j\pi(k-1)/N}}N\,\mathrm{drcl}(\pi(k-1)/N, N) - \frac{e^{-j\pi(k+1)}}{e^{-j\pi(k+1)/N}}N\,\mathrm{drcl}(\pi(k+1)/N, N)\right]$$

The CTFT of $\mathrm{x}(t)$, evaluated at the discrete frequencies kf_s/N is

$$X\left(k\underbrace{f_s/N}_{=1/T}\right) = (jAT/2)\left[\operatorname{sinc}\left(\underbrace{Tkf_s/N}_{=k}+1\right)e^{-j\pi(Tkf_s/N+1)} - \operatorname{sinc}\left(\underbrace{Tkf_s/N}_{=k}-1\right)e^{-j\pi(Tkf_s/N-1)}\right]$$

$$X(k/T) = (jAT/2)\left[\operatorname{sinc}(k+1)e^{-j\pi(k+1)} - \operatorname{sinc}(k-1)e^{-j\pi(k-1)}\right]$$

k	$\cdots$	-3	-2	-1	0	1	2	3	$\cdots$
$X(k/T)$	$\cdots$	0	0	$jAT/2$	0	$-jAT/2$	0	0	$\cdots$

The product of $X[k]$ and T_s is

$$T_s X[k] = \frac{A\overbrace{NT_s}^{=T}}{j2}\left[\frac{e^{-j\pi(k-1)}}{e^{-j\pi(k-1)/N}}\operatorname{drcl}(\pi(k-1)/N,N) - \frac{e^{-j\pi(k+1)}}{e^{-j\pi(k+1)/N}}\operatorname{drcl}(\pi(k+1)/N,N)\right]$$

and

k	$\cdots$	-3	-2	-1	0	1	2	3	$\cdots$
$T_s X[k]$	$\cdots$	0	0	$jAT/2$	0	$-jAT/2$	0	0	$\cdots$

For this special case, the use of the DFT to approximate the CTFT yields an exact result. This is not true in general.